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DYNAMICAL SYSTEMS OF CHARACTERISTIC 0^+

SHAIR AHMAD

The main purpose of this paper is to classify the dynamical systems on the plane which satisfy a certain type of stability criterion. Such flows are referred to as dynamical systems of characteristic 0^+. The classification is based on the consideration of three mutually exclusive and exhaustive cases: Dynamical systems of characteristic 0^+ which have no critical points. Those whose critical points form nonempty compact sets, and those whose critical points do not form compact sets.

Dynamical systems of characteristic 0^+ are those dynamical systems in which all closed positively invariant sets are positively D-stable, i.e., stable in Ura's sense (see [11]). If the phase space of a flow is regular, then a closed positively invariant set, which is positively stable in Liapunov's sense, is also positively D-stable. Thus, some simple examples of flows of characteristic 0^+ are those where the phase spaces are regular and all closed invariant sets are positively stable in Liapunov's sense.

In § 2 we give some of the basic definitions and notations that are used throughout the paper. In § 3 we prove some results of a more general nature which are later applied to flows of characteristic 0^+ on the plane. It is proved that if the phase space X of a flow is normal and connected and a closed invariant set F is globally $+$ asymptotically stable, then F is connected. Further, if the phase space X of a flow of characteristic 0^+ is connected and locally compact, then a compact subset M of X is a positive attractor implies that M is globally $+$ asymptotically stable.

In § 4 we discuss flows of characteristic 0^+ on the plane. It is shown that if the set of critical points S of such a flow is empty, then the flow is parallelizable. If S is compact, then it either consists of a single point which is a Poincaré center, or it is globally $+$ asymptotically stable. If S is not compact, then either $R^2 = S$, or S is $+$ asymptotically stable; S and the region of positive attraction $A^+(S)$ of S each has a countable number of components. Further, each component of $A^+(S)$ is homeomorphic to R^2. At the end of this section, we summarize all the results of this section in the form of a complete classification of such flows.

In § 5 we discuss flows of characteristic $0^\pm$ on the plane, i.e., those in which every closed invariant set is positively and negatively stable in Ura's sense. We prove that such a flow is either parallelizable, or it has a single critical point which is a global Poincaré center, or all points are critical points.

2. **Notations and definitions.** Let R, R^+, and R^- denote the sets of real numbers, nonnegative, and nonpositive real numbers, respectively. Given a topological space X and a mapping π of the product space $X \times R$ into X, we say (X, π) defines a *dynamical system* or *flow* on the phase space X if the following conditions are satisfied.

1. *Identity axiom*: $\pi(x, 0) = x$.
2. *Homomorphism axiom*: $\pi(\pi(x, t), s) = \pi(x, s + t)$.
3. *Continuity axiom*: π is continuous on $X \times R$.

For brevity, we denote $\pi(x, t)$ by xt. For each $x \in X$, we let $C(x)$ denote the *trajectory* or *orbit* through x, i.e., $C(x) = xR$. Similarly, the *positive* and *negative semi-trajectories* through x are represented by $C^+(x)$ and $C^-(x)$, respectively, i.e., $C^+(x) = xR^+$ and $C^-(x) = xR^-$. We let $L^+(x)$ denote the *positive* (or ω-) *limit set* of x, i.e., $L^+(x) = \bigcap\{\overline{C^+(xt)}: t \in R\}$. Similarly, $L^-(x)$ denotes the *negative* (or α-) *limit set* of x. A point x is called a *critical* or *rest* point if $xR = x$. A subset M of X is said to be *invariant* if $C(M) = M$, and *positively* (negatively) *invariant* if $C^+(M) = M(C^-(M) = M)$. A closed invariant set M is *minimal* if it has no proper subset which is closed and invariant.

Throughout this paper, we use ∂M and $\bar{M}$ to represent the boundary and closure of M. Given a Jordan curve C on the plane R^2, we let int (C) denote the bounded component of $R^2 - C$. Let $(R^2)^* = R^2 \cup \{\omega\}$ be the one point compactification of the plane.

A closed positively invariant set M is said to be *positively Liapunov stable*, or more simply, *positively stable*, if for every neighborhood U of M, there exists a neighborhood V of M such that $C^+(V) \subset U$. M is said to be a *positive attractor* if there exists a neighborhood U of M such that $\varphi \neq L(x) \subset M$ for all x in U. The largest such neighborhood U is called the *region of positive attraction of* M and will be denoted by $A^+(M)$. M is said to be $+$ *asymptotically stable* if it is both positively stable and a positive attractor. It is said to be globally $+$ asymptotically stable if it is $+$ asymptotically stable and $A^+(M) = X$.

For each $x \in X$, the (first) *positive* (*negative*) *prolongation* $D^+(x)$ $(D^-(x))$ of x is given by

$$D^+(x) = \bigcap_{N \in \eta(x)} \{\overline{C^+(N)}\} \qquad (D^-(x) = \bigcap_{N \in \eta(x)} \{\overline{C^-(N)}\}) ,$$

where $\eta(x)$ is the neighborhood filter of x.

