REMARKS ON SPLITTING EXTENSIONS

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1. Introduction. If N is a normal subgroup of the finite group G we call G an *extension* of N. Such an extension G over N is said to *split* if there exists a *complement* of N in G, that is, if there exists a subgroup of G which contains exactly one element from each coset of G modulo N. A frequently used criterion for splitting is provided by a theorem of Schur, namely, *if N has order prime to its index in G, then G splits over N.* W. Gaschütz [1] has recently given a generalization of this theorem for the case when $N = A$ is abelian, which states that (i) *G splits over A if and only if there is for each prime p a p-Sylow subgroup S of G which splits over $S \cap A$,*[1] *and* (ii) *there exists a subgroup $U < G$ such that $G = AU$ if and only if there exists for some prime p a p-Sylow subgroup S of G and a subgroup V of S such that*

$$S = [S \cap A]V, \text{ and } \mathfrak{n}_G(V \cap A) = S \cap A.^1$$

Here $\mathfrak{n}_G(V \cap A)$ denotes the subgroup generated by all the conjugates to $V \cap A$ in G.

In § 2 of this note we apply part (i) of the theorem of Gaschütz to establish a generalization of the theorem of Schur for non-abelian extension. In § 3 we apply (ii) to obtain a characterization of extensions G over N such that N is contained in the Frattini subgroup. The remaining two sections are concerned with the question of conjugacy of complements.

NOTATIONS. Group will always mean finite group unless the contrary is explicitly stated. For H a subgroup of a group G, $[G:H]$ = index of H in G. For Y a set of elements of G, $\{Y\}$ = subgroup generated by the elements of Y. If A and B are groups, $A \times B$ denotes their direct product. $A \leq B$ means A is con-

[1] Since any two p-Sylow subgroups of G are conjugate, this condition is satisfied by all p-Sylow subgroups whenever it is satisfied by any one of them. The condition is automatically satisfied by those p-Sylow subgroups of G for which p does not divide both the order and the index of the normal subgroup.

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tained in B, while $A < B$ means proper inclusion. $A \cap B$ = set theoretic intersection of A and B.

2. A subgroup C of G is a complement for the extension G over N if and only if $G = NC$ and $1 = N \cap C$.

THEOREM 1. *A subgroup C of a group G is a complement for the extension G over N if and only if C is minimal with respect to the property $G = NC$, and there exists for each prime p a p-Sylow subgroup S of G, and a complement of $N \cap S$ in S which is part of C.*[1]

Proof. Assume that C is a complement of N in G. Then clearly C is minimal with respect to the property $G = NC$. If P is a p-Sylow subgroup of C, and if S is a p-Sylow subgroup of G such that $P \leq S$, then P is a complement of $S \cap N$ in S. For, since $P \leq C$, $N \cap P \leq N \cap C = 1$. And since $P \leq S$, $[S \cap N]P \leq S$. But $S \cap N$ is a p-Sylow subgroup of N, and P is a p-Sylow subgroup of C, from which it follows that $[S \cap N]P$ is a p-Sylow subgroup of G. Hence $[S \cap N]P = S$. We have proved the necessity of the condition of the theorem.

Now assume conversely that this condition is satisfied. Let P be a Sylow subgroup of $M = N \cap C$, x an element of C. Since M is a normal subgroup of C, P^x is also a Sylow subgroup of M for the same prime. Hence there is an element y in M such that $P^{xy} = P$. Then xy is in the normalizer T of P in C, that is x is in MT. Hence $C = MT$, so that $G = NC = NMT = NT$. Hence by the minimality property of C, $T = C$. We have shown that each Sylow subgroup of M is normal in C, that is, that M is nilpotent.[2] We must prove that $M = 1$.

