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INTERSECTIONS OF THE SPACE OF SKEW-SYMMETRIC MAPS WITH ITS TRANSLATES

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For a quadratic space V over a field K, let $\mathscr{L} \subseteq \mathrm{End}(V)$ be the space of all maps which are skew-symmetric with respect to the inner product. For $g \in \mathrm{GL}(V)$, let $\mathscr{D}(g) = \dim(\mathscr{L} \cap g\mathscr{L})$. In this paper we determine the largest few values possible for $\mathscr{D}(g)$, and we classify the maps g which achieve these values. The restriction of this result to maps g in the orthogonal group $\mathcal{O}(V)$ generalizes the characterization of $\pm$ symmetries originally proved by Botta and Pierce.

1. Introduction. Let K be a field of characteristic not two and let (V, B) be a quadratic space of dimension n over K. That is, V is a K-vector space of dimension n and $B: V \times V \to K$ is a non-degenerate symmetric bilinear form. Let $G = \mathcal{O}(V, B)$ be the orthogonal group of this space. Let $\mathscr{L}$ be the space of all elements of $\mathrm{End}(V)$ which are skew-symmetric with respect to B. For $g \in G$, the space $g\mathscr{L} = \{g \circ h \mid h \in \mathscr{L}\}$ may be considered as the tangent space to G at the point g.

For $g \in \mathrm{End}(V)$, let

$$\mathscr{D}(g) := \dim(\mathscr{L} \cap g\mathscr{L}).$$

This dimension $\mathscr{D}(g)$ is viewed as a measure of how far g is from being a scalar map. For $a \in K^\times$, $\mathscr{D}(a1) = \dim \mathscr{L} = \frac{1}{2}n(n-1)$. The following theorem, due to Botta and Pierce, shows that the non-scalar elements $g \in G$ which have the largest value for $\mathscr{D}(g)$ are the $\pm$ symmetries. For an anisotropic $v \in V$, the *symmetry* (or hyperplane reflection) corresponding to the line Kv is the map $\tau_v \in G$ which sends v to $-v$ and which pointwise fixes the hyperplane $(Kv)^\perp$.

THEOREM 1.1. [**1**, Prop. 7.1.]. *Suppose $n \geqq 3$ and $g \in G, g \neq \pm 1$. Then*

$$\mathscr{D}(g) \leqq \frac{(n-1)(n-2)}{2}$$

with equality holding if and only if $\pm g$ is a symmetry.

The purpose of the present paper is to determine the next largest values for $\mathscr{D}(g)$, and to determine which maps g attain these values. The value of $\mathcal{D}(g)$ is computed in terms of the Jordan form of g, by interpreting $\mathcal{D}(g)$ as the dimension of the kernel of the linear map $g\otimes 1-1\otimes g$ on $\Lambda^2 V$. The Jordan forms of all $g\in \mathrm{GL}(V)$ having $\mathcal{D}(g)\geqq \frac{1}{2}(n-2)(n-3)$ are found, using some manipulations with partitions. Then, we utilize results of Milnor [2] to restrict attention back to elements g in the orthogonal group. The final result is stated below.

We assume K is algebraically closed here, but this is not much loss of generality (see Remark 1.5). The equalities stated in the theorem mean that, with respect to some basis of V, the matrix of g equals the indicated matrix. The symbol, $\perp$, stands for orthogonal direct sum. We write $J_r(1)$ for the $r\times r$ matrix in Jordan form which corresponds to the elementary divisor $(x-1)^r$. A subscript indicates the order of a matrix.

THEOREM 1.2. *Let K be an algebraically closed field. Suppose $n\geqq 4$ and $g\in \mathcal{O}(V,B)$, over K.*

(1) $\mathcal{D}(g)=\dfrac{n(n-1)}{2}$ *iff* $\pm g=1_n$.

(2) $\mathcal{D}(g)=\dfrac{(n-1)(n-2)}{2}$ *iff* $\pm g=$ *symmetry* $=1_{n-1}\perp(-1_1)$.

(3) $\mathcal{D}(g)=1+\dfrac{(n-2)(n-3)}{2}$ *iff either*

(i) $\pm g=$ *2-plane reflection* $=1_{n-2}\perp(-1_2)$;

or

(ii) $\pm g=1_{n-4}\perp(J_2(1)\oplus J_2(1))$;

or

(iii) $n=4$ *and* $g=\begin{pmatrix} a1_2 & 0\\ 0 & a^{-1}1_2\end{pmatrix}$, *for some scalar* $a\neq 0,1,-1$.

