THE RECIPROCITY THEOREM FOR DEDEKIND SUMS

L. CARLITZ

1. Introduction. Let $((x)) = x - [x] - 1/2$, where $[x]$ denotes the greatest integer $\leq x$, and put

$$\bar{s}(h, k) = \sum_{r (\mathrm{mod}\ k)} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right), \tag{1.1}$$

the summation extending over a complete residue system (mod k). Then if $(h, k) = 1$, the sum $\bar{s}(h, k)$ satisfies (see for example [4])

$$12hk\{\bar{s}(h, k) + \bar{s}(k, h)\} = h^2 + 3hk + k^2 + 1. \tag{1.2}$$

Note that $\bar{s}(h, k) = s(h, k) + 1/4$, where $s(h, k)$ is the sum defined in [4].

In this note we shall give a simple proof of (1.2) which was suggested by Redei's proof [5]. The method also applies to Apostol's extension [1]; [2].

2. A formula for $\bar{s}(h, k)$. We start with the easily proved formula

$$\left(\left(\frac{r}{k}\right)\right) = -\frac{1}{2k} + \frac{1}{k}\sum_{s=1}^{k-1}\frac{\rho^{-rs}}{\rho^s - 1} \qquad (\rho = e^{2\pi i/k}), \tag{2.1}$$

which is equivalent to a formula of Eisenstein. (Perhaps the quickest way to prove (2.1) is to observe that

$$\sum_{r=0}^{k-1}\left(\left(\frac{r}{k}\right)\right)\rho^{rs} = \begin{cases} 1/(\rho^s - 1) & (k \nmid s) \\ -1/2 & (k \mid s); \end{cases}$$

inverting leads at once to (2.1)).

Now substituting from (2.1) in (1.1) we get

Received August 11, 1952.
Pacific J. Math. 3 (1953), 523-527

$$\overline{s}(h,k)=\sum_r\left\{-\frac{1}{2k}+\frac{1}{k}\sum_{t=1}^{k-1}\frac{\rho^{-ts}}{\rho^t-1}\right\}\left\{-\frac{1}{2k}+\frac{1}{k}\sum_{s=1}^{k-1}\frac{\rho^{-hrs}}{\rho^s-1}\right\}$$

$$=\frac{1}{4k}+\frac{1}{k^2}\sum_{s,\,t=1}^{k-1}\frac{1}{(\rho^s-1)(\rho^t-1)}\sum_{r=0}^{k-1}\rho^{-r(sh+t)}.$$

Since the inner sum vanishes unless $s+ht\equiv 0 \pmod{k}$, we get

$$\overline{s}(h,k)=\frac{1}{4k}+\frac{1}{k}\sum_{k=1}^{k-1}\frac{1}{(\rho^{-s}-1)(\rho^{hs}-1)},$$

or, what is the same thing,

$$\overline{s}(h,k)=\frac{1}{4k}+\frac{1}{k}\sum_{\zeta\neq 1}\frac{1}{(\zeta^{-1}-1)(\zeta^h-1)}, \tag{2.2}$$

where ζ runs through the kth roots of unity distinct from 1.

3. Proof of (1.2) In the next place consider the equation

$$(x^h-1)f(x)+(x^k-1)g(x)=x-1, \tag{3.1}$$

where $f(x)$, $g(x)$ are polynomials, $\deg f(x)<k-1$, $\deg g(x)<h-1$. Then if ζ has the same meaning as in (2.2), it is clear from (3.1) that

$$(\zeta^h-1)f(\zeta)=\zeta-1.$$

Thus by the Lagrange interpolation formula

$$f(x)=(x^k-1)\left\{\frac{f(1)}{k(x-1)}+\frac{1}{k}\sum_{\zeta\neq 1}\frac{\zeta}{x-\zeta}\,\frac{\zeta-1}{\zeta^h-1}\right\}. \tag{3.2}$$

