

SUBGROUPS OF FREE PRODUCTS

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1. Introduction. It was first shown by Kurosch [3] that a subgroup H of a free product

$$G = \prod_{\nu}^* A_{\nu}$$

is itself a free product

$$H = F * \prod_j^* x_j^{-1} U_j x_j$$

of a free group F and conjugates $x_j^{-1} U_j x_j$ of subgroups U_j of free factors A . The original proof of Kurosch involved constructing the free factors of H one at a time, and both the proof and the construction depended on transfinite induction. A later proof by Baer and Levi [1] was topological in nature. More recently a proof has been given by Kuhn [2].

The present paper gives a new proof of this theorem, which, apart from the use of well-ordering, is purely algebraic. It is shorter and simpler than the Kurosch proof. In terms of a semi-alphabetical ordering of G , a set K of elements generating H is found, and it is then shown from the properties of K that H is a free product of factors as stated above.

2. The theorem of Kurosch. The result is the following:

THEOREM OF KUROSCH. *A subgroup $H \neq 1$ of a free product*

$$G = \prod_{\nu}^* A_{\nu}$$

is itself a free product

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$$H = F * \prod_j^* x_j^{-1} U_j x_j,$$

where F is a free group and each $x_j^{-1} U_j x_j$ is the conjugate of a subgroup U_j of one of the free factors A_ν of G .

Proof. The elements of the free factors of G may be well ordered by beginning with the identity, then taking an ordering of the free factors, and within a free factor taking an ordering of the elements $\neq 1$. Based on this ordering we define an *alphabetical ordering* for the elements of G . Write

$$(1) \quad g = a_1 a_2 \cdots a_t$$

as the reduced form of an element g of G . The void product is the identity; and for $g \neq 1$ each a_i is an element $\neq 1$ of one of the free factors A_ν , and no two consecutive terms a_i, a_{i+1} ($i = 1, \dots, t-1$) belong to the same free factor A_ν . The length $l(g)$ of an element g is defined as zero for $g = 1$, and for $g \neq 1$ as the number t of terms in its reduced form (1). We define the alphabetical ordering of elements by ordering successively on

- 1) the length of g ;
- 2.1) the order of the first term a_1 if $g = a_1 a_2 \cdots a_t$ is its reduced form;
- 2.2) the order of a_2 ;
- · · · ·
- 2.t) the order of a_t .

This is clearly a well ordering of the elements of G .

We now define a second ordering for the elements of G , the *semi-alphabetical ordering*. For this we write an element g of even length $t = 2r$ in the form $g = \alpha \beta^{-1}$, where $l(\alpha) = l(\beta) = r$; and an element g of odd length $t = 2s + 1$ in the form $g = \alpha a_{s+1} \beta^{-1}$, where $l(\alpha) = l(\beta) = s$. The semi-alphabetical ordering for elements g is determined successively by:

- 1) the length of g ;
- 2) for $g = \alpha \beta^{-1}$ of even length by 2.1) the alphabetical order of α , and by 2.2) the alphabetical order of β ;
- 3) for $g = \alpha a_{s+1} \beta^{-1}$ of odd length by 3.1) the alphabetical order of α , by 3.2) the alphabetical order of β , and by 3.3) the order of a_{s+1} .

The proof that the subgroup H of G is a free product will be carried out by selecting, in terms of the semi-alphabetical ordering, a subset K of the elements of H and showing 1) that the elements of K generate H and then 2) that the elements K generate a free product

$$F * \prod_j^* x_j^{-1} U_j x_j,$$

where F is a free group and each U_j is a subgroup of some free factor A .

The set K of elements shall consist of all elements $k \neq 1$ such that 1) $k \in H$, and 2) k does not belong to the group generated by the elements of H which precede k in the semi-alphabetical ordering.

Since $H \neq 1$, the first $h \neq 1$ of H belongs to the set K , and so K is not vacuous. Consider the group $[K]$ generated by the set K . Clearly $[K] \subseteq H$. If $[K] \neq H$, there must be a first $h \in H$ such that $h \notin [K]$. Such an h does not belong to K , and so is a product of elements h_i preceding h and belonging to H . But these h_i belong to $[K]$, and so h as a product of these h_i 's also belongs to $[K]$. Hence $[K] = H$, and this covers the first part of the proof.

