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SOME INCLUSIONS IN MULTIPLIERS

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G is a compact abelian group. The main result of this paper is that if T is a $(p, 1)$ multiplier, $1 < p \leqq 2$, then T is a (p, s) multiplier for all s in the range $1 \leqq s < p$ and also an (r, r) multiplier for $p < r < p'$ (p' conjugate of p).

An operator T defined on $L^p(G)$, whose range lies in the set of measurable functions on G is said to be of *weak type* (p, q) if there is a number A such that

$$m(\{x \in G \colon |Tf(x)| > t\}) \leqq \left(\frac{A\|f\|_p}{t}\right)^q$$

for all $f \in L^p$ and all $t > 0$. (m is Haar measure.) T need not be linear.

A linear operator, defined on $L^p(G)$ is said to be of *strong type* (p, q) if there exists a number A such that

$$\|Tf\|_q \leqq A\|f\|_p .$$

If T is of strong type (p, q) and commutes with translations (or equivalently with convolutions), then T is called a (p, q) *multiplier*. In this case we write $T \in M_p^q$. The Banach space of (p, p) multipliers is denoted M_p.

If $T \in M_p^q$, then there is a function φ on the dual of G such that $(Tf)^\wedge = \varphi\hat{f}$, for all $f \in L^p$, where $^\wedge$ denotes the Fourier transformation. T and φ are in one-to-one correspondence and we shall often write T_φ for T.

Using a deep theorem of E. M. Stein [3] on limits of sequences of operators we prove the following theorems:

THEOREM 1. *If $T \in M_p^1, 1 \leqq p \leqq 2$ then T is of weak type (p, p).*

THEOREM 2. (*converse.*) *Let T be a linear map of $L^p, 1 < p \leqq 2$ which commutes with translation and is of weak type (p, p). Then T is of strong type (p, s) for all s in the range $1 \leqq s < p$.*

These theorems imply the following corollaries.

COROLLARY 1. *If $T \in M_p^1, 1 < p \leqq 2$, then $T \in M_p^s$ for all s in the range $1 \leqq s < p$.*

COROLLARY 2. *If $T \in M_p^1, 1 \leqq p \leqq 2$, then $T \in M_r$ for all r in the* *range* $p < r < p'$ (p' *conjugate of* p).

LEMMA 1. *Let* $T = T_\varphi \in M_p^1, 1 < p \leqq 2$ *and let* $g \in L^1$. *Then* $T_{\hat{g}\varphi}$ *is of weak type* (p, p)

Proof. The operator $T_{\hat{g}\varphi}: L^p \to L^1$ is defined by $(T_{\hat{g}\varphi} f)\hat{} = \hat{g}\varphi\hat{f} = \hat{g}(Tf)\hat{}$. Let g_n be the function g cut off at n and put $T_n = T_{\hat{g}_n\varphi}: L^p \to L^1$. If $f \in L^p$, then $T_n f = g_n * Tf$ and since g_n is bounded, then $T_n f \in L^\infty \subset L^p$. Thus T_n is of strong type (p, p), by the closed graph theorem.

We shall show that for any $f \in L^p$ we have

$$\limsup |T_n f(x)| < \infty \quad \text{for almost every } x . \tag{1}$$

In fact, by Fubini's theorem

$$|g| * |Tf|(g) < \infty \quad \text{on a set } E \text{ of measure } 1 .$$

By dominated convergence

$$T_n f(x) = g_n * Tf(x) \longrightarrow g * Tf(x) \quad x \in E \text{ which proves (1)} .$$

A theorem of Stein [3] states that (1) implies that the operator T^* defined by

$$T^* f(x) = \sup |T_n f(x)|$$

is of weak type (p, p). A fortiori the operator $T_{\hat{g}\varphi}$ is of weak type (p, p) and the lemma is proved.

LEMMA 2. (*Varopoulos-Johnson-Rieffel.*) *Let* $f_n \in L^p, f_n \to 0$ in L^p. *Then there are* $g \in L^1$ *and* $g_n \in L^p$ *such that* $f_n = g * g_n$ *and* $g_n \to 0$ *in* L^p.