The (first) *positive* (*negative*) *prolongational limit set* of x is given by

$$J^+(x) = \bigcap_{t \in R} \{D^+(xt)\} \qquad (J^-(x) = \bigcap_{t \in R} \{D^-(xt)\}) \,.$$

It is known and easy to verify that $L^+(x) \subset J^+(x)$. Further, if X is a Hausdorff space, then $D^+(x) = C^+(x) \cup J^+(x)$.

A closed positively invariant set M is said to be *positively D-stable* if $D^+(M) = M$.[1]

It is easy to verify that if X is regular and a closed positively invariant set M is positively stable (i.e., stable in Liapunov's sense as defined above), it is also positively D-stable. The converse is false.

The following theorem, which we use several times in this paper, is due to Ura [11].

THEOREM (*Ura*). *Let (X, π) be a dynamical system on a locally compact space X, and let M be a compact subset of X. Then M is positively stable if and only if it is positively D-stable.*

REMARK. The statement "X is locally compact" is used in the Bourbaki sense throughout this paser, i.e., X is assumed to be a Hausdorff space.

3. Flows of characteristic 0^+. Before discussing flows of characteristic 0^+, we prove a lemma and a proposition concerning flows in general.

LEMMA 1. *Let (X, π) be any dynamical system. If $x \in X$ and $y_1, y_2 \in L^+(x)$, then $y_1 \in D^+(y_2)$ and $y_2 \in D^+(y_1)$.*

Proof. We note that

$$D^+(y_1) = \bigcap_{N \in \eta(y_1)} \{\overline{C^+(N)}\} \,,$$

where $\eta(y_1)$ denotes the neighborhood filter of y_1. Since $y_1, y_2 \in L^+(x)$, for each $N \in \eta(y_1)$ and $M \in \eta(y_2)$, there exist $t_1, t_2 \in R^+$ with $xt_1 \in N$ and $(xt_1)t_2 = x(t_1 + t_2) \in M$. Hence $y_2 \in \overline{C^+(N)}$, and consequently, $y_2 \in D^+(y_1)$. Similarly, $y_1 \in D^+(y_2)$.

PROPOSITION 3.1. *Let (X, π) be a dynamical system on a normal (and Hausdorff) connected topological space X. If a closed invariant subset F of X is globally $+$ asymptotically stable, then F is connected.*

Proof. Suppose F is not connected. Then there exist two non-

[1] The theory of prolongation and D-stability is due to Ura (see [**11**], [**12**], and [**13**]). Ura [**11**] refers to D-stability as *stability* and to Liapunov stability as L-*stability*.

empty disjoint closed sets F_1 and F_2 such that $F = F_1 \cup F_2$. Since X is normal, there exist two disjoint open neighborhoods U_1 and U_2 of F_1 and F_2, respectively. On the other hand, since F is positively stable, corresponding to the neighborhood $U = U_1 \cup U_2$ of F, there is an open neighborhood V of F such that $C^+(V) \subset U$. Therefore, if we let $V_i = V \cap U_i$, $i = 1, 2$, then for each $x \in V_i$, $C^+(x) \subset U_i$ since $C^+(x)$ is connected. Thus, $L^+(x) \subset F_i$ i.e., $V_i \subset A^+(F_i)$ since $\overline{U}_i \cap F_j = \varnothing$, $i \neq j$. Hence, we have shown that F_1 and F_2 are positive attractors; consequently $A^+(F_1)$ and $A^+(F_2)$ are open, since the boundary of each is closed and invariant. But this contradicts the assumption that X is connected, since $X = A^+(F) = A^+(F_1) \cup A^+(F_2)$, where $A^+(F_1)$ and $A^+(F_2)$ are clearly nonempty disjoint open sets. This completes the proof of Proposition 3.1.

DEFINITION 3.1. A dynamical system (X, π) is said to have characteristic 0^+ if and only if $D^+(x) = \overline{C^+(x)}$ for all $x \in X$.

The above definition is equivalent to saying that (X, π) has characteristic 0^+ if and only if every closed positively invariant subset of X is positively D-stable.

It follows that if the phase space X of a flow of characteristic 0^+ is a Hausdorff space, then $D^+(x) = C^+(x) \cup L^+(x)$, for all $x \in X$.

LEMMA 2. *Let (X, π) be a flow of characteristic 0^+. If $x \in X$ such that $L^-(x) \neq \varnothing$, then $x \in L^-(x)$.*

Proof. Suppose $L^-(x) \neq \varnothing$ and let $y \in L^-(x)$. Then, $y \in D^-(x)$, and hence $x \in D^+(y) = \overline{C^+(y)}$. On the other hand, $y \in L^-(x)$ implies that $\overline{C^+(y)} \subset L^-(x)$, since $L^-(x)$ is a closed invariant set. Therefore, $x \in L^-(x)$.