We shall use the sign $<$ for numerical inequalities and for both the alphabetical and semi-alphabetical orderings. It will be clear from the context which meaning is appropriate, the semi-alphabetical ordering applying to entire words, the alphabetical to beginnings or endings of words. Writing $u \neq 1$ in the form $u = \alpha \beta^{-1}$ or $u = \alpha a \beta^{-1}$, we cannot have $\beta = \alpha$ for words of even length since $\alpha \alpha^{-1} = 1$. For elements of odd length, $\beta = \alpha$ is possible; and those elements of H of the form $\alpha a \alpha^{-1}$ for fixed α , and a 's belonging to some fixed A_ν , together with the identity, form a subgroup $\alpha B \alpha^{-1}$ conjugate to $B \subseteq A_\nu$. Let us call elements $\alpha a \alpha^{-1}$ *transforms*. Let us extend the set K to a larger set T which consists of K and, for each α and A_ν , those transforms $\alpha a' \alpha^{-1}$, $a' \in A_\nu$, generated by transforms $\alpha a \alpha^{-1}$, $a \in A_\nu$, belonging to K . Hence T consists of elements of H not generated by their predecessors and transforms $\alpha a' \alpha^{-1}$ generated by earlier transforms of the same kind.

An element $h \in H$ can be written in the form

$$(2) \quad h = u_1 u_2 \cdots u_t,$$

where $u_i \in T$ or T^{-1} (the set of inverses of elements in T). Moreover we can take (2) so that a) $u_i u_{i+1} \neq 1$ ($i = 1, \dots, t-1$) and b) no two consecutive u_i, u_{i+1} belong to the same conjugate group $\alpha B \alpha^{-1}$, $B \subseteq A_\nu$. If these conditions are satisfied then we shall say that $u_1 \cdots u_t$ is in half-reduced form.

The theorem will follow immediately if it can be shown that any nonvacuous half-reduced form cannot be the identity. For then it will follow that the elements of K which are not transforms generate a free group F , and that H is the free product of F and the conjugates $\alpha B \alpha^{-1}$, $B \subseteq A_\nu$.

If u is an element of K , and $u^{-1} \neq u$, then $u < u^{-1}$ since $u = (u^{-1})^{-1}$ and u^{-1} cannot be a predecessor of u . Also if $u \neq v$ are elements of K , then $w = u^\epsilon v^\eta$ ($\epsilon, \eta = \pm 1$) will follow both u and v , since any two of u, v, w generate the third and by the choice of K neither u nor v is generated by predecessors. These two principles are the main tools in studying the way in which the elements of T and T^{-1} combine. In reducing a product $a_1 a_2 \cdots a_m$ in G , where each a_i belongs to one of the free factors, we say that a_i and a_{i+1} *amalgamate* into a'_i if a_i and a_{i+1} belong to the same free factor A and $a_i a_{i+1} = a'_i$, and that they *cancel* if $a_i a_{i+1} = 1$.

LEMMA 1. *If $u = \alpha \beta^{-1}$ or $\alpha a \beta^{-1} \in T$, and $\beta \neq \alpha$, then $\alpha < \beta$.*

Proof. Since $\beta \neq \alpha$, we have $u \in K$; and if $\beta < \alpha$, we would have $u^{-1} < u$. Thus the elements of T are of three kinds:

- 1) $l(u)$ even, $u = \alpha \beta^{-1}$, $\alpha < \beta$, $u \in K$;
- 2) $l(u)$ odd, $u = \alpha a \beta^{-1}$, $\alpha < \beta$, $u \in K$;
- 3) $l(u)$ odd, $u = \alpha a \alpha^{-1}$, generated by transforms of the same kind in K .

LEMMA 2. *If $u \neq v$ belong to T and are not both in the same conjugate $\alpha B \alpha^{-1}$, and w is any one of $u^\epsilon v^\eta$ or $v^\eta u^\epsilon$ ($\epsilon, \eta = \pm 1$), then w follows both u and v in the semi-alphabetical ordering. This leads to the following restrictions on cancellation and amalgamation in the product w :*

- 1) *If $u = \alpha \beta^{-1}$, β does not cancel, and if α cancels, then the adjacent term of β^{-1} does not amalgamate;*
- 2) *if $u = \alpha a \beta^{-1}$, $\alpha < \beta$, α and a do not cancel, and if β cancels, then a does not amalgamate;*
- 3) *if $u = \alpha a \alpha^{-1} \in \alpha B \alpha^{-1}$, α and a do not cancel, and if $v^\eta = \alpha a' \sigma$, with $a, a' \in A_\nu$, then a' is the earliest element in the coset Ba' .*

Proof. Of the two different elements u and v belonging to T , let the letter u represent the earlier, so that $u < v$. If w does not follow both u and v , then $w < v$. Here the possibility $w = v$ may be eliminated at once since it would imply $u = 1$, which cannot hold, or $u = v^2$ or v^{-2} . Now the square of a transform v is 1 or is a similarly transform, while if v is not a transform then $l(v^2) > l(v)$.

In either case $u = v^2$ or v^{-2} is impossible.