For a proof see Rieffel [2].

THEOREM 1. *If* $T_\varphi \in M_p^1, 1 \leqq p \leqq 2$ *then* T *is of weak type* (p, p).

Proof. Assume T_φ is not of weak type (p, p). Then, to every positive integer n there corresponds a real number t_n and $f_n \in L^p$ such that

$$m(\{x: |T_\varphi f_n(x)| > t_n\}) > \left(\frac{n \| f_n \|_p}{t_n}\right)^p . \tag{1}$$

Multiplying t_n and f_n by an appropriate constant we may suppose that $\| f_n \|_p = n^{-1/2}$. We now apply Lemma 2, writing

$$f_n = g*g_n\ , \qquad g \in L^1$$

$$(2) \qquad g_n \longrightarrow 0 \quad \text{in } L^p\ .$$

By Lemma 1, there is a constant $A = A_g$ such that

$$m(\{x\colon |T_{\hat{g}\varphi}g_n(x)| > t_n\}) \leqq \left(\frac{A\,||g_n||_p}{t_n}\right)^p .$$

This is

$$m(\{x\colon |T_\varphi f_n(x)| > t_n\}) \leqq \left(\frac{A\,||g_n||_p}{t_n}\right)^p .$$

Hence, by (1)

$$\frac{n\cdot n^{-1/2}}{t_n} = \frac{n\,||f_n||_p}{t_n} \leqq \frac{A\,||g_n||_p}{t_n}$$

$$n^{1/2} \leqq A\,||g_n||_p\ .$$

This contradicts (2). Therefore T_φ is of weak type (p, p) and the theorem is proved

THEOREM 2. (*converse.*) *Let T be a linear map of L^p, $1 < p \leqq 2$, which is of weak type (p, p) and which commutes with translation. Then T is of strong type (p, s) i.e., $T \in M_p^s$ for all s in the interval $1 \leqq s < p$.*

Proof. We first show that if $f \in L^p$, then $Tf \in L^s$. For, there is a constant A such that for every $f \in L^p$ and every positive t we have

$$(1) \qquad m(\{x\colon |Tf(x)| > t\}) \leqq \left(\frac{A\,||f||_p}{t}\right)^p .$$

Now it is well known (see, e.g., [1] 13.7.3) that for any nonnegative measurable function g we have

$$\int_G g^s = \int_0^\infty st^{s-1}m(\{x \in G\colon g(x) > t\})dt\ .$$

Then, by (1), for $1 \leqq s < p$

$$\int_G |Tf|^s \leqq \int_0^1 sm(G)dt + \int_1^\infty st^{s-1}\left(\frac{A\,||f||_p}{t}\right)^p dt < \infty\ .$$

This means $Tf \in L^s$. By the closed graph theorem we deduce easily $T \in M_p^s$.

COROLLARY 1. *If $1 < p \leqq 2$, $1 \leqq s < p$, then $M_p^1 = M_p^s$.*

This is an immediate consequence of Theorems 1 and 2 and of the trivial inclusion $M_p^s \subset M_p^1$.

COROLLARY 2. *If* $1 \leqq p < 2$, *then* $M_p^1 \subset \bigcap_{p<r\leqq 2} M_r$.

Proof. Let $T \in M_p^1$. By Theorem 1, T is of weak type (p, p). T if also strong type $(2, 2)$. The Marcinkiewicz interpolation theorem shows that T is of strong type (r, r) for each r satisfying $p < r \leqq 2$. Since T commutes with translation, then $T \in M_r$, for $p < r \leqq 2$.

COROLLARY 3. *If* $1 \leqq p \leqq 2$ *then* $\bigcap_{p<r\leqq 2} M_r = \bigcap_{p<r\leqq 2} M_r^1$.

Proof. One part of the inclusion is due to $M_r \subset M_r^1$. The other part is a consequence of Corollary 2.

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