PROPOSITION 3.2. *Let (X, π) be a flow of characteristic 0^+ on a connected locally compact space X. If M is a compact positively invariant subset of X and M is a positive attractor, then M is globally $+$ asymptotically stable.*

Proof. Since M is a closed positively invariant set, we have $D^+(M) = M$. Therefore, M is positively stable by Ura's theorem. It is sufficient to show that $\partial A^+(M) = \varnothing$. Suppose that $\partial A^+(M) \neq \varnothing$, and let $x \in \partial A^+(M)$. Let $\eta_A(x)$ be the trace of the neighborhood filter $\eta(x)$ of x on $A \equiv A^+(M)$. Then, for each $N_A \in \eta_A(x)$, $\varnothing \neq L^+(N_A) \subset M$. Since M is compact, the cluster set of the filter base $\{L^+(N_A) \mid N_A \in \eta_A(x)\}$ is a nonempty subset of M; hence $J^+(x) \cap M \neq \varnothing$. However, this contradicts the assumption that (X, π) has characteristic 0^+, since $\partial A^+(M)$ is a closed invariant set disjoint with M. Therefore, $\partial A^+(M) = \varnothing$ and the proof of Proposition 3.2 is complete.

4. Flows of characteristic 0^+ on the plane. Throughout this section, we assume the phase space to be the plane R^2 and (R^2, π) to be a fixed flow of characteristic 0^+. We let S denote the set of rest points of this flow.

LEMMA 3. *For each $x \in X$, if $L^+(x) \neq \varnothing$, then $L^+(x)$ is either a periodic orbit or it consists of a single rest point.*

Proof. If $L^+(x)$ contains a rest point s_0, then $L^+(x) = \{s_0\}$. For, $y \in L^+(x)$ implies that $y \in D^+(s_0) = \{s_0\}$, by Lemma 1. Suppose that $L^+(x)$ consists of regular points only. Then, to complete the proof of the lemma, it is sufficient to prove that $L^+(x)$ is compact. We note that if $y \in L^+(x)$, then $\overline{C^+(y)} = L^+(x)$. For, $z \in L^+(x)$ implies that $z \in D^+(y) = \overline{C^+(y)}$. Also, $\overline{C^+(y)} \subset L^+(x)$ since $L^+(x)$ is a closed invariant set, and hence $\overline{C^+(y)} = L^+(x)$. Since $\overline{C^+(y)} \subset \overline{C(y)} \subset L^+(x)$, we have $\overline{C(y)} = L^+(x)$. Therefore, $L^+(x)$ is a minimal set. We recall that if M is a minimal subset of R^2 which is not compact, then for each $m \in M$, $L^\pm(m) = \varnothing$ (c.f. p. 37 of [6]). Suppose that $L^+(x)$ is not compact, and let y_1 and y_2 be two distinct points in $L^+(x)$. Then, $y_1 \in D^+(y_2) = C^+(y_2)$ and $y_2 \in D^+(y_1) = C^+(y_1)$. But, if t_1 and t_2 are positive numbers such that $y_1 = y_2t_1$ and $y_2 = y_1t_2$, then $y_1 = y_1(t_1 + t_2)$; showing that $C^+(y_1)$ is a periodic orbit. Hence, $L^+(x)$ is a periodic orbit, since $L^+(x) = C^+(y_1)$, as it is a minimal set; thus contradicting the assumption that $L^+(x)$ is not compact.

For a proof of the following theorem see [5].

THEOREM (*Bhatia*). *A flow F on a metric space X is dispersive if and only if for each $x \in X$, $D^+(x) = C^+(x)$ and there are no rest points or periodic orbits.*

THEOREM 4.1. *If $S = \varnothing$, then the flow (R^2, π) is parallelizable.*

Proof. We note that for each $x \in R^2$, $L^+(x) = \varnothing$, and hence $D^+(x) = \overline{C^+(x)} = C^+(x)$. For, if $L^+(x) \neq \varnothing$, then by Lemma 3, it must be a periodic orbit since it consists of regular points only. But this is impossible since the bounded component of a periodic orbit contains a rest point. Thus, the proof of our assertion follows from Bhatia's Theorem, stated above (c.f. Auslander [2]) and the fact that the notions of parallelizability and dispersiveness are equivalent for a flow on the plane (see Antosiewicz and Dugundji [1]).

THEOREM 4.2. *If R^2 contains a periodic point, then S is a singleton. Further, if $S = \{s_0\}$, then one of the following holds.*

1. *s_0 is a global Poincaré center.*[2]

2. *s_0 is a local Poincaré center. The neighborhood N of s_0, consisting of s_0 and periodic orbits surrounding s_0, is a globally + asymptotically stable simply connected continuum. Further, if $x \notin N$, then $L^+(x) = \partial N$.*