Similarly, if η runs through the hth roots of unity,

$$g(x)=\left\{\frac{g(1)}{h(x-1)}+\frac{1}{h}\sum_{\eta\neq 1}\frac{\eta}{x-\eta}\,\frac{\eta-1}{\eta^k-1}\right\}. \tag{3.3}$$

Now it follows from (3.1) that $hf(1)+kg(1)=1$; hence substituting from (3.2) and (3.3) in (3.1) we get the identity

$$(3.4)\qquad \frac{1}{k}\sum_{\zeta\neq 1}\frac{\zeta}{x-\zeta}\,\frac{\zeta-1}{\zeta^h-1}+\frac{1}{h}\sum_{\eta\neq 1}\frac{\eta}{x-\eta}\,\frac{\eta-1}{\eta^k-1}$$

$$=\frac{x-1}{(x^k-1)(x^h-1)}-\frac{1}{hk(x-1)}.$$

Next put $x = 1 + t$ in (3.4) and expand both members in ascending powers of t. We find without difficulty that the right member of (3.4) becomes

$$(3.5)\qquad -\frac{h+k-2}{2hk}+\frac{h^2+3hk+k^2-3h-3k+1}{12hk}\,t+\cdots.$$

Comparison of coefficients of t in both sides of (3.4) leads at once to

$$-\frac{1}{k}\sum_{\zeta\neq 1}\frac{\zeta}{\zeta-1}\,\frac{1}{\zeta^h-1}-\frac{1}{h}\sum_{\eta\neq 1}\frac{\eta}{\eta-1}\,\frac{1}{\eta^k-1}$$

$$=\frac{h^2+3hk+k^2-3h-3k+1}{12hk}.$$

Therefore by (2.2) and the corresponding formula for $s(k, h)$, we have

$$\bar{s}(h,k)+\bar{s}(k,h)=\frac{h^2+3hk+k^2+1}{12hk},$$

which is the same as (1.2).

4. The generalized reciprocity formula. The identity (3.4) implies a good deal more than (1.2). For example, for $x = 0$, we get

$$(4.1)\qquad \frac{1}{k}\sum_{\zeta\neq 1}\frac{\zeta-1}{\zeta^h-1}+\frac{1}{h}\sum_{\eta\neq 1}\frac{\eta-1}{\eta^k-1}=1-\frac{1}{hk},$$

while if we use the constant term in (3.5), we find that

$$(4.2)\qquad \frac{1}{k}\sum_{\zeta\neq 1}\frac{\zeta}{\zeta^h-1}+\frac{1}{h}\sum_{\eta\neq 1}\frac{\eta}{\eta^k-1}=\frac{h+k-2}{2hk}.$$

Again if we multiply by x and let $x \longrightarrow \infty$, we get

$$(4.3)\qquad \frac{1}{k}\sum_{\zeta\neq 1}\zeta\,\frac{\zeta-1}{\zeta^h-1}+\frac{1}{h}\sum_{\eta\neq 1}\eta\,\frac{\eta-1}{\eta^k-1}=-\frac{1}{hk}\,.$$

More generally, expanding (3.4) in descending powers of x, we have

$$(4.4)\qquad \frac{1}{k}\sum_{\zeta\neq 1}\zeta^r\,\frac{\zeta-1}{\zeta^h-1}+\frac{1}{h}\sum_{\eta\neq 1}\eta^r\,\frac{\eta-1}{\eta^k-1}=\begin{cases}-\dfrac{1}{hk} & (1\leq r<h+k-1)\\[2ex] 1-\dfrac{1}{hk} & (r=h+k-1).\end{cases}$$

