ON BROWNIAN MOTION IN A HOMOGENEOUS RIEMANNIAN SPACE

Kôsaku Yosida

1. **Introduction.** Let R be an n-dimensional, orientable, infinitely differentiable Riemannian space such that the group G of isometric transformations S^* of R onto R constitutes a Lie group transitive on R. Consider a temporally homogeneous Markoff process in R, and let $P(t, x, E)$ be the transition probability that the point $x \in R$ is, by this process, transferred into a Borel set $E \subseteq R$ after the lapse of t units of time, $t > 0$. We assume that $P(t, x, E)$ is, for fixed (t, x), countably additive for Borel sets E and, for fixed (t, E), Borel measurable in x. Then we must have the probability condition

$$P(t, x, E) \geq 0, \; P(t, x, R) = 1, \tag{1.1}$$

and Smoluchowski's equation

$$P(t+s, x, E) = \int_R P(t, x, dy)\, P(s, y, E) \qquad (t, s > 0). \tag{1.2}$$

We further assume that the process is spatially homogeneous:

$$P(t, x, E) = P(t, S^*x, S^*E) \qquad \text{for every } S^* \in G. \tag{1.3}$$

The purpose of the present note is to prove the following:

THEOREM 1. *Let x_0 be any point of R and assume that the Lie subgroup $\{S^* \in G;\, S^*x_0 = x_0\}$ of G is compact*[1]. *Let us denote by $d(x, y)$ the distance of two points $x, y \in R$. Then the continuity condition*:

$$\lim_{t \to 0+} \frac{1}{t} \int_{d(x,y) > \epsilon} P(t, x, dy) = 0 \qquad \text{for any} \quad \epsilon > 0, \tag{1.4}$$

implies the condition of Lindeberg's type:

[1] At first, this condition was overlooked. Mr. Seizô Itô kindly remarked that this condition is necessary for the convergence of the integral (2.11) below.

Received August 30, 1951.

Pacific J. Math. 2 (1952), 263-270

$$\overline{\lim_{t \to 0+}} \frac{1}{t} \int_R \frac{d(x, y)^2}{1 + d(x, y)^2} P(t, x, dy) < \infty. \tag{1.5}$$

From this theorem we may deduce:

THEOREM 2. *The finite limits* $(x = (x^1, x^2, \cdots, x^n))$

$$a^i(x) = \lim_{t \to 0+} \frac{1}{t} \int_{d(x, y) \leq \epsilon} (y^i - x^i) P(t, x, dy), \tag{1.6}$$

$$b^{ij}(x) = \lim_{t \to 0+} \frac{1}{t} \int_{d(x, y) \leq \epsilon} (y^i - x^i)(y^j - x^j) P(t, x, dy) \tag{1.7}$$

exist, independently of the sufficiently small $\epsilon > 0$. *Moreover, if a real-valued function* $f_0(x)$ *be such that* $f_0(x)$, $\partial f_0/\partial x^i$, $\partial^2 f_0/\partial x^i \partial x^j$ *are bounded and uniformly continuous on* R, *then*

$$\lim_{t \to 0+} \frac{1}{t} \left(\int_R f_0(y) P(t, x, dy) - f_0(x) \right) = a^i(x) \frac{\partial f_0}{\partial x^i} + b^{ij}(x) \frac{\partial^2 f_0}{\partial x^i \partial x^j}. \tag{1.8}$$

REMARK. In the literature [4; 2; 6], (1.8) is derived by assuming the condition of Lindeberg's type:

$$\lim_{t \to 0+} \left(\int_R d(x, y)^3 P(t, x, dy) / \int_R d(x, y)^2 P(t, x, dy) \right) = 0 \tag{1.5$'$}$$

and some differentiability hypothesis concerning $P(t, x, E)$. Considering the Brownian motion on the real line, Seizô Itô raised the question whether, under the condition of the spatial homogeneity (1.3), "the (almost sure) continuity of the sample motions of the temporally homogeneous Markoff process" which is equivalent to the continuity condition (1.4), would be sufficient to derive Theorem 2. And he proved Theorem 2 in the special case where $R = G$ and G is a maximally almost periodic Lie group. The present note gives an extension of his result to general homogeneous space, without the hypothesis of the maximal almost periodicity of the group G of motions of the space R. Thus we may define the Brownian motions in a homogeneous Riemannian space R as temporally and spatially homogeneous Markoff processes satisfying the condition (1.4) of continuity.