(4) $\mathcal{D}(g)=\dfrac{(n-2)(n-3)}{2}$ *iff either*

(i) $\pm g=1_{n-2}\perp\begin{pmatrix} a & 0\\ 0 & a^{-1}\end{pmatrix}$, *for some scalar* $a\neq 0,1,-1$;

or

(ii) $\pm g=1_{n-3}\perp J_3(1)$;

or

(iii) $n=6$ *and* $g=1_3\perp(-1_3)$;

or

(iv) $n = 6$ *and* $g = \begin{pmatrix} a1_3 & 0 \\ 0 & a^{-1}1_3 \end{pmatrix}$, *for some scalar* $a \neq 0, 1, -1$.

(5) $\mathscr{D}(g) < \frac{(n-2)(n-3)}{2}$ *otherwise.*

PROPOSITION 1.3. *Let* $g \in G$.

(1) *Suppose* $n = 2$. *Then* $\mathscr{D}(g) = 1$ *if* $g = \pm 1$ *and* $\mathscr{D}(g) = 0$ *otherwise.*

(2) *Suppose* $n = 3$. *Then* $\mathscr{D}(g) = 3$ *if* $g = \pm 1, \mathscr{D}(g) = 1$ *if* $\pm g =$ *symmetry, and* $\mathscr{D}(g) = 0$ *otherwise.*

REMARK 1.4. If g is a product of two symmetries, then either g is a 2-plane reflection, as in part (3)(i), or g lies either in part (4)(i) or (4)(ii) of the theorem.

REMARK 1.5. Similar results over a general field K (of characteristic not 2) quickly follow. The only parts of Theorem 1.2 that need to be changed are the forms of the special matrices in parts (3)(iii), and (4)(i), (ii), (iv). This generalization easily follows from Proposition 3.1 below, and the fact that two matrices over K which are similar over the algebraic closure of K must already be similar over K.

We would like to thank A. Wadsworth for his helpful comments which led to a substantial improvement of the theorem.

2. Computation of $\mathscr{D}(g)$. Throughout this section, we use a fixed $g \in \mathrm{GL}(V)$. We will return to the orthogonal group in Section 3.

The non-degenerate bilinear form B induces an adjoint involution, $\sim$, on End(V), defined by: $B(\tilde{f}(u), v) = B(u, f(v))$. If an orthonormal basis of V is chosen, and matrices are used, then this involution is the transpose map.

By definition, $\mathscr{L} = \{h \in \mathrm{End}(V) \mid \tilde{h} = -h\}$. An element $h \in \mathscr{L}$ lies in $\mathscr{L} \cap g\mathscr{L}$ iff $gh = h\tilde{g}$. Let $T\colon \mathscr{L} \to \mathrm{End}(V)$ be defined:

$$T(h) := gh - h\tilde{g}.$$

Then $\mathscr{D}(g) = \dim(\mathscr{L} \cap g\mathscr{L}) = \dim(\ker T)$.

By the usual method, the bilinear form B gives an identification of V with its dual space V^*, so that $\mathrm{End}(V) \cong V \otimes V^*$ is identified with $V \otimes V$. Furthermore, for $l, k \in \mathrm{End}(V)$, the operation $h \mapsto lh\tilde{k}$ on End(V) is identified with the usual operation of $l \otimes k$ on $V \otimes V$, and the adjoint involution, $\sim$, is identified with the switch operation

$v \otimes w \mapsto w \otimes v$ on $V \otimes V$. Therefore, $\mathcal{L} \subseteq \mathrm{End}(V)$ becomes $\Lambda^2 V \subseteq V \otimes V$, and the map T above becomes the restriction to $\Lambda^2 V$ of $g \otimes 1 - 1 \otimes g$.