Proof. Let x_0 be any periodic point, and let $S_0 = \text{int}\,(C^+(x_0)) \cap S$. We note that $\text{int}\,(C^+(x_0)) \neq S_0$ since S is closed; and for each regular point x in $\text{int}\,(C^+(x_0))$, $C^+(x)$ is a periodic orbit, by virtue of Lemma 2.[3] Let $(B_\alpha)_{\alpha \in I}$ be the family of all periodic orbits such that for each $\alpha \in I$, $\text{int}\,(B_\alpha) \cap S = S_0$. Let $B = \bigcup_{\alpha \in I} \text{int}\,(B_\alpha)$. If $\partial B = \emptyset$, then $B = R^2$. Suppose that $\partial B \neq \emptyset$. Then ∂B is a closed invariant set since B is invariant. Further, $\partial B \cap S = \emptyset$. For, if $b_0 \in \partial B \cap S$, then one can choose a simple closed curve C such that $\text{int}\,(C) \cap S_0 = \emptyset$, since $S_0 \subset \text{int}\,(C^+(x_0)) \subset B$ and S_0 is closed. Clearly, there is no neighborhood W of b_0 with $C^+(W) \subset \text{int}\,(C)$, since $x \in W \cap B - S_0$ would imply that x is a periodic point, by Lemma 2, and $\text{int}\,(C^+(x)) \cap S_0 \neq \emptyset$. But this contradicts the fact that $\{b_0\}$ is positively stable, as $D^+(b_0) = \{b_0\}$; thus showing that $\partial B \cap S = \emptyset$. This also shows that ∂B is not a singleton since it is invariant and consists of regular points.

We note that if $x \in B$ and $x \notin S_0$, then x is a periodic point, by Lemma 2, with $C^+(x) \subset B$ and $\text{int}\,(C^+(x)) \cap S_0 \neq \emptyset$. For, x belongs to $\text{int}\,(B_\alpha)$ for some $\alpha \in I$. Thus, $x \notin S$ since $\text{int}\,(B_\alpha) \cap S = S_0$. Further $L^-(x) \neq \emptyset$ and $C^+(x) \subset B$ since x is surrounded by the periodic orbit B_α. Thus, x is a periodic point with $\text{int}\,(C^+(x)) \cap S_0 \neq \emptyset$ since $C^+(x) \subset \text{int}\,(B_\alpha)$ and $\text{int}\,(B_\alpha) \cap S = S_0$. Now we wish to show that ∂B is a periodic orbit. In order to accomplish this, we consider two cases.

Case 1. Suppose $\partial B \cap C^+(x_0) \neq \emptyset$. Then, since ∂B is invariant, we must have $C^+(x_0) \subset \partial B$. On the other hand, $\partial B \subset C^+(x_0)$. For, assume $\partial B \not\subset C^+(x_0)$, and let $b \in \partial B - C^+(x_0)$. Then, $b \notin \text{int}\,(C^+(x_0))$ since $\text{int}\,(C^+(x_0)) \subset B$. Thus, one can choose a neighborhood U of b such that $U \cap \overline{\text{int}\,(C^+(x_0))} = \emptyset$ since $b \notin \overline{\text{int}\,(C^+(x_0))}$, as $b \notin C^+(x_0)$ and

[2] s_0 is a global Poincaré center if for each $x \neq s_0$, $C(x)$ is a periodic orbit surrounding s_0. It is a local Poincaré center if it has a neighborhood M such that for each $x \in M - \{s_0\}$, $C(x)$ is a periodic orbit surrounding s_0.

[3] It is a known fact about flows on the plane that a point is positively (or negatively) Poisson stable if and only if it is either a rest point or a periodic point (see [10]).

$b \notin \text{int}(C^+(x_0))$. Let $x \in U \cap B$. Then, $x \notin S_0$ since $S_0 \subset \text{int}(C^+(x_0))$. Thus $C^+(x)$ is a periodic orbit. Since $\overline{\text{int}(C^+(x_0))}$ is connected, $\text{int}(C^+(x)) \cap \overline{\text{int}(C^+(x_0)} \neq \emptyset$, as $\text{int}(C^+(x)) \cap S_0 \neq \emptyset$ and

$$\partial\, \text{int}(C^+(x)) \cap \overline{\text{int}(C^+(x_0)} = C^+(x) \cap \overline{\text{int}(C^+(x_0))} = \emptyset\ ,$$

it follows that $\overline{\text{int}(C^+(x_0))} \subset \text{int}(C^+(x))$. But, $C^+(x_0) \subset \text{int}(C^+(x)) \subset B$ contradicts the assumption that $\partial B \cap C^+(x_0) \neq \emptyset$, as B is open; hence $\partial B = C^+(x_0)$.