2. **Preliminaries.** Let us denote by $C(R)$ the totality of real-valued bounded functions $f(x)$ on R which are uniformly continuous on R. The space $C(R)$ is a Banach space by the norm

$$\|f\| = \sup_x |f(x)|. \tag{2.1}$$

We define, for any $f \in C(R)$,

$$(T_t f)(x) = \int_R P(t, x, dy) f(y); \tag{2.2}$$

then we have, by (1.1),

$$\sup_x |(T_t f)(x)| \leq \sup_x |f(x)|. \tag{2.3}$$

We have, by (1.3),

$$(T_t f)(S^* x) = \int_R P(t, S^* x, dy) f(y) = \int_R P[t, S^* x, d(S^* y)] f(S^* y) \tag{2.3}$$
$$= \int_R P(t, x, dy) f(S^* y),$$

and hence the commutativity

$$T_t S = S T_t, \tag{2.4}$$

where S is defined by

$$(Sf)(x) = f(S^* x), \qquad S^* \in G. \tag{2.5}$$

Thus, if $S^* \in G$ be such that $S^* x = x'$, we have

$$(T_t f)(x) - (T_t f)(x') = (T_t f)(x) - (S T_t f)(x) = T_t(f - Sf)(x). \tag{2.6}$$

By the uniform continuity of $f(x)$, and by (2.3) and (2.6), we see that $(T_t f)(x)$ is bounded and uniformly continuous in x. Hence T_t defines a bounded linear transformation on $C(R)$ into $C(R)$ such that

$$\|T_t\| = \sup_{\|f\| = 1} \|T_t f\| = 1. \tag{2.7}$$

We have, from (1.2),

$$T_{t+s} = T_t T_s \qquad (t, s > 0). \tag{2.8}$$

We have also, from (1.1),

$$(T_t f)(x) - f(x) = \int_R P(t, x, dy) [f(y) - f(x)]$$
$$= \int_{d(x,y) \leq \epsilon} P(t, x, dy) [f(y) - f(x)] + \int_{d(x,y) \geq \epsilon} P(t, x, dy) [f(y) - f(x)].$$

Thus, in view of conditions (1.4) and (1.1), and the uniform continuity of $f(x)$, we have

$$(2.9) \qquad \lim_{t \to 0+} (T_t f)(x) = f(x) \qquad \text{boundedly in } x.$$

Hence T_t is weakly continuous in t, and therefore, by (2.8) and N. Dunford's theorem [1], T_t is strongly continuous in t and

$$(2.9)' \qquad \text{strong} \lim_{t \to 0+} T_t f = f; \text{ that is, } \lim_{t \to 0+} \| T_t f - f \| = 0.$$

Therefore we may apply the theory [3; 5] of one-parameter semigroups of bounded linear operators to the semigroup $\{T_t\}$. In particular, we have the result:

(2.10) strong $\lim_{t \to 0+} (T_t f - f)/t = Af$ exists, for those f which constitute a linear subset $D(A)$ of $C(R)$ which is dense in $C(R)$. Moreover, A is a closed linear operator defined on $D(A) \subseteq C(R)$ with values in $C(R)$.

LEMMA. *Let $g(x) \in C(R)$ vanish outside a compact set. Then the convolution*

$$(2.11) \qquad (f \otimes g)(x) = \int_G f(S_y^* x)\, g(S_y^* x_0)\, dy$$

belongs to $D(A)$ if f belongs to $D(A)$. Here S_y^* is a general element of G, dy is a right invariant Haar measure of G, and x_0 is any fixed point of R.

Proof. The integral may be approximated by the Riemann sum

$$(2.12) \qquad \sum_{i=1}^{m} f(S_{y_i}^* x)\, c_i$$

uniformly in x. This we see by the uniform continuity of $f(x)$ and the fact that $g(x)$ vanishes outside a compact set. We know, from (2.4), that A is commutative with every S_y:

$$(2.13) \qquad f \in D(A) \text{ implies } S_y f \in D(A) \text{ and } S_y A f = A S_y f.$$

Hence (2.12) belongs to $D(A)$, and we have

$$(2.14) \qquad A\Big(\sum_{i=1}^{m} f(S_{y_i}^* x)\, c_i\Big) = A\Big(\sum_{i=1}^{m} (S_{y_i} f)(x)\, c_i\Big) = \sum_{i=1}^{m} (S_{y_i} h)(x)\, c_i,$$

where $h = Af$. Therefore, since $h \in C(R)$, we see that (2.14) converges, when

$m \longrightarrow \infty$, to a function $\in C(R)$ uniformly in x. Since A is a closed operator, we must have $(f \otimes g)(x) \in D(A)$.