We work with the Jordan decomposition of V with respect to $g \in \mathrm{GL}(V)$. Let the elementary divisors of g be $(x - b_i)^{l_i}$, for $i = 1, \cdots, p$. Then, $V = V_1 \oplus \cdots \oplus V_p$, where V_i is the cyclic submodule corresponding to $(x - b_i)^{l_i}$. There may be repetitions among the eigenvalues b_i; let $a_1, \cdots, a_t$ be the *distinct* eigenvalues. Define $V((a)) :=$ the sum of all V_i such that $b_i = a$, so that

$$V((a)) = \{v \in V \mid (g - a1)^k(v) = 0, \ \text{for some } k\}.$$

Then, $V = V((a_1)) \oplus \cdots \oplus V((a_t))$. Let $n_j := \dim V((a_j))$ so that $n = n_1 + \cdots + n_t$. For each $j = 1, \cdots, p$, the various l_i's corresponding to $b_i = a_j$ add up to n_j. They give a partition P_j of the number n_j. The eigenvalues a_j and partitions P_j completely determine the Jordan form of g. We will compute $\mathcal{D}(g)$ using these partitions. The idea of using this interpretation of $\mathcal{D}(g)$ and some computations with Jordan forms is already present in **[1]**.

DEFINITION 2.1. Suppose $P = (m_1, \cdots, m_k)$ is a partition of n into k parts. That is, $n = m_1 + \cdots + m_k$, where each m_i is an integer and $1 \leqq m_1 \leqq \cdots \leqq m_k$. Define

$$\mathcal{D}(P) := \sum_{j=1}^{k} (k - j) m_j.$$

PROPOSITION 2.2. *For $g \in \mathrm{GL}(V)$ as above, with partitions P_i of n_i, $\mathcal{D}(g) = \mathcal{D}(P_1) + \cdots + \mathcal{D}(P_t)$.*

Proof. For $v, w \in V$, define $v \wedge w := \frac{1}{2}(v \otimes w - w \otimes v)$ and $v \circ w := \frac{1}{2}(v \otimes w + w \otimes v)$. For subspaces $U, W \subseteq V$, define $\Lambda^2(U, W) := \mathrm{Span}\{u \wedge w \mid u \in U, w \in W\}$ and $S^2(U, W) := \mathrm{Span}\{u \circ w \mid u \in U, w \in W\}$. Then, if $U \cap W = 0$, $\dim \Lambda^2(U, W) = \dim S^2(U, W) = (\dim U) \cdot (\dim W)$. If $U = W$ has dimension l, then $\Lambda^2(U) := \Lambda^2(U, U)$ has dimension $l(l-1)/2$ while $S^2(U) := S^2(U, U)$ has dimension $l(l+1)/2$.

Since $V = V_1 \oplus \cdots \oplus V_p$, we have $\Lambda^2 V$ is the direct sum of the $\Lambda^2(V_r, V_s)$, for $r \leqq s$, and similarly for $S^2 V$. Since $T(v \wedge w) = g(v) \circ w - v \circ g(w)$, the map T is the direct sum of the maps

$$T_{rs}: \Lambda^2(V_r, V_s) \to S^2(V_r, V_s),$$

for $r \leqq s$. Therefore, $\ker T$ is the direct sum of the $\ker(T_{rs})$, for $r \leqq s$.

Let $r \leqq s$ be fixed. Choose a basis $\{v_i \mid 1 \leqq i \leqq l_r\}$ of V_r corresponding to the Jordan form; that is (setting $v_0 = 0$), $g(v_i) = b_r v_i + v_{i-1}$. Choose a similar basis $\{w_j \mid 1 \leqq j \leqq l_s\}$ of V_s. It follows that

$$T(v_i \wedge w_j) = (b_r - b_s) v_i \circ w_j + v_{i-1} \circ w_j - v_i \circ w_{j-1}.$$

We will compute $\dim \ker(T_{rs})$ by working with $\operatorname{coker}(T_{rs})$, which is $S^2(V_r, V_s) \bmod \operatorname{im}(T_{rs})$. In the cases $r \neq s$, $\ker(T_{rs})$ and $\operatorname{coker}(T_{rs})$ have the same dimension.

Case 1. $b_r \neq b_s$. Then, in $\operatorname{coker}(T_{rs})$ we have $(b_r - b_s) v_i \circ w_j \equiv - v_{i-1} \circ w_j + v_i \circ w_{j-1}$, and an inductive argument shows that every $v_i \circ w_j \equiv 0$. Hence, $\dim \ker(T_{rs}) = \dim \operatorname{coker}(T_{rs}) = 0$.

Case 2. $b_r = b_s$ but $r \neq s$. Suppose $l_r \leqq l_s$. The space $\operatorname{coker}(T_{rs})$ is spanned by elements $v_i \circ w_j$, which satisfy the relations $v_{i-1} \circ w_j \equiv v_i \circ w_{j-1}$, for $1 \leqq i \leqq l_r, 1 \leqq j \leqq l_s$. Then, every $v_i \circ w_j$ reduces either to 0 or to some $v_k \circ w_{l_s}$. Therefore these l_r elements span the space, so $\dim \operatorname{coker}(T_{rs}) \leqq l_r$.

