

ON THE IDEAL STRUCTURE OF THE RING OF ENTIRE FUNCTIONS

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1. **Introduction.** Let R be the ring of entire functions, and let K be the complex field. The ring R consists of all functions from K to K differentiable everywhere (in the usual sense).

The algebraic structure of the ring of entire functions seems to have been investigated extensively first by O. Helmer [1].

The ideals of R are herein classified as in [2]: an ideal I is called *fixed* if every function in it vanishes at at least one common point; otherwise, I is called *free*. The structure of the fixed ideals was determined in [1]. The structure of the free ideals is determined below.

While examples of free ideals are easily given, transfinite methods seem to be needed to construct maximal free ideals. The latter are characterized below, and it is shown that the residue class field of a maximal free ideal is always isomorphic to K ; the field theory of E. Steinitz [5] is used.

2. **Elementary properties.** Many expositions of the elementary properties of entire functions are available; see [3]. Some of these properties will be repeated below for the sake of completeness.

DEFINITION 1. If $f \in R$, let $A(f) \equiv \{z \in K \mid f(z) = 0\}$.

NOTE. As in [1], we take $A(f)$ as an "algebraic set." That is, if z is a zero of multiplicity m of f , the z appears m times in $A(f)$. The union and intersection of two such sets is taken in the same sense.

DEFINITION 2. If I is any subset of R , let $A(I) \equiv \{A(f) \mid f \in I\}$.

(2.1) If f is any nonzero element of R , then $A(f)$ is a closed, discrete (and hence finite or countably infinite) subset of K in the natural topology of K . Conversely, given any closed, discrete subset D of K , there is an $f \in R$ such that $D = A(f)$. Note that any nonempty closed discrete subset of K either is

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finite or is a sequence $\{z_n\}_{n=1}^{\infty}$ such that $\lim_{n \rightarrow \infty} z_n = \infty$.

(2.2) R is an integral domain.

(2.3) If $D = \{z_n\}_{n=1}^{\infty}$ is any closed, discrete subset of K , and if $\{w_n\}_{n=1}^{\infty}$ is any sequence of complex numbers, then there is an $f \in R$ such that $f(z_i) = w_i$; $i = 1, 2, \dots$ (see [3] p. 33, Exercise 3).

(2.4) If $A(f) = A(g)$, then $f = gh$, where h is a unit (element with inverse) of R .

3. Ideals of R . It will be seen below that the nature of the ideals I of R is completely determined by the sets $A(I)$. Since an element of R is a unit if and only if it vanishes nowhere, every function in an ideal vanishes somewhere. Hence the ideals of R are classified as in [2].

DEFINITION 3. An ideal I of R is called *fixed* if $\bigcap_{f \in I} A(f)$ is nonempty. Otherwise I is called a *free* ideal.

As the definition indicates, the structure of the fixed ideals is very simple. Helmer determined their structure in the course of his investigation of the arithmetic properties of entire functions. He noted that if S is any subset of R , then any function d such that

$$A(d) = \bigcap_{f \in S} A(f),$$

is a greatest common divisor (unique to within a unit factor) of the functions of S . Moreover, Helmer showed that if the set S is finite, its elements being f_1, \dots, f_n , then there exist elements e_1, \dots, e_n of R such that $d = e_1 f_1 + \dots + e_n f_n$. Hence we have:

THEOREM 1 (Helmer). *If f_1, \dots, f_n is any finite set of elements of R , there exists a function d and elements e_1, \dots, e_n of R such that*

$$A(d) = \bigcap_{k=1}^n A(f_k)$$

and $d = e_1 f_1 + \dots + e_n f_n$. Hence any ideal I of R with finite basis is principal, and so is fixed.

Proof. The proof of the first part is given in [1, Theorem 9]. Clearly if the ideal I is generated by f_1, \dots, f_n , then $\bigcap_{f \in I} A(f) = A(d)$, where d is a greatest common divisor of f_1, \dots, f_n .