Case 2. Suppose $\partial B \cap C^+(x_0) = \emptyset$, and let $b_1, b_2 \in \partial B$. First we show that $b_2 \in D^+(b_1)$ and $b_1 \in D^+(b_2)$. In order to show that $b_2 \in D^+(b_1)$, it is sufficient to show that if C_1 and C_2 are any simple closed curves with $b_1 \in \text{int}(C_1)$ and $b_2 \in \text{int}(C_2)$, then there exist $x_1 \in \text{int}(C_1)$ and $t_1 \in R^+$ such that $x_1t_1 \in \text{int}(C_2)$. Let $y_1 \in \text{int}(C_1) \cap B - \text{int}(C^+(x_0))$, so that y_1 is a periodic point with $\text{int}(C^+(y_1)) \cap S = S_0$. Since B is open and $b_1, b_2 \in \partial B$, there exists a point $y_2 \in \text{int}(C_2) \cap B \cap (R^2 - \overline{\text{int}(C^+(y_1))})$. Then, y_2 is a periodic point with $C^+(y_2) \subset R^2 - \overline{\text{int}(C^+(y_1))}$ and $\text{int}(C^+(y_2)) \cap S_0 \neq \emptyset$. Since $\text{int}(C^+(y_2)) \cap \overline{\text{int}(C^+(y_1))} \neq \emptyset$, $\overline{\text{int}(C^+(y_1))}$ is connected and $\partial\,\text{int}(C^+(y_2)) \cap \overline{\text{int}(C^+(y_1))} = \emptyset$, we must have $\overline{\text{int}(C^+(y_1))} \subset \text{int}(C^+(y_2))$. This implies that $\text{int}(C_1) \cap \text{int}(C^+(y_2)) \neq \emptyset$. It is also clear that $\text{int}(C_1) \cap (R^2 - \overline{\text{int}(C^+(y_2))} \neq \emptyset$ since $b_1 \in \partial B$ and B is open. Therefore, $C^+(y_2) \cap \text{int}(C_1) \neq \emptyset$ since $\text{int}(C_1)$ is connected. Certainly, for each $x_1 \in C^+(y_2) \cap \text{int}(C_1)$, there exists $t_1 \in R^+$ such that $x_1t_1 \in \text{int}(C_2)$ since $C^+(x_1) = C^+(y_2)$ and y_2 is a periodic point. This shows that $b_2 \in D^+(b_1)$. Similarly, $b_1 \in D^+(b_2)$. If $L^+(b_1) \neq \emptyset$, then it is a periodic orbit, by Lemma 3, since $\partial B \cap S = \emptyset$ and $L^+(b_1) \subset \partial B$. That $L^+(b_1) \subset \partial B$ follows from the fact that ∂B is a closed invariant set, as B is invariant. Further, $\partial B \subset L^+(b_1)$, since $b \in \partial B$ and $y \in L^+(b_1)$ implies $b \in D^+(y) = \overline{C^+(y)} = L^+(b_1)$, as $L^+(b_1)$ is a periodic orbit contained in ∂B. Therefore $\partial B = L^+(b_1)$ is a periodic orbit. Similarly, if $L^+(b_2) \neq \emptyset$, then ∂B is a periodic orbit. Suppose $L^+(b_1) = L^+(b_2) = \emptyset$. Then we must have $b_1 \in C^+(b_2)$ and $b_2 \in C^+(b_1)$, which again implies that $C^+(b_1)$ is a periodic orbit containing b_2 (see proof of Lemma 3). Thus, we conclude that ∂B is a periodic orbit.

Let $N = \partial B \cup \text{int}(\partial B)$. We wish to show that $N = \bar{B}$. Since S is closed, one can choose a simple closed curve C such that $N \subset \text{int}(C)$ and $(\text{int}(C) - N) \cap S = \emptyset$. We note the N is positively stable since $D^+(N) = N$. Thus, there exists a neighborhood V of N such that $C^+(V) \subset \text{int}(C)$. It follows that $(V - N) \cap B = \emptyset$. For, if $x \in (V - N) \cap B$, then x is a periodic point, by Lemma 2, since x is surrounded by some periodic orbit B_α. Therefore, we must have $\partial B \subset \text{int}(C^+(x))$, since $C^+(x) \subset \text{int}(C)$ and $(\text{int}(C) - N) \cap S = \emptyset$. But, it is impossible to have

$\partial B \subset \operatorname{int}(C^+(x))$ since $\operatorname{int}(C^+(x)) \subset B$. Thus, we have established that $(V - N) \cap B = \varnothing$, and hence $\operatorname{int}(\partial B) \cap B \neq \varnothing$, since $\partial B \cap B = \varnothing$, as B is open. We note that B is connected since it is the union of the family of connected sets $(\operatorname{int}(B_\alpha))_{\alpha \in I}$ with $\varnothing \neq S_0 \subset \bigcap_{\alpha \in I} \operatorname{int}(B_\alpha)$. Therefore, $B \subset \operatorname{int}(\partial B)$ since $B \cap \partial(\operatorname{int}(\partial B)) = B \cap \partial B = \varnothing$. Now, suppose $\operatorname{int}(\partial B) \neq B$. Then, clearly, $\operatorname{int}(\partial B) \cap B$ is a nonempty open set. Also, $\operatorname{int}(\partial B) - B$ is a nonempty open set. For, $x \in \operatorname{int}(\partial B) - B$ implies that $x \notin \partial B$ and $x \notin B$; hence $x \notin \bar{B}$. Let V be a neighborhood of x such that $V \cap \bar{B} = \varnothing$. Then $U = V \cap \operatorname{int}(\partial B)$ is a neighborhood of x and $U \subset \operatorname{int}(\partial B) - B$. Hence, $\operatorname{int}(\partial B)$ is disconnected; a contradiction to the Jordan Curve Theorem. We have thus shown that $N = \partial B \cup B$.