By continuing the expansion of (3.5) we can also show that

$$h\sum_{\zeta\neq 1}\frac{\zeta}{(\zeta-1)^r(\zeta^h-1)}+k\sum_{\eta\neq 1}\frac{\eta}{(\eta-1)^r(\eta^k-1)}\qquad (r\geq 1)$$

is a polynomial in h, k, but the explicit expression seems complicated. A more interesting result can be obtained as follows. First we divide both sides of (3.4) by $x-1$ so that the left member becomes

$$\frac{1}{k}\sum_{\zeta}\frac{\zeta}{\zeta^h-1}\left(\frac{1}{x-\zeta}-\frac{1}{x-1}\right)+\frac{1}{h}\sum_{\eta}\frac{\eta}{\eta^k-1}\left(\frac{1}{x-\eta}-\frac{1}{x-1}\right)$$

$$=\frac{1}{k}\sum_{\zeta}\frac{\zeta}{\zeta^h-1}\,\frac{1}{x-\zeta}+\frac{1}{h}\sum_{\eta}\frac{\eta}{\eta^k-1}\,\frac{1}{x-\eta}-\frac{h+k-2}{2hk(x-1)}$$

by (4.2). We now put $x=e^t$. Transposing the last term above to the right we find that the right member has the expansion

$$(4.5)\qquad \frac{1}{hk}\sum_{m=0}^{\infty}\frac{(Bh+Bk)^m t^{m-2}}{m!}+\frac{h+k}{2hk}\sum_{m=0}^{\infty}\frac{B_m t^{m-1}}{m!}+\frac{1}{hk}\sum_{m=0}^{\infty}\frac{(m-1)B_m t^{m-2}}{m!},$$

where the B_m are the Bernoulli numbers. In the left member we put

$$\frac{1-\zeta}{e^t-\zeta}=\sum_{m=0}^{\infty}H_m(\zeta)\,\frac{t^m}{m!},$$

where the $H_m(\zeta)$ are the so-called "Eulerian numbers"; we thus get

$$(4.6)\qquad \frac{1}{k}\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{\zeta}\frac{H_m(\zeta^{-1})}{(\zeta-1)(\zeta^{-h}-1)}+\frac{1}{h}\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{\eta}\frac{H_m(\eta^{-1})}{(\eta-1)(\eta^{-k}-1)}.$$

But by [3, formula (6.6)], for p odd > 1,

$$\frac{p}{k^p}\sum_{\zeta}\frac{H_{p-1}(\zeta)}{(\zeta-1)(\zeta^{-h}-1)}=s_p(h,k)$$

where [1]

$$s_p(h,k)=\sum_{r(\mathrm{mod}\ k)}\bar{B}_1\left(\frac{r}{k}\right)\bar{B}_p\left(\frac{hr}{k}\right),$$

and $\bar{B}_r(x)$ is the Bernoulli function. Thus the coefficient of $t^{p-1}/(p-1)!$ in (4.6) is

$$(4.7)\qquad \frac{1}{p}\left\{k^{p-1}s_p(h,k)+h^{p-1}s_p(k,h)\right\},$$

while the corresponding coefficient in (4.5) is

$$(4.8)\qquad \frac{1}{p(p+1)hk}(Bh+Bk)^{p+1}+\frac{1}{(p+1)hk}B_{p+1}.$$

Hence equating (4.7) and (4.8) we get Apostol's formula [1, Theorem 1]:

$$(p+1)\{hk^p s_p(h,k)+kh^p s_p(k,h)\}=(Bh+Bk)^{p+1}+pB_{p+1}$$

for p odd > 1. Note that $s_1(h,k)=\bar{s}(h,k)$.

References

1. T. M. Apostol, *Generalized Dedekind sums and transformation formulae for certain Lambert series*, Duke Math. J. **17** (1950), 147-157.

2. ———, *Theorems on generalized Dedekind sums*, Pacific J. Math. **2** (1950), 1-9.

3. L. Carlitz, *Some theorems on generalized Dedekind sums*, Pacific J. Math. 3(1953), 513-522.

4. H. Rademacher and A. Whiteman, *Theorems on Dedekind sums*, Amer. J. Math. 63 (1941), 377-407.

5. L. Redei, *Elementarer Beweis und Verallgemeinerung einer Reziprozitätsformel von Dedekind*, Acta Sci. Math. Szeged **12**, Part B (1950), 236-239.

Duke University