COROLLARY 1. *No free ideal of R has a finite basis.*

COROLLARY 2. *No polynomial can belong to a free ideal.*

Proof. If I is an ideal of R and if $f, g \in I$, then by Theorem 1 we have $d \in I$ where $A(d) = A(f) \cap A(g)$. Hence any finite number of elements of I have common zeros. But a polynomial p has only a finite number of zeros. Hence if $p \in I$, then all the functions of I would have a common zero, whence I would be a fixed ideal.

Helmer gave an example of a fixed ideal without finite basis (see [1, proof of Theorem 8]).

The example below shows that free ideals exist.

EXAMPLE. Let $\{z_n\}_{n=1}^{\infty}$ be any sequence of complex numbers such that $\lim_{n \rightarrow \infty} z_n = \infty$. Let $\{S_N\} = \{z_N, z_{N+1}, \dots\}$. By (2.1), we can construct for each natural number N an entire function F_N vanishing precisely on S_N . Let I be the ideal generated by the F_N . It is easily seen that I is a free ideal.

Free ideals are characterized in terms of the families $A(I)$ as in [2, Theorem 36]:

THEOREM 2. *Let $\{A_\alpha\}_{\alpha \in A}$ (where A is an index set) be a family of sequences of complex numbers $\{z_n^{(\alpha)}\}$ such that $\lim_{n \rightarrow \infty} z_n^{(\alpha)} = \infty$, for all $\alpha \in A$. Moreover, suppose that:*

- (i) *The family $\{A_\alpha\}_{\alpha \in A}$ is closed under finite set intersection.*
- (ii) *$\bigcap_{\alpha \in A} A_\alpha$ is empty.*

If $F_\alpha = [f \in R \mid f(z_\alpha) = 0, \text{ all } z_\alpha \in A_\alpha]$, then the ideal I generated by the families F_α is free. Conversely, if $I = [f_\alpha]$ is any free ideal of R , and $A_\alpha = [z \in K \mid f_\alpha(z) = 0]$, then the family $\{A_\alpha\}$ satisfies (i) and (ii) above.

Proof. Suppose I is defined as above. Since (i) holds, I is closed under subtraction. If $f \in I$, $g \in R$, then f is in some F_α , whence fg is in the same F_α . So I is an ideal, and (ii) ensures that it is free.

Conversely, if I is any free ideal, Theorem 1 ensures that (i) holds, and (ii) follows from the definition of a free ideal.

4. Maximal ideals and their residue class fields. If z_0 is any fixed element of K , let $I(z_0) \equiv [f \in R \mid f(z_0) = 0]$. Clearly $I(z_0)$ is a fixed ideal of R . Moreover, the mapping $f(z) \rightarrow f(z_0)$ is clearly a homomorphism of R upon K whose kernel is $I(z_0)$. Hence $I(z_0)$ is a maximal fixed ideal. Conversely, if I is a fixed ideal and if $\bigcap_{f \in I} A(f)$ contains two points z_1, z_2 (not necessarily distinct), then I is properly contained in $I(z_1)$ or $I(z_2)$. Hence we have proved:

THEOREM 3. *Every maximal fixed ideal of R is of the form*

$$I(z_0) = [f \in R \mid f(z_0) = 0]$$

for some $z_0 \in K$. Moreover, the residue class field of every maximal fixed ideal is the complex field K .

The maximal free ideals are not so simple in structure. They may, however, be characterized as follows:

THEOREM 4. *A free ideal M of R is maximal if and only if $A(M)$ satisfies:*

- (iii) *If $D = \{z_n\}_{n=1}^{\infty}$ is any infinite, closed, discrete set of K such that $D \cap A(f)$ is nonempty for every $f \in M$, then $D \in A(M)$.*

Proof. Suppose M is a free ideal and (iii) holds. If M is not maximal, then there is an ideal N properly containing M . Suppose $g \in N$ and apply (i) of Theorem 2 to $A(N)$. Then $A(g) \cap A(f)$ is nonempty for every $f \in N$, and hence for every $f \in M$. Hence $g \in M$ by (iii). Hence M is a maximal free ideal.