N is a simply connected continuum, by Schoenflie's Theorem. We wish to show that N is globally + asymptotically stable. In view of Proposition 3.2, it is sufficient to show that N is a positive attractor. Since N is compact and S is closed, we can choose a compact neighborhood U_0 of N such that $U_0 \cap (S - S_0) = \varnothing$. Then, there exists a neighborhood V_0 of N such that $C^+(V_0) \subset U_0$. For each $x \in V_0 - N$, $L^+(x) \neq \varnothing$ and $L^+(x) \cap S = \varnothing$. Hence, $L^+(x)$ is a periodic orbit and $S_0 \subset \operatorname{int}(L^+(x))$. Similarly, if $y \in \operatorname{int}(L^+(x)) - N$, then $S_0 \subset \operatorname{int}(L^+(y))$. It follows from the way N was constructed that $L^+(x) = \partial N$.

We note that if $B = R^2$, then $S = S_0$. Also, if $B \neq R^2$, then $S = S_0$ since $N \cap (S - S_0) = \varnothing$ and N is a globally + asymptotically stable neighborhood of S_0. In particular, since x_0 was an arbitrary periodic point, it follows that S is contained in the interior of every periodic orbit. Now, we wish to show that S is a singleton. This will complete the proof of the theorem, since $B = R^2$ will then imply the first and $B \neq R^2$ the second assertion of the theorem. Let $D = \bigcap_{\alpha \in I} \operatorname{int}(B_\alpha)$. Then, we have $S \subset D$. Suppose that D contains a regular point d. Then, $L^-(d) \neq \varnothing$ since d is surrounded by periodic orbits, and hence $C^+(d)$ is a periodic orbit (see footnote 3). But this would imply that $d \in \operatorname{int}(C^+(d))$, which is impossible. For, as we pointed out above, $S = S_0$ and S_0 is contained in the interior of every periodic orbit. Hence every periodic orbit belongs to the family $(B_\alpha)_{\alpha \in I}$ and, consequently, D is contained in the interior of every periodic orbit. Therefore, $D = S$. Let $d_1 \in \partial D$, and suppose that D contains a point d_2 distinct from d_1. Let C_1 be a simple closed curve such that $d_1 \in \operatorname{int}(C_1)$ and $d_2 \notin \operatorname{int}(C_1)$. Since $\{d_1\}$ is positively stable, there exists a neighborhood W_1 of d_1 with $C^+(W_1) \subset \operatorname{int}(C_1)$. But, if x is a regular point in $W_1 \cap B$, then we must have $D \subset \operatorname{int}(C^+(x))$, and in particular, $d_2 \in \operatorname{int}(C^+(x))$, which is impossible. This completes the proof of Theorem 4.2.

For flows of characteristic 0^+, the following theorem is a rather strong generalization of Bendixson's theorem (see [4]), which states that for every isolated critical point s on the plane, either there exists a point $y \neq s$ such that $L^+(y) = \{s\}$ or $L^-(y) = \{s\}$, or every neighborhood of s contains a periodic orbit surrounding s.

THEOREM 4.3. *If S has a compact component S_0 which is isolated from $S - S_0$, then one of the following holds.*[4]

(1) *S is a singleton and one of the two assertions of Theorem 4.2 holds.*

(2) *S_0 is globally + asymptotically stable, and consequently, $S_0 = S$.*

Proof. Let V be a compact neighborhood of S_0 such that $V \cap (S - S_0) = \emptyset$. Since $D^+(S_0) = S_0$, S_0 is positively stable. Let U be a neighborhood of S_0 such that $C^+(U) \subset V$. Then, for each $x \in U$, $L^+(x) \neq \emptyset$. If a periodic orbit exists, then the proof follows from Theorem 4.2. If there are no periodic orbits, then for each $x \in U$, $L^+(x)$ consists of a single rest point, by Lemma 3. Further, $L^+(x) \subset S_0$ since $L^+(x) \subset V$. Therefore, S_0 is globally + asymptotically stable, by Proposition 3.2, and hence $S_0 = S$.

COROLLARY. *If S contains a point s_0 which is isolated from $S - \{s_0\}$, then $S = \{s_0\}$.*

THEOREM 4.4. *If S is compact, then either S is a singleton and one of the two assertions of Theorem 4.2 holds, or S is globally + asymptotically stable.*

Proof. Let C be a simple closed curve such that $S \subset \mathrm{int}(C)$. Since S is positively stable, as $D^+(S) = S$, there exists a neighborhood V of S such that $C^+(V) \subset \mathrm{int}(C)$. Therefore, for each $x \in V, L^+(x) \neq \emptyset$. If a periodic orbit exists, then the proof follows from Theorem 4.2. If there are no periodic orbits, then $L^+(x)$ consists of a single rest point, by Lemma 3. Hence, S is globally + asymptotically stable, by Proposition 3.2.