Also, the space $\operatorname{im}(T_{rs})$ is spanned by the $l_r l_s$ elements $T(v_i \wedge w_j)$. For every k with $1 \leqq k \leqq l_r$, there is a relation $T(v_k \wedge w_1 + v_{k-1} \wedge w_2 + \cdots + v_1 \wedge w_k) = 0$, so that at least l_r of these generators are redundant. Hence, $\dim \operatorname{im}(T_{rs}) \leqq l_r l_s - l_r$, so $\dim \operatorname{coker}(T_{rs}) = l_r l_s - \dim \operatorname{im}(T_{rs}) \geqq l_r$. Consequently, $\dim \ker(T_{rs}) = \dim \operatorname{coker}(T_{rs}) = l_r = \min(l_r, l_s)$.

Case 3. $r = s$. Counting dimensions using the exact sequence

$$0 \to \ker(T_r) \to \Lambda^2(V_r) \to S^2(V_r) \to \operatorname{coker}(T_r) \to 0,$$

we see that $\dim \ker(T_r) + l_r = \dim \operatorname{coker}(T_r)$. The space $\operatorname{coker}(T_r)$ is spanned by elements $v_i \circ v_j$, with relations $v_{i-1} \circ v_j \equiv v_i \circ v_{j-1}$, for $1 \leqq i, j \leqq l_r$. Then, as in Case 2, every $v_i \circ v_j$ reduces either to 0 or to some $v_k \circ v_{l_r}$. Hence, at most l_r generators are needed, so $\dim \operatorname{coker}(T_r) \leqq l_r$. Therefore, by the equation above, $\dim \ker(T_r) = 0$.

By Case 1, $\mathcal{D}(g)$ is the sum of the $\mathcal{D}$-values of the restrictions of g to the eigenspaces $V((a_j))$. For fixed j, suppose that $(x - a_j)^{m_1}, \cdots, (x - a_j)^{m_k}$ are the elementary divisors of g with eigenvalue a_j, where $1 \leqq m_1 \leqq \cdots \leqq m_k$. Then P_j is the partition $(m_1, \cdots, m_k)$. By Cases 2 and 3, the dimension of the kernel of T restricted to $\Lambda^2(V((a_j)))$ is

$$\sum_{1 \leqq r < s \leqq k} \min(m_r, m_s) = \sum_{r < s} m_r = (k-1)m_1 + (k-2)m_2 + \cdots + m_{k-1} = \mathcal{D}(P_j).$$

This completes the proof.

In order to determine which partitions have large $\mathcal{D}$-values, we determine the maximal $\mathcal{D}$-value of a partition of n into k parts.

DEFINITION 2.3. For integers $1 \leqq k \leqq n$, define $F_n(k)$ as follows:

$$\text{if } n = kq + r, \qquad 0 \leqq r < k,$$

$$F_n(k) := \frac{k(k-1)}{2}q + \frac{r(r-1)}{2}.$$

LEMMA 2.4. *The maximal $\mathcal{D}$-value of all partitions of n into k parts is $F_n(k)$.*

Proof. Suppose the partition $n = m_1 + \cdots + m_k$ has the maximal $\mathcal{D}$-value. If $m_k - m_1 \geqq 2$, then there exist $i < j$ such that either $m_i < m_{i+1} \leqq m_{j-1} < m_j$ or $j = i + 1$ and $m_j - m_i \geqq 2$. In either case, define a new partition of n into k parts by replacing m_i by $m_i + 1$ and m_j by $m_j - 1$. This new partition has larger $\mathcal{D}$-value, contrary to hypothesis. Therefore, $m_k - m_1 \leqq 1$, and the partition looks like $n = q + q + \cdots + q + (q+1) + \cdots + (q+1)$, for some q, where there are, say, r of the $(q+1)$ terms $(0 \leqq r < k)$. Then $n = (k-r)q + r(q+1) = kq + r$, and the (maximal) $\mathcal{D}$-value for this partition is

$$\begin{aligned}\mathcal{D} &= [(k-1) + \cdots + r]q + [(r-1) + \cdots + 1](q+1)\\ &= \frac{k(k-1)}{2}q + \frac{r(r-1)}{2},\end{aligned}$$

as claimed.