If p is a prime, there exists by our assumption a p-Sylow subgroup S of G, and a complement U of $S \cap N$ in S which is part of C. Since $U \leq S$, U is a p-subgroup of C. If Q is a p-Sylow subgroup of C such that $U \leq Q$, then U is a complement of $M \cap Q$ in Q. For, let P be a p-Sylow subgroup of G such that $Q \leq P$. Then there is an element x in G such that

$$P = S^x = [S \cap N]^x U^x = [P \cap N] U^x.$$

Hence, since $1 = U \cap N$, $P = [P \cap N] U$, so that

$$Q = Q \cap P = [Q \cap N] U = [Q \cap M] U.$$

[2]The condition that C be minimal with respect to the property $G = NC$ is equivalent to the condition $M = N \cap C \leq \phi(C)$. We may infer the nilpotency of M from the nilpotency of $\phi(C)$ (c.f. § 3).

For X a subgroup of G, set $\overline{X} = M'X/M'$, where M' denotes the commutator subgroup of M. Then $\overline{Q}$ is a p-Sylow subgroup of $\overline{C}$, and since $M'U \cap M = M'[U \cap M] = M'$, $\overline{U}$ is a complement of $\overline{Q} \cap \overline{M}$ in C. Hence, since $\overline{M}$ is abelian, there exists by part (i) of the theorem of Gaschütz a complement $\overline{D} = D/M'$ of $\overline{M}$ in $\overline{C}$. But then $C = MD$ and $M' = M \cap D$. Since M is nilpotent, $M \neq 1$ implies $M \cap D = M' < M = M \cap C$, that is $D < C$. Since $G = NC = NMD = ND$, this contradicts the minimality property of C. Hence $M = 1$, which proves the sufficiency of the condition.

COROLLARY (Schur's theorem). *If N has order prime to its index in G, then G splits over N.*

REMARK. Theorem 1 does not, of course, settle the question of the necessity of the hypothesis that N be abelian for the theorem of Gaschütz. [3]

The following example shows that in a splitting extension G over N, not every subgroup C which is minimal with respect to the property $G = NC$ need be a complement, even when N is abelian.

EXAMPLE. Let $M \neq 1$ be an abelian normal subgroup of the group C, and assume that M is contained in the Frattini subgroup $\phi(C)$ of C (*c.f.* §3). Since $\phi(C)$ is nilpotent it will have a center $\neq 1$; we may take, for instance, M = the center of $\phi(C)$. By a theorem of Artin [2, p. 103] there exists a free abelian group A of finite rank, and an (infinite) group G such that if we set $N = M \times A$, then

1. G is a splitting extension of N.
2. $G = NC$
3. $M = N \cap C$.

By the choice of M and C, no proper subgroup of C satisfies 2.

Let now m be the order of M. Since N is abelian, N^m [= the totality of mth powers of elements of N] is a characteristic subgroup of N, and hence is normal in G. Furthermore, $N^m \cap M = 1$. Since N is abelian of finite rank, and since G/N is finite, as an isomorphic image of the finite group C/M, G/N^m is finite. Set $\overline{G} = G/N^m$, $\overline{N} = N/N^m$ and $\overline{C} = N^mC/N^m$. Since the extension G over N splits, so does $\overline{G}$ over $\overline{N}$. $\overline{C}$ is minimal with respect to the property $\overline{G} = \overline{N}\overline{C}$, but

$$\overline{C} \cap \overline{N} = MN^m/N^m \simeq M \neq 1.$$

THEOREM 2. *For an extension G over N the following five conditions are*

[3] That this hypothesis actually is necessary has been shown by Professor Zassenhaus. See the note at the end of this paper.

equivalent.

