ON MATRICES HAVING THE SAME CHARACTERISTIC EQUATION

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1. **Introduction.** Let A and B be $n \times n$ matrices whose elements lie in an infinite perfect[1] field F. Alfred Brauer [1] and W. V. Parker [2] have considered the question: "When do A and B have the same characteristic equation?" Their results have been sufficiency conditions with special forms of A and B. W. T. Reid [3] has considered a related problem.

The present paper is concerned with the following theorem that contains the results of Brauer and Parker as special cases.

THEOREM. *A necessary and sufficient condition for matrices A and B to have the same characteristic equation is that there exist a nonsingular matrix P (with elements in F) such that for $N = A - P^{-1}BP$:*

Every polynomial g in A and N, each term of which contains N at least once, is nilpotent.

We introduce a special canonical form in §2 and give the proof in §3.

2. **Canonical forms.** For any matrix A, there exists a nonsingular matrix P_1, with elements in F, such that

$$P_1^{-1} A P_1 = A_1 \dotplus A_2 \dotplus \cdots \dotplus A_k , \tag{2.1}$$

where the characteristic equation of A_i is $[p_i(x)]^{\alpha_i} = 0$, and $p_i(x)$ is an irreducible polynomial over F. Moreover, for each A_i we have the decomposition by the nonsingular matrix P_{2i} with elements in F:

$$P_{2i}^{-1} A_i P_{2i} = A_{i1} \dotplus A_{i2} \dotplus \cdots \dotplus A_{ik_i} , \tag{2.2}$$

in which each $A_{i\mu}$ is nonderogatory with characteristic equation $[p_i(x)]^{\alpha_{i\mu}} = 0$ and is of the form [4, p. 750]

[1] Every irreducible equation over F is separable.

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$$
(2.3) \qquad A_{i\mu} \equiv \begin{pmatrix} C_i & I_i & 0 & \cdots & 0 \\ 0 & C_i & I_i & \cdots & 0 \\ \cdot & & \ddots & \ddots & \vdots \\ \cdot & & & \ddots & I_i \\ 0 & \cdot & \cdot & \cdot & C_i \end{pmatrix};
$$

where C_i is the companion matrix of $p_i(x)$, and occurs $\alpha_{i\mu}$ times down the main diagonal; I_i is the identity matrix of order the degree of $p_i(x)$. Clearly

$$
\sum_{\mu=1}^{k_i} \alpha_{i\mu} = \alpha_i.
$$

Letting $P_2 = P_{21} \dotplus P_{22} \cdots \dotplus P_{2k}$ and $P = P_1 P_2$, we have a direct sum decomposition of A into matrices $A_{i\mu}$ of form (2.3). We shall indicate this by

$$
(2.4) \qquad P^{-1} A P = \dotplus \sum_{i=1}^{k} \sum_{\mu=1}^{k_i} A_{i\mu}.
$$

It should be pointed out that the existence of the canonical form (2.3) depends only on the perfectness of the field F.

3. **Proof of the theorem.** *Necessity.* Suppose A and B have the same characteristic equation

$$
m(x) = \prod [p_i(x)]^{\alpha_i} = 0.
$$

We may then find matrices P_a and P_b (see §2) such that

$$
(3.1) \qquad P_a^{-1} A P_a = \dotplus \sum_{i=1}^{k} \sum_{\mu=1}^{k_i} A_{i\mu},
$$

$$
P_b^{-1} B P_b = \dotplus \sum_{i=1}^{k} \sum_{\mu=1}^{h_i} A_{i\mu}^*;
$$

where $A_{i\mu}$ and $A_{i\mu}^*$ (for the same subscript i) are of the form (2.3) and thus have the same blocks C_i on the main diagonal. Moreover $\dotplus \sum_{\mu=1}^{k_i} A_{i\mu}$ and $\dotplus \sum_{\mu=1}^{h_i} A_{i\mu}^*$ have the same order since A and B have the same characteristic equation.

Clearly $\dotplus \sum_{\mu=1}^{k_i} A_{i\mu}$ is contained in the algebra of all $\alpha_i \times \alpha_i$ matrices, with elements in the field $F(C_i)$, whose elements below the main diagonal are zero. Moreover,

$$N_i = \dot{+} \sum_{\mu=1}^{k_i} A_{i\mu} - \dot{+} \sum_{\mu=1}^{h_i} A_{i\mu}^*$$

is in the radical of this algebra since all elements on or below the main diagonal are zero. Thus $g(\dot{+} \sum_{\mu=1}^{k_i} A_{i\mu}, N_i)$, for g satisfying the conditions of the theorem, is a radical element and thus nilpotent. Hence, letting

$$N^1 = N_1 \dot{+} N_2 \dot{+} \cdots \dot{+} N_k = P_a^{-1} A P_a - P_b^{-1} B P_b,$$

we see that $g(P_a^{-1} A P_a, N^1)$ is nilpotent. Finally, letting

$$P = P_b P_a^{-1} \quad \text{and} \quad N = P_a N^1 P_a^{-1} = A - P^{-1} B P,$$

we have the result that

$$P_a \, g(P_a^{-1} A P_a, N^1) P_a^{-1} = g(A, N) \tag{3.2}$$

is nilpotent. This completes the proof of the necessity.

Sufficiency. Assume that a P exists such that every polynomial g, satisfying the conditions of the theorem, is nilpotent. Define

$$A_\theta = A - \theta N \qquad (N = A - P^{-1} B P),$$

$m_\theta(\lambda) = |\lambda I - A_\theta| \equiv \lambda^n + a_1(\theta)\lambda^{n-1} + \cdots + a_{n-1}(\theta)\lambda + a_n(\theta)$; where θ is an indeterminate and $a_i(\theta)$ $(i = 1, 2, \cdots, n)$ are polynomials in θ with coefficients in F.

Clearly, $m_0(\lambda) = 0$ and $m_1(\lambda) = 0$ are the characteristic equations of $A_0 \equiv A$ and $A_1 \equiv P^{-1} B P$, respectively.

If we now let θ assume values from F we have

$$m_0(A_\theta) = m_0(A) + h_\theta(A, N) = h_\theta(A, N);$$

moreover $h_\theta(A, N)$ contains N in each term and is nilpotent by hypothesis.

The characteristic roots of $m_0(A_\theta)$ are $m_0(\alpha_\theta^i)$ $(i = 1, \cdots, n)$, where the α_θ^i are the characteristic roots of A_θ. Since $m_0(A_\theta)$ is nilpotent we must have

$$m_0(\alpha_\theta^i) = 0 \qquad (i = 1, \cdots, n). \tag{3.3}$$

From (3.3) it is clearly seen that there can be only a finite number of different characteristic equations $m_\theta(\lambda) = 0$, since all the characteristic roots of A_θ are roots of $m_0(\lambda) = 0$. Since F is assumed to be infinite, this implies that $a_i(\theta)$ is a constant independent of θ. Thus $m_0(\lambda) \equiv m_1(\lambda)$, and the proof of the sufficiency is complete.

References

1. Alfred Brauer, *On the characteristic equations of certain matrices*, Bull. Amer. Math. Soc. 53 (1947), 605-607.

2. W. V. Parker, *On matrices whose characteristic equations are identical*, Proc. Amer. Math. Soc. **1** (1950), 464-466.

3. W. T. Reid, *A note on the characteristic polynomials of certain matrices*, Proc. Amer. Math. Soc. **1** (1950), 584-585.

4. John Williamson, *The idempotent and nilpotent elements of a matrix*, Amer. J. Math. 58 (1936), 747-758.

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