REMARK. By a similar argument it follows that, for fixed $n \geqq 1$, $F_n(k)$ is a strictly increasing function for $1 \leqq k \leqq n$.

From the following result, we can easily find the Jordan forms of all $g \in \mathrm{GL}(V)$ with $\mathcal{D}(g) \geqq \frac{1}{2}(n-2)(n-3)$.

PROPOSITION 2.5. *Let $n = n_1 + \cdots + n_t$, where $1 \leqq n_1 \leqq \cdots \leqq n_t$, and let P_i be a partition of n_i. Let $\mathcal{D} = \mathcal{D}(P_1) + \cdots + \mathcal{D}(P_t)$. The following list gives all cases when $\mathcal{D} \geqq \frac{1}{2}(n-2)(n-3)$.*

I. $t = 1$

(i) $P_1 = (1, \cdots, 1)$ *and* $\mathcal{D} = \frac{1}{2}n(n-1)$.

(ii) $P_1 = (1, \cdots, 1, 2)$ *and* $\mathcal{D} = \frac{1}{2}(n-1)(n-2)$.

(iii) $P_1 = (1, \cdots, 1, 2, 2)$ *and* $\mathcal{D} = 1 + \frac{1}{2}(n-2)(n-3)$.

(iv) $P_1 = (1, \cdots, 1, 3)$ *and* $\mathcal{D} = \frac{1}{2}(n-2)(n-3)$.

(v) $n = 6, P_1 = (2, 2, 2)$ *and* $\mathcal{D} = 6$.

II. $t = 2$

(i) $P_1 = (1), P_2 = (1, \cdots, 1)$ *and* $\mathcal{D} = \frac{1}{2}(n-1)(n-2)$.

(ii) $P_1 = (1,1), P_2 = (1,\cdots,1)$ *and* $\mathcal{D} = 1 + \frac{1}{2}(n-2)(n-3)$.
(iii) $P_1 = (1), P_2 = (1,\cdots,1,2)$ *and* $\mathcal{D} = \frac{1}{2}(n-2)(n-3)$.
(iv) $P_1 = 2, P_2 = (1,\cdots,1)$ *and* $\mathcal{D} = \frac{1}{2}(n-2)(n-3)$.
(v) $n = 6, P_1 = (1,1,1), P_2 = (1,1,1)$ *and* $\mathcal{D} = 6$.

III. $t = 3$
(i) $P_1 = (1), P_2 = (1), P_3 = (1,\cdots,1)$ *and* $\mathcal{D} = \frac{1}{2}(n-2)(n-3)$.

Proof. I. $t = 1$. By Proposition (2.4), $\frac{1}{2}(n-2)(n-3) \leqq \mathcal{D}(P_1) \leqq F_n(k)$, where the partition P_1 has k parts. If $n = kq + r, 0 \leqq r < k$, then

$$n^2 - 5n + 6 \leqq 2F_n(k) = k(k-1)q + r(r-1) = (k-1)n - (k-r)r$$

so $n^2 + 6 \leqq (k+4)n$ and therefore $k \geqq n-3$. Suppose $P_1 = (m_1, m_2,\cdots, m_k)$ and $m_1 = m_2 = \cdots = m_l = 1$, while $2 \leqq m_{l+1} \leqq \cdots \leqq m_k$. Then $n \geqq l + 2(k-l) = 2k - l \geqq 2n - 6 - l$, so $l \geqq n-6$. Finding all such partitions and computing their $\mathcal{D}$-values is now easily done. Note that, for any partition P, $\mathcal{D}(P) \leqq \frac{1}{2}n(n-1)$ with equality iff $P = (1,\cdots,1)$.

II. $t = 2$. By part I, $\mathcal{D} \leqq \frac{1}{2}n_1(n_1-1) + \frac{1}{2}n_2(n_2-1)$. The function $f(x) = \frac{1}{2}x(x-1)$ satisfies the following shift property: if $x \leqq y$ then $f(x) + f(y) < f(x-1) + f(y+1)$. Suppose $n_1 \geqq 3$ (so that $n \geqq 6$) and shift to get $\mathcal{D} \leqq 3 + \frac{1}{2}(n-3)(n-4)$. If $n > 6$, this value is less than $\frac{1}{2}(n-2)(n-3)$. Then $n = 6$, and since equalities hold, no shifts could have occurred. Therefore, $n_1 = n_2 = 3$, and since $\mathcal{D}(P_i) = 3$, we have $P_1 = P_2 = (1,1,1)$.