(1) *N has order prime to its index in G.*

(2) *if H is a subgroup of G, then*

(a) *there exists a complement of $N \cap H$ in H.*

(b) *if either $H/[N \cap H]$ or $N \cap H$ is solvable, then any two complements of $N \cap H$ in H are conjugate in H.*

(3) *for each prime divisor p of the order of N, there exists a p-Sylow subgroup S of N such that if T denotes the normalizer of S in G,*

(a) *there exists a complement of $N \cap T$ in T.*

(b) *if H is a nilpotent subgroup of T, then any two complements of $N \cap H$ in H are conjugate in H.*

(4) *if H is a nilpotent subgroup of G, then*

(a) *there exists a complement of $N \cap H$ in H.*

(b) *any two complements of $N \cap H$ in H are conjugate in H.*

(5) *if H is a nilpotent subgroup of G, then there exists a subgroup C of H such that for each subgroup U of H, $U = [U \cap N] \times [U \cap C]$.*

Proof. Assume that N has order prime to its index in G, then clearly the same is true of the normal subgroup $N \cap H$ of H, for any subgroup H of G. Hence H splits over $N \cap H$ by the theorem of Schur. Furthermore, by a theorem of Zassenhaus [2, p. 132] if either $H/[N \cap H]$ or $N \cap H$ is solvable then any two complements for this extension are conjugate in H. Thus (2) is a consequence of (1).

Conditions (3) and (4) are immediate consequences of (2).

Next we shall prove that (3) and (4) each imply (1). Assume that the extension G over N satisfies (3), and assume that p is a prime which divides both the order and the index of N. Then by (3), (a) there exists a p-Sylow subgroup S of N, and a subgroup C such that if T denotes the normalizer of S in G, C is a complement of $N \cap T$ in T. But $G = NT$, so that C is a complement of N in G. Thus $[C:1] = [G:N]$, hence since p divides $[G:N]$, there exists an element x in C of order p. Since x is in T, and since p divides the order of N, there exists an element z of order p in $S \cap N$ such that $xz = zx$. Since x is not in N, $H = \{x, z\} = \{x\} \times \{z\}$ and $N \cap H = \{z\}$, whereby it follows from (3), (b), that $\{x\}$ and $\{xz\}$ are conjugate in H. Since this is impossible, (3) implies (1).

Now assume (4), and suppose again that p is a prime which divides both the order and the index of N. If S is a p-Sylow subgroup of G, there exists by

(4), (a), a complement C of $S \cap N$ in S. Since p divides $[G:N]$, there exists an element x in C of order p. Since p divides the order of N, $S \cap N$ is a non-trivial normal subgroup of S. Now a repetition of the construction of the preceding paragraph leads to a contradiction with (4), (b), proving that (4) implies (1). We have proved the equivalence of the first four conditions.

If H is a nilpotent subgroup of G, (2) implies the existence of a complement C of $N \cap H$ in H, and (1) implies that the orders of $N \cap H$ and C are relatively prime. Now (5) is a consequence of a property of nilpotent groups. Thus (5) is implied by the equivalent conditions (1) and (2). Conversely, if S is a p-Sylow subgroup of G, (5) implies the existence of a subgroup C of S such that $U = [U \cap N] \times [U \cap C]$ for each subgroup U of S. But it is well known that this implies that $S \cap N$ and C have relatively prime orders. Hence one of $S \cap N$ and C is trivial. This proves that (5) implies (1), completing the proof of Theorem 2.

3. The Frattini subgroup $\phi(G)$ of the group G is the intersection of G with all its maximal subgroups. In this section we shall note a characterization of those normal subgroups N of G which are contained in $\phi(G)$. It is well known that

(a) *$N \leq \phi(G)$ if and only if $G = NC$, C a subgroup of G implies $G = C$.*

Hence part (ii) of the theorem of Gaschütz has an equivalent statement

(b) *the abelian normal subgroup A of G is contained in $\phi(G)$ if and only if for each prime p there is a p-Sylow subgroup S of G such that $S = [S \cap A]V$, V a subgroup, implies $S \cap A = \mathfrak{n}_G(V \cap A)$.*[1]

Using (a) it is easy to verify that

(c) *if M is a normal subgroup of G such that $M \leq N$, then $N \leq \phi(G)$ if and only if $M \leq \phi(G)$ and $N/M \leq \phi(G/M)$.*

Since $\phi(G)$ is nilpotent [2, p. 122; this can be proved using (a) together with the first part of the argument of the sufficiency proof of Theorem 1] it will suffice for the purposes of determining the normal subgroups N which are contained in $\phi(G)$ to consider the case in which N has prime power order.