REMARK. If S is + asymptotically stable, then for each $s \in \partial S$, there is a regular point y with $L^+(y) = \{s\}$. For, if x is a regular point, then it follows from Lemma 2 and Theorem 4.2 that $C^-(x)$ is unbounded. Thus, if C is a simple closed curve surrounding s, then one can choose sequences $\{x_n\}$ and $\{t_n\}$ in R^2 and R^-, respectively, such that $\{x_n\}$ converges to s and $\{x_n t_n\}$ converges to some point $x_0 \in C$. But this would imply that $x_0 \in D^-(s)$ or $s \in D^+(x_0)$, and hence $L^+(x_0) = \{s\}$.

[4] S_0 is isolated from $S - S_0$ if S_0 has a neighborhood disjoint from $S - S_0$.

LEMMA 4. *If S is $+$ asymptotically stable, then $A^+(S)$ is an open set.*

Proof. We note that $\partial A^+(S)$ is a closed invariant set, since $A^+(S)$ is invariant. Thus, for each $x \in \partial A^+(S)$, $L^+(x) \subset \partial A^+(S)$. But, $\partial A^+(S) \cap S = \varnothing$ since S is $+$ asymptotically stable. Therefore, $\partial A^+(S) \cap A^+(S) = \varnothing$, and hence $A^+(S)$ is open.

THEOREM 4.5. *If S is unbounded, then the following hold.*

(1) *Either $S = R^2$, or $R^2 - S$ is unbounded.*

(2) *If $S \neq R^2$, then S is $+$ asymptotically stable. Further, if S is disconnected, then it is not globally $+$ asymptotically stable.*

(3) *$x \notin A^+(S)$ implies that $L^\pm(x) = \varnothing$.*

Proof. The first assertion follows from the fact that there are no periodic orbits, and consequently, if x is a regular point, then $C^-(x)$ is unbounded. To prove (2), let $s \in \partial S$ and let C be a simple closed curve such that $s \in \operatorname{int}(C)$. Since $\{s\}$ is positively stable, there exists a neighborhood U of s such that $C^+(U) \subset \operatorname{int}(C)$. Therefore, for each $x \in U$, $L^+(x) \neq \varnothing$, and hence $L^+(x) \subset S$ since there are no periodic orbits. The last assertion of (2) follows from Proposition 3.1. Statement (3) follows from Lemma 4 and the fact that $\partial A^+(S)$ is positively invariant and there are no periodic orbits.

THEOREM 4.6. *If $S \neq R^2$ and S is unbounded, then $A^+(S)$ has a countable number of components. The boundary of each component is constituted by a countable number of orbits $C(x)$ such that $L^\pm(x) = \varnothing$.*

Proof. Since by Lemma 4, $A^+(S)$ is open, the first statement follows immediately from the fact that the components of $A^+(S)$ form a collection of mutually disjoint open subsets of R^2. To prove the second assertion, let K be any component of $A^+(S)$. We note that ∂K is invariant and is thus constituted by whole trajectories. For each $x \in \partial K$, $L^\pm(x) = \varnothing$, since x cannot belong to any component of $A^+(S)$ and there are no periodic orbits. Thus, $C_x = C(x) \cup \{\omega\}$ constitutes a simple closed curve in $(R^2)^*$ and K is contained in one of the components of $(R^2)^* - C_x$. Let K_x denote the component of $(R^2)^* - C_x$ which is disjoint from K, i.e., $K_x \cap K = \varnothing$. Then we must have $K_x \cap \partial K = \varnothing$. If $y \in \partial K - C_x$, then $K_x \cap K_y = \varnothing$. For, suppose $K_x \cap K_y \neq \varnothing$. Then, $K_x \cap \partial K_y = K_x \cap C_y = \varnothing$ since $y \in \partial K$, $\partial K \cap K_x = \varnothing$ and ∂K is invariant. Hence, $K_x \subset K_y$. Similarly, $K_y \subset K_x$ and thus $K_x = K_y$. Now, $y \notin C_x$ and $y \notin K_x$ since $K_x \cap \partial K = \varnothing$. Therefore, the component $(R^2)^* - (K_x \cup C_x)$

must be a neighborhood of y. But this is a contradiction to $y \in \partial K_y$ since $(R^2)^* - (K_x \cup C_x)$ contains no point of $K_x = K_y$. This shows that $K_x \cap K_y = \varnothing$. The second assertion of Theorem 4.6 now follows from the fact that $(R^2)^*$ is a Lindëlof of space, and hence the collection $(K_x)_{C(x) \subset \partial K}$ is countable.

THEOREM 4.7. *If $S \neq R^2$ and S is unbounded, then every component of $A^+(S)$ is homeomorphic to R^2.*

Proof. Let K_0 be any component of $A^+(S)$. Since K_0 is an open subset of R^2, it is sufficient to show that K_0 is simply connected. Let C_0 be any simple closed curve such that $C_0 \subset K_0$. If x is a regular point in $\operatorname{int}(C_0)$, then $L^-(x) = \varnothing$ since there are no periodic orbits. Therefore, $C^-(x) \cap C_0 \neq \varnothing$. But $x_0 \in C^-(x) \cap C_0$ implies that $x_0 \in A^+(S)$, and hence $x \in A^+(S)$ since $x \in C^+(x_0)$. This shows that $\operatorname{int}(C_0) \subset A^+(S)$, since $S \subset A^+(S)$. Since $\operatorname{int}(C_0)$ is connected, $\operatorname{int}(C_0) \subset K_0$, i.e., C_0 is retractible.