COROLLARY 1. *The convolution* $(f \otimes g)(x)$ *is infinitely differentiable if* $g(x)$ *is infinitely differentiable.*

Proof. It is possible, for sufficiently small $d(x, x_0)$, to choose $S^*(x) \in G$ such that

$$S^*(x)x = x_0 \text{ and } S^*(x)x_0 \text{ depends analytically on } x^1, \cdots, x^n. \tag{2.15}$$

This we see from the fact that the set $\{S_y^* \in G;\ S_y^* x = x_0\}$ forms an analytic submanifold of G; it is one of the cosets of G with respect to the Lie subgroup $\{S_y^* \in G;\ S_y^* x_0 = x_0\}$. Hence, by the right invariance of dy, we have

$$\begin{aligned}(f \otimes g)(x) &= \int_G f(S_y^* S^*(x)x)\,(g(S_y^* S^*(x)x_0)\,dy \\ &= \int_G f(S_y^* x_0)\, g(S_y^* S^*(x)x_0)\,dy.\end{aligned} \tag{2.16}$$

The right side is infinitely differentiable in the vicinity of x_0, and

$$\begin{aligned}&\frac{\partial^{q_1 + \cdots + q_n}(f \otimes g)(x)}{\partial(x^1)^{q_1} \cdots \partial(x^n)^{q_n}} \\ &\qquad = \int_G f(S_y^* x_0)\,\frac{\partial^{q_1 + \cdots + q_n} g(S_y^* S^*(x)x_0)}{\partial(x^1)^{q_1} \cdots \partial(x^n)^{q_n}}\,dy\end{aligned} \tag{2.17}$$

belongs to $C(R)$.

COROLLARY 2. (i) *There exist infinitely differentiable functions* $F^1(x)$, $F^2(x), \cdots, F^n(x) \in D(A)$ *such that the Jacobian*

$$\frac{\partial(F^1(x), \cdots, F^n(x))}{\partial(x^1, \cdots, x^n)} \text{ does not vanish at } x = x_0. \tag{2.18}$$

(ii) *There exists an infinitely differentiable function* $F_0(x) \in D(A)$ *such that*

$$(x^i - x_0^i)(x^j - x_0^j)\,\frac{\partial^2 F}{\partial x_0^i\,\partial x_0^j} \geq \sum_{i=1}^{n} (x^i - x_0^i)^2. \tag{2.19}$$

Proof. In (2.16), f belongs to $D(A)$, which is dense in $C(R)$; and $g(x) \in C(R)$

is arbitrary except that $g(x)$ must vanish outside a compact set. Thus, by taking $F(x) = (f \otimes g)(x)$ suitably, we may prove (i) and (ii).

3. **Proof of Theorem 1.** Because of their functional independence, we may take $F^1(x), \cdots, F^n(x)$ as local coordinates of the points x which satisfy $d(x, x_0) < \epsilon$ for sufficiently small $\epsilon > 0$. Since $F^i(x) \in D(A)$,

$$(3.1) \quad \text{a finite } \lim_{t \to 0+} \frac{1}{t} \int_R (F^i(x) - F^i(x_0))\, P(t, x_0, dx) \text{ exists} \quad (i = 1, \cdots, n).$$

Because of (1.4), this limit is equal to

$$(3.1)' \qquad \lim_{t \to 0+} \frac{1}{t} \int_{d(x, x_0) \leq \epsilon} (F^i(x) - F^i(x_0))\, P(t, x, dx),$$

independently of the positive constant ϵ. We shall denote these new local coordinates $F^1(x), F^2(x), \cdots, F^n(x)$ by the letters $x^1, x^2, \cdots, x^n$. Then

$$(3.1)'' \qquad \lim_{t \to 0+} \frac{1}{t} \int_{d(x, x_0) \leq \epsilon} (x^i - x_0^i)\, P(t, x, dx) = a^i(x_0) \text{ exists} \qquad (i = 1, \cdots, n),$$

independently of $\epsilon > 0$. The function $F_0(x)$ belongs to $D(A)$; hence, by (1.4),

$$(3.2) \qquad (AF_0)(x_0) = \lim_{t \to 0+} \frac{1}{t} \int_{d(x, x_0) \leq \epsilon} (F_0(x) - F_0(x_0))\, P(t, x_0, dx),$$

independently of $\epsilon > 0$. This limit is equal to

$$\lim_{t \to 0+} \left[\frac{1}{t} \int_{d(x, x_0) \leq \epsilon} (x^i - x_0^i) \frac{\partial F_0}{\partial x_0^i} P(t, x_0, dx) \right.$$
$$\left. + \frac{1}{t} \int_{d(x, x_0) \leq \epsilon} (x^i - x_0^i)(x^j - x_0^j) \left(\frac{\partial^2 F_0}{\partial x^i\, \partial x^j} \right)_{x = x_0 + \theta(x - x_0)} P(t, x_0, dx), \right]$$
$$0 < \theta < 1.$$