$N^{(i)}$ denotes the ith derived subgroup of N, $N^{(0)} = N$, $N^{(1)} = N'$. For X a subgroup of G, $\mathfrak{n}_G(X)$ denotes the subgroup generated by all the conjugates to X in G.

THEOREM 3. *Let N be a normal subgroup of the group G, and assume that N has p-power order, p a prime. Then $N \leq \phi(G)$ if and only if there exists a* *p-Sylow subgroup S of G such that for all $i \geq 0$, $S = N^{(i)}V$, V a subgroup, implies*

$$N^{(i)} = N^{(i+1)}\eta_G(V \cap N^{(i)}).^{1}$$

Proof. Assume first that $N \leq \phi(G)$. For X a subgroup of G, write $\overline{X} = N^{(i+1)}X/N^{(i+1)}$. Then $\overline{N^{(i)}}$ is an abelian normal subgroup of G with p-power order. Furthermore, by (c), $\overline{N^{(i)}} \leq \phi(\overline{G})$. Let $\overline{\zeta}$ be the p-Sylow subgroup of $\overline{G}$ whose existence is inferred by (b) (indeed, any p-Sylow subgroup will do[1]). Then $S = S/N^{(i+1)}$, S a p-Sylow subgroup of G. If $S = N^{(i)}V$, V a subgroup, then $S = \overline{N^{(i)}}\,\overline{V}$. Hence by (b) it follows that $\overline{N^{(i)}} = \eta_G(\overline{V} \cap \overline{N^{(i)}})$. But

$$\overline{N^{(i)}} \cap \overline{V} = N^{(i)}/N^{(i+1)} \cap N^{(i+1)}V/N^{(i+1)} = N^{(i+1)}[N^{(i)} \cap V]/N^{(i+1)},$$

from which it is easily verified that

$$\eta_{\overline{G}}(\overline{V} \cap N^{(i)}) = N^{(i+1)}\eta_G(V \cap N^{(i)})/N^{(i+1)}.$$

Hence $N^{(i)} = N^{(i+1)}\eta_G(V \cap N^{(i)})$. We have proved the necessity of the condition of the theorem.

Assume conversely that this condition is satisfied. We prove $N \leq \phi(G)$ by induction on the order of N. If $N = 1$ there is nothing to prove. Otherwise, since N is a p-group, $N' < N$, and since the condition of the theorem is clearly satisfied by N' whenever it is satisfied by N, it follows from the induction hypothesis that $N' \leq \phi(G)$. $\overline{S} = S/N'$ is a p-Sylow subgroup of $\overline{G} = G/N'$. If $\overline{V} = V/N'$ is a subgroup of $\overline{G}$ such that $\overline{S} = \overline{N}\,\overline{V}$, $\overline{N} = N/N'$, then $S = NV$. Now the condition of the theorem implies

$$\overline{N} = N/N' = N'\eta_G(V \cap N)/N' = \eta_G(V \cap N)/N' = \eta_{\overline{G}}(\overline{V} \cap \overline{N}).$$

Hence, since N is an abelian p-group it follows from (b) that $\overline{N} \leq \phi(\overline{G})$. Hence $N \leq \phi(G)$ by (c).