Suppose $n_1 = 2$. If $P_1 = (2)$, then $\mathcal{D} = \mathcal{D}(P_2) \leqq \frac{1}{2}(n-2)(n-3)$, so equality must hold. Therefore, by part I, $P_2 = (1,\cdots,1)$. If $P_1 = (1,1)$, then $\mathcal{D} = 1 + \mathcal{D}(P_2) \leqq 1 + \frac{1}{2}(n-2)(n-3)$. If $P_1 \neq (1,\cdots,1)$ then, by part I, $n-2 \geqq 2$ and $\mathcal{D}(P_2) \leqq \frac{1}{2}(n-3)(n-4)$. Then $\frac{1}{2}(n-2)(n-3) \leqq \mathcal{D} \leqq 1 + \frac{1}{2}(n-3)(n-4)$, and this implies $n \leqq 4$. Hence, $n = 4$ and $P_2 = (2)$. This case is already covered, after switching P_1 and P_2.

If $n_1 = 1$, then $\frac{1}{2}(n-2)(n-3) \leqq \mathcal{D} = \mathcal{D}(P_2)$. By part I, both $P_2 = (1,\cdots,1)$ and $P_2 = (1,\cdots,1,2)$ will do, but all other cases are eliminated.

III. $t \geqq 3$. By the shift property stated in part II, we can increase $\mathcal{D}$ by cutting $n_1, n_2,\cdots, n_{t-1}$ down to 1 and increasing n_t to $n - t + 1$. Hence, $\mathcal{D} \leqq \frac{1}{2}(n-t+1)(n-t)$. If $t \geqq 4$, this value is too small. If $t = 3$, equality must hold, so that no shifts could have occurred. Then $n_1 = n_2 = 1$ and $n_3 = n-2$. Again, since equality holds, P_3 must be $(1,\cdots,1)$. This completes the proof.

3. Specializing to orthogonal group elements. The theorem (1.2) will follow from the results of the previous section once we know what the Jordan form of an element $g \in \mathcal{O}(V,B)$ can be. This information is found in a more general setting in **[2]**. We quote the relevant results.

Let $g \in \mathcal{O}(V,B)$ be fixed. Then V is a $K[x]$-module, in the usual way, and we examine the decomposition of this module into cyclic submodules. Recall that for an eigenvalue a of g, $V((a))$ is the submodule consisting of all v with $(g-a)^k(v)=0$, for large k.

PROPOSITION 3.1. *Let $g \in \mathcal{O}(V,B)$. Then*: (1) *$V((a))$ and $V((b))$ are orthogonal, unless $ab=1$.*

(2) *The cyclic decompositions of $V((1))$ and $V((-1))$ can be chosen so that cyclic pieces of different dimensions are orthogonal.*

Proof. This is a special case of Lemma 3.1 and Theorem 3.2 of **[2]**.

Let $g \in \mathrm{End}(V)$. For each eigenvalue a_i of g, let $n_i = \dim V((a_i))$ and let P_i be the partition of n_i given by the degrees of the elementary divisors of g corresponding to a_i.

PROPOSITION 3.2. (1) *Suppose $g \in \mathcal{O}(V,B)$ and $a_i = \pm 1, P_i = (m_1,\cdots,m_k)$. If an even number occurs among the m_i, it must occur an even number of times.*

(2) *Suppose $g \in \mathcal{O}(V,B)$ and $a_i \neq \pm 1$. Then, for some j, $a_i^{-1} = a_j$, $n_i = n_j$, and the partitions P_i and P_j are the same.*

(3) *If $g \in \mathrm{End}(V)$ has Jordan form satisfying the conditions in* (1) *and* (2), *then there exists a nondegenerate symmetric bilinear form B on V with $g \in \mathcal{O}(V,B)$.*

Proof. This is all a special case of Theorem 3.4 of **[2]** and his discussion on pp. 94–95, 97.

The proofs of Theorem (1.2) and Proposition (1.3) are now easily done. If $g \in \mathcal{O}(V,B)$ and $\mathcal{D}(g) \geqq \frac{1}{2}(n-2)(n-3)$, then the possible partitions induced by the Jordan form of g are known, by Propositions (2.5) and (3.2). Eigenvalues can then be assigned to these partitions, subject to the restrictions in Proposition (3.2). Then, the possible Jordan forms for g are known, and Proposition (3.1) tells which pieces may be taken to be orthogonal. We omit the details.

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