The first term in [ ] has the limit

$$a^i(x_0) \frac{\partial F_0}{\partial x_0^i},$$

and hence the second term has a limit. Thus, by virtue of (1.1) and (2.19),

$$\overline{\lim_{t\to 0+}}\ \frac{1}{t}\int_{d(x,\,x_0)\le\epsilon}\ \sum_{i=1}^{n}(x^i-x_0^i)^2\,P(t,\,x_0,\,dx)<\infty. \tag{3.3}$$

Hence, by (1.1) and Schwarz's inequality,

$$\frac{1}{t}\int_{d(x,\,x_0)\le\epsilon}(x^i-x_0^i)\,(x^j-x_0^j)\,P(t,\,x_0,\,dx)\ \text{is bounded in}\ t>0. \tag{3.4}$$

Therefore, by (1.4), we obtain (1.5).

4. **Proof of Theorem 2.** Since $c_{ij}(\epsilon)$ is of order ϵ, we have

$$\begin{aligned}\frac{(T_tf_0)(x_0)-f_0(x_0)}{t} &= \frac{1}{t}\int_{d(x,\,x_0)\le\epsilon}(x^i-x_0^i)\,P(t,\,x_0,\,dx)\,\frac{\partial f_0}{\partial x_0^i}\\ &+\frac{1}{t}\int_{d(x,\,x_0)\le\epsilon}(x^i-x_0^i)\,(x^j-x_0^j)\,P(t,\,x_0,\,dx)\,\frac{\partial^2 f_0}{\partial x_0^i\,\partial x_0^j}\\ &+\frac{1}{t}\int_{d(x,\,x_0)\le\epsilon}(x^i-x_0^i)\,(x^j-x_0^j)\,c_{ij}(\epsilon)\,P(t,\,x_0,\,dx)\\ &+\frac{1}{t}\int_{d(x,\,x_0)\ge\epsilon}(f_0(x)-f_0(x_0))\,P(t,\,x_0,\,dx)\\ &= I_1(t,\,\epsilon)+I_2(t,\,\epsilon)+I_3(t,\,\epsilon)+I_4(t,\,\epsilon).\end{aligned} \tag{4.1}$$

Now

$$\lim_{t\to 0+} I_1(t,\,\epsilon)=a^i(x_0)\,\frac{\partial f_0}{\partial x_0^i}\ \text{by (3.1)}'';\quad \lim_{\epsilon\to 0+} I_3(t,\,\epsilon)=0\ \text{by (3.4)}; \tag{4.2}$$

$$\lim_{t\to 0+} I_4(t,\,\epsilon)=0\ \text{by (1.4)}.$$

On the other hand, by (1.4) and (3.4), the finite limits

$$\overline{\lim_{t\to 0+}}\ \frac{1}{t}\int_{d(x,\,x_0)\le\epsilon}(x^i-x_0^i)\,(x^j-x_0^j)\,P(t,\,x_0,\,dx)=b_1^{ij}(x_0),$$

$$\underline{\lim_{t\to 0+}}\ \frac{1}{t}\int_{d(x,\,x_0)\le\epsilon}(x^i-x_0^i)\,(x^j-x_0^j)\,P(t,\,x_0,\,dx)=b_2^{ij}(x_0)$$

exist and are independent of $\epsilon > 0$. Let us, in place of $f_0(x)$, take $F_0(x)$ of the form $(f \otimes g)(x)$. We may choose $F_0(x)$ such that $\partial F_0/\partial x_0^i \, \partial x_0^j$ assumes values arbitrarily near to given constants α_{ij} $(i, j = 1, \cdots, n)$. Thus, by (4.1) and (4.2) and the fact $F_0(x) \in D(A)$, we see that $b_1^{ij}(x_0)$ must be equal to $b_2^{ij}(x_0)$. Hence (1.7) is proved.

Therefore, by (1.4), (3.1)$''$, and (4.2), we obtain (1.8).

References

1. Nelson Dunford, *On one-parameter groups of linear transformations*, Ann. of Math. 39 (1938), 569-573.

2. W. Feller, *Zur Theorie der stochastischen Prozesse*, Math. Ann. **113** (1936), 113-160.

3. E. Hille, *Functional analysis and semi-groups*, American Mathematical Society, New York, 1948.

4. A. Kolmogoroff, *Zur Theorie der stetigen zufälligen Prozesse*, Math. Ann. **108** (1933), 149-160.

5. K. Yosida, *On the differentiability and the representation of one-parameter semi-groups of linear operators*, J. Math. Soc. Japan **1** (1948), 15-21.

6. ———, *An extension of Fokker-Planck's equation*, Proc. Japan Acad. 25 (1949), No. 9, 1-3.

MATHEMATICAL INSTITUTE,
NAGOYA UNIVERSITY.