4. In this section we assume that $N = A$ is abelian, and consider the problem of the conjugacy of complements of A in G. A complement C of A in G is in particular a set of representatives for G over A; C consists of exactly one element $c(X)$ from each coset X in G/A. If D is a second complement, $d(X) = D \cap X$, then the function t from G/A to A defined by $d(X) = t(X)c(X)$ satisfies

(1) $$1 = t(Y)^X\, t(XY)^{-1}\, t(X)$$

for all X, Y in G/A. (Since A is abelian, all the elements x in X induce the same automorphism of A. We write $a^X = a^x$ for a in A).

Conversely, if t is any function from G/A to A which satisfies (1), then the totality D of elements $d(X) = t(X)c(X)$ for X in G/A is a complement of A in G. Moreover

(2) *two complements C and D which are related by t are conjugate subgroups of G if and only if there is an element a in A such that $t(X) = a^{1-X}$ for X in G/A.*[3]

Let H be a subgroup of G such that $A \leq H$, and set $m = [G:H]$.

THEOREM 4. *If m is prime to the order of A, if the function t from G/A to A satisfies (1), and if c is an element of A such that $t(Y) = c^{1-Y}$ for all Y in H/A, then there is an element a in A such that $t(X) = a^{1-X}$ for all X in G/A.*[4]

Proof. The function f defined by $f(X) = t(X)c^{X-1}$ satisfies (1), and has the property that $f(Y) = 1$ for all Y in H/A. Choose a system L of left representatives for C/A over H/A so that each X in G/A has (uniquely) the form $X = \overline{X}\underline{X}$, with $\overline{X}$ in L, $\underline{X}$ in H/A. By (1) we have

$$1 = f(\underline{X})^{\overline{X}} f(X)^{-1} f(\overline{X}) = f(X)^{-1} f(\overline{X})$$

that is $f(X) = f(\overline{X})$. Hence

$$f(X) = f(Y)^{-X} f(XY) = f(Y)^{-X} f(\overline{XY}).$$

Taking the product over all Y in L we have

$$(3) \qquad f(X)^m = \prod_{Y \in L} f(Y)^{-X} \prod_{Y \in L} f(\overline{XY}).$$

Since m is prime to the order of A, the mapping $\alpha : a \longrightarrow a^m$ is an automorphism of A (which commutes with every other automorphism of A). Hence

[3] In terms of the cohomology theory of groups this means that the number of classes of conjugate complements of A in G is the order of the first cohomology group of G/A by A.

[4] This result is a consequence of the 1-dimensional case, whereas (i) of the Gaschutz theorem is a consequence of the 2-dimensional case, of a general theorem in the cohomology theory of groups (see B. Eckmann, *Cohomology groups and transfer*, Ann. of Math., 58 (1953), 481-493.

$$b = \left\{ \prod_{Y \in L} f(Y) \right\} \alpha^{-1}$$

is an element of A. As Y runs through L, so does $\overline{XY}$, hence, applying α^{-1} to (3) we have $f(X) = b^{-X} b = b^{1-X}$. Thus

$$t(X) = f(X) c^{1-X} = b^{1-X} c^{1-X} = (bc)^{1-X}.$$

Theorem 4 is now proved with $a = bc$.

COROLLARY 1. *If m is prime to the order of A, then two complements C and D of A in G are conjugate in G if and only if $C \cap H$ and $D \cap H$ are conjugate in H.*

Proof. Let t be the function relating C and D. The subgroups $C \cap H$ and $D \cap H$ are complements of A in H, and are related by the restriction of t to H/A. If C and D are conjugate in G, then by (2) there exists an element a in A such that $t(X) = a^{1-X}$ for all X in G/A, and hence in particular for X in H/A. Hence by (2), $C \cap H$ and $D \cap H$ are conjugate in H.

If on the other hand $C \cap H$ and $D \cap H$ are conjugate subgroups of H, then it follows by (2) that there is an element c in A such that $t(Y) = c^{1-y}$ for all y in H/A. Hence by Theorem 4 there exists a in A such that $t(X) = a^{1-X}$ for all X in G/A. Hence by (2), C and D are conjugate in G. This proves the corollary.