Since any two of u, v, w generate the third, w must be the third if v belongs to K . Thus we need only consider cases with $v = \alpha a \alpha^{-1} \in \alpha B \alpha^{-1}$ a transform. Now with $u < v$, since $u \notin \alpha B \alpha^{-1}$, we also have $u < v^*$, where v^* is any transform in $\alpha B \alpha^{-1}$. Hence if the canceling between u and $v = \alpha a \alpha^{-1}$ involves only α (or α^{-1}), the same will hold for u and some $v^* = \alpha a^* \alpha^{-1} \in K$, yielding a product $w^* = u^\epsilon v^*$ or $v^* u^\epsilon$ with $w^* < v^*$ contrary to $v^* \in K$. Hence the canceling between u and $v = \alpha a \alpha^{-1}$ involves all of α and cancellation or amalgamation with the center term a . Thus $u^\epsilon = \alpha a'' \sigma^{-1}$, where a'' amalgamates or cancels with a . Since $u < v = \alpha a \alpha^{-1}$, either $l(\sigma) < l(\alpha)$ or $l(\sigma) = l(\alpha)$; and $u = \sigma a''^{-1} \alpha^{-1}$, with $\sigma < \alpha$. In either event, u and $\sigma(a''^{-1} a^* a'') \sigma^{-1}$ precede and generate a $v^* = \alpha a^* \alpha^{-1} \in K$, a contradiction. Thus in all cases we reach a contraction if $w \leq v$, and so w follows both u and v .

In consequence of the fact that all eight products $u^\epsilon v^\eta$ and $v^\eta u^\epsilon$ follow both u and v , we have the restrictions on canceling and amalgamating listed in the theorem. These say explicitly that not more than half of either u or v cancels, and that in cases where canceling and amalgamating with one replaces an initial (or final) segment of the other with another segment of the same length, the result is an element later in the ordering.

LEMMA 3. *In a product $u_1 u_2 \dots u_t$ with $u_i \in T \cup T^{-1}$ ($i = 1, \dots, t$), $u_i u_{i+1} \neq 1$ ($i = 1, \dots, t-1$), and u_i, u_{i+1} not both in the same group $\alpha B \alpha^{-1}$ ($B \subseteq A_\nu$), the reduced form will end as follows:*

- 1) β^{-1} if $u_t = \alpha \beta^{-1}$,
- 2) $b^* \alpha^{-1}$ if $u_t = (\alpha \beta^{-1})^{-1}$,
- 3) $a^* \beta^{-1}$ if $u_t = \alpha a \beta^{-1}$, $\alpha < \beta$,
- 4) $a^{-1} \alpha^{-1}$ if $u_t = (\alpha a \beta^{-1})^{-1}$, $\alpha < \beta$,
- 5) $a^* \alpha^{-1}$ if $u_t = \alpha a \alpha^{-1}$.

Here b^* in 2) and a^* in 5), are either the term immediately preceding in u_t or are amalgamations with a similar term in u_{t-1} . In 3), a^* can involve amalgamation with u_{t-1} and u_{t-2} .

Proof. This lemma will be proved by induction on t , being trivial for $t = 1$. For $t = 2$, the results come directly from Lemma 2 with the added observation that for $u = \alpha \beta^{-1}$ or $\alpha a \beta^{-1}$ the cancellation in u^2 does not go through α or β .

In proving the induction from t to $t + 1$, we need only apply Lemma 2 to each of the five cases listed above, as well as each of the five possibilities for u_{t+1} , using only one additional property not an immediate consequence of Lemma 2. This is as follows: It may happen that $u_t = \alpha a \alpha^{-1}$, that α cancels, and that a amalgamates with $u_{t-1} = \sigma a^{1-1} \alpha^{-1}$, and similarly with $u_{t+1} = \alpha a'' \lambda$. Now by Lemma 2 each of a' and a'' is the earliest element in its own coset Ba' , Ba'' . If $a'^{-1} a a'' = 1$ this would mean that a' and a'' were in the same coset, and so $a' = a''$, $a = 1$, $u_t = 1$, a contradiction. Hence $a'^{-1} a a'' \neq 1$, and the reduced form of $u_{t-1} u_t u_{t+1}$ is $\sigma(a'^{-1} a a'') \lambda$. This is the only way in which amalgamation can involve as many as three consecutive terms in any product $u_1 u_2 \cdots u_m$ which is half reduced.

In establishing the ending of the reduced form for the half reduced expression $h = u_1 u_2 \cdots u_t$, we have shown *à fortiori* that $h \neq 1$, and hence that H is the free product of the infinite cyclic groups generated by the elements $\alpha \beta^{-1}$ and $\alpha a \beta^{-1}$ ($a < \beta$) and the conjugates $\alpha B \alpha^{-1}$ of subgroups B of free factors A .

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