Conversely, suppose M is a maximal free ideal. If there were an infinite, closed, discrete set D violating (iii), then any $g \in R$ such that $A(g) = D$ would together with M generate an ideal N properly containing M . Hence (iii) must hold.

NOTE. This result is similar to a theorem of Hewitt on maximal ideals of rings of real valued continuous functions; see [2, Theorem 36].

Since maximal free ideals are complicated in structure, it is natural to expect the same of their residue class fields. First we show:

THEOREM 5. *If M is a maximal free ideal, then R/M contains a subfield isomorphic to the field $R(z)$ of all rational functions of a complex variable z .*

Proof. By Corollary 2 of Theorem 1, M can contain no polynomial. Hence if p_1, p_2 are two distinct polynomials, then $p_1 \not\equiv p_2 \pmod{M}$. Hence R/M contains as a subring all polynomials in z . So R/M contains $R(z)$ as a subfield.

COROLLARY. *The field K is subfield of R/M . If R is considered as an algebra over K , then the residue class field R/M may be considered as a division algebra containing K as a proper subfield.*

Proof. If R is considered to be an algebra over K , the homomorphism of R upon the quotient algebra R/M is assumed to keep complex numbers fixed. Hence K is a proper division subalgebra of R/M .

If one does not insist that the complex numbers stay fixed under a homo-

morphism of R upon R/M , we have:

THEOREM 6. *If M is a maximal free ideal, then R/M is isomorphic (as a ring) to K .*

We shall establish two lemmas before proving the theorem.

LEMMA 1. *The field R/M is algebraically closed.*

Proof. Note first that if $f \in M$, then M contains all functions vanishing on the distinct points of $A(f)$. This is true since the maximality of M ensures that M contains a function g with simple zeros at the distinct points of $A(f)$; and by (2.4) it contains all such functions. Now let

$$\Phi(z, X) = f_0(z) + f_1(z)X + \cdots + f_n(z)X^n$$

be any polynomial with coefficients $f_0, f_1, \dots, f_n \in R$, where f_n is not in M , $n > 0$. Choose any sequence $\{z_k\} \in A(M)$. Now for any fixed k , the function $\Phi(z_k, X)$ is a polynomial with coefficients in K and hence has n complex roots. Choose any such root and call it r_k . Then construct, by (2.3), a function $g \in R$ such that $g(z_k) = r_k$ ($k = 1, 2, 3, \dots$). Clearly, by the above, $\Phi(z, g(z)) \equiv 0 \pmod{M}$. Hence R/M is algebraically closed.

LEMMA 2. *The field R/M contains c elements, where c is the cardinal number of the continuum.*

Proof. Since K contains a countable dense subset, there are only c continuous functions, and hence only c entire functions. Hence R/M has at most c elements. But all complex numbers are incongruent \pmod{M} , so R/M has at least c elements. Hence R/M has precisely c elements.

Proof of Theorem 6. Steinitz has shown [5, p. 125, Section 6] that any algebraically closed field of characteristic 0 and of degree of transcendence c over its prime field is isomorphic to K . Since R/M contains K , it has degree of transcendence at least c . By Lemma 2, R/M can have degree of transcendence at most c . Hence R/M has degree of transcendence c . By Lemma 1, R/M is algebraically closed. Hence R/M is isomorphic to K .

5. Topological considerations. In [4], Schilling investigated the ring R of entire functions as a topological ring under the topology of uniform convergence on compact sets. He showed [4, p. 949, Lemma 3] that any closed ideal of R is principal. Hence, in particular, no free ideal is closed. He also stated [4, p. 952, Lemma 6] that a maximal ideal M of R is nonclosed if and only if R/M is a proper extension of K . This is in apparent contradiction with our Theorem 6.

This apparent discrepancy is easily resolved. Although it is nowhere explicitly stated, Schilling considered R , R/M , and K as algebras over K . He then proved the equivalent of our Theorem 5. Thus R/M is a commutative division algebra containing K properly, which is isomorphic as a ring to K .

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