THEOREM 4.8. *If $S \neq R^2$ and S is unbounded, then S has a countable number of components, each being simply connected. Further, the set of critical points in each component of $A^+(S)$ form a component of S.*

Proof. We note that $S \subset A^+(S)$, and by Theorem 4.6, $A^+(S)$ is partitioned into a countable number of components. Therefore, in order to prove the first assertion, it is sufficient to show that if K_0 is any component of $A^+(S)$ and $S_0 = K_0 \cap S$, then S_0 is a component of S. To show that S_0 is a component of S, it is sufficient to show that S_0 is connected. For, it follows from the proof of Theorem 4.6 that $\partial K_0 \cap S = \varnothing$, and consequently, the component of S containing S_0 is contained in K_0. However, we note that S_0 is $+$asymptotically stable, globally, in K_0. Therefore, the fact that S_0 is connected follows from Proposition 3.1.

To prove that components of S are simply connected, let S_1 be any component of S and let C_1 be any simple closed curve such that $C_1 \subset S_1$. Suppose $\operatorname{int}(C_1)$ contains a regular point x. Then $L^-(x) \neq \varnothing$ since x is surrounded by the simple closed curve C_1 consisting of rest points. But this implies that x is a periodic point (see footnote on page 10). Therefore, $\operatorname{int}(C_1)$ consists of rest points and is hence contained in S_1, since S_1 is a maximal connected subset of S. This completes the proof.

It follows from Theorem 4.6 and the proof of Theorem 4.7 that each component of S is isolated from other points of S. Thus, using Theorem 4.3, we have the following sharpening of Theorem 4.3.

THEOREM 4.9. *If S has a compact component, then one of the two possibilities stated in Theorem* 4.3 *holds.*

We now summarize the results of this section.

Case 1. $S = \varnothing$ and (R^2, π) is parallelizable.

Case 2. S is compact implies one of the following.
(a) $S = \{s_0\}$ is a singleton and s_0 is a global Poincaré center.
(b) $S = \{s_0\}$ is a singleton and s_0 is a local Poincaré center. Further, the set N consisting of s_0 and periodic orbits surrounding s_0, is a globally $+$ asymptotically stable simply connected continuum.
(c) S is a globally $+$asymptotically simply connected continuum.

Case 3. If S is unbounded, then either (A) $S = R^2$ or (B) the following hold.
(a) $R^2 - S$ is unbounded.
(b) S is $+$asymptotically stable.
(c) $A^+(S)$ has a countable number of components each being homeomorphic to R^2 and unbounded.
(d) S has a countable number of components, each being non-compact and simply connected. For each $s \in \partial S$, there is a regular point y with $L^+(y) = \{s\}$.
(e) $A^+(S_0)$ is a component of $A^+(S)$ if and only if S_0 is a component of S.
(f) For each $x \in R^2$, $L^+(x)$ is either empty or consists of a single rest point. Further, $L^+(x) = \varnothing$ for all $x \notin A^+(S)$ and $L^-(x) = \varnothing$ for all $x \in R^2 - S$.

The above theorems indicate that imposing characteristic 0^+ on a dynamical system on R^2 is a fairly strong restriction. However, for more general phase spaces the situation is different. By way of illustration, we give the following example.

EXAMPLE 1. Consider the subspace of R^3 consisting of the xy-plane and the negative z-axis. Consider the flow in which the origin 0 is a rest point, points on the xy-plane are periodic whose trajectories surround 0 and points on the negative z-axis tend positively to 0, i.e., $L^+(x) = 0$ for all x on the negative z-axis.

We have clearly defined a flow of characteristic 0^+ which has only one rest point, and yet none of the conditions of Theorems 4.2 or 4.3 hold.

5. Flow of characteristic $0^\pm$ on the plane.

DEFINITION 5.1. A flow (R^2, π) on the plane is of characteristic $0^\pm$ if for each $x \in R^2$, $D^+(x) = \overline{C^+(x)}$ and $D^-(x) = \overline{C^-(x)}$.

The above definition is equivalent to saying that a flow is of characteristic $0^\pm$ if and only if every closed invariant subset M of R^2 is positively and negatively D-stable (i.e., $D^+(M) = D^-(M) = M$). The following theorem completely classifies such flows. The proof of this theorem follows immediately from the previous section and is hence omitted.

THEOREM 5.1. *Let (R^2, π) be a dynamical system of characteristic $0^\pm$ on the plane. Then one of the following holds.*
(1) *$S = \varnothing$ and the flow is parallelizable.*
(2) *$S = R^2$.*
(3) *$S = \{s_0\}$ is a singleton and s_0 is a global Poincaré center.*

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OKLAHOMA STATE UNIVERSITY