By part (i) of the theorem of Gaschütz the extension G over A splits if and only if there is for each prime p a p-Sylow subgroup S of G which splits over $S \cap A$. By Theorem 4 we have

COROLLARY 2. *Let G be a splitting extension of A. If for each prime p there is a p-Sylow subgroup S of G such that any two complements of $S \cap A$ in S are conjugate in S, then any two complements of A in G are conjugate in G.*

Proof. We must prove that for each function t satisfying (1) there is an element a in A such that $t(X) = a^{1-X}$ for all X in G/A. Let p_i be the prime divisors of the order of A and let A_i be the corresponding primary components of A $(i = 1, 2, \cdots, k)$. Then $A = A_1 \times \cdots \times A_k$, and each A_i, being characteristic in A, is a normal subgroup of G. For each X in G/A, $t(X)$ has (uniquely) the form

$$t(X) = \prod_{i=1}^{k} t_i(X),$$

with $t_i(X)$ in A_i. Define $T_i(A_i x) = t(Ax)$ for x in G, and let S_i be a p_i-Sylow subgroup of G. We have assumed that S_i may be chosen in such a way that there is an element b_i in A_i with

$$T_i(A_i y) = b_i^{1-A_i y}$$

for all y in S_i (indeed, any p_i-Sylow subgroup will do). By Theorem 4, there exists a_i in A such that

$$T_i(A_i x) = a_i^{1-A_i x}$$

for all x in G. Hence

$$t_i(Ax) = T_i(A_i x) = a_i^{1-A_i x} = a_i^{1-Ax},$$

whereby

$$t(Ax) = \prod_{i=1}^{k} t_i(Ax) = \prod_{i=1}^{k} a_i^{1-Ax} = \left\{\prod_{i=1}^{k} a_i\right\}^{1-Ax}$$

for all x in G, with $a = \prod_{i=1}^{k} a_i$ an element of A.

5. It has been conjectured that if N has order prime to its index in G, then any two complements of N in G are conjugate. The following theorem shows that this conjecture is equivalent to

(+) *if G is a group, and Γ a group of automorphisms of G such that the orders of Γ and G are relatively prime, then for each prime p, there exists a p-Sylow subgroup of G which is mapped onto itself by every automorphism in Γ.*

Thus the theorem of Zassenhaus [2, p. 132] suffices to prove (+) in case either G or Γ is solvable.

THEOREM 5. *For an extension G over N such that N has order prime to its index in G, the following are equivalent statements.*

(a) *if C and D are complements of N in G, then they are conjugate in $\{C, D\}$.*

(b) *for each subgroup H of G such that $G = NH$, and for each pair C, D of complements of $N \cap H$ in H, there exists an automorphism α of H such that $C = D^{\alpha}$.*

(c) *for each subgroup H of G such that $G = NH$, for each complement C of $N \cap H$ in H, and for each prime p, there exist a p-Sylow subgroup S of $N \cap H$ such that C is part of the normalizer of S.*

Proof. Clearly (a) implies (b). Assume (b), and let H be a subgroup of G such that $G = NH$. Let P be a p-Sylow subgroup of $N \cap H$, and let T be the normalizer of P in H. Then $H = [N \cap H]T$. Hence, since the order of $N \cap H$ is prime to its index in H, there exists by the theorem of Schur a complement D of $N \cap H$ in H which is part of T, that is, which normalizes P. If now C is any complement of $N \cap H$ in H, there exists by (b) an automorphism of H such that $C = D^{\alpha}$. Hence C normalizes the p-Sylow subgroup $S = P^{\alpha}$ of $N \cap H$. Thus (b) implies (c).

Now assume (c), and let C and D be two complements of N in G. Assume that if $\{U, V\}$ is a pair of complements of N in G such that the order of $\{U, V\}$ is less than the order of $H = \{C, D\}$, then U and V are conjugate in $\{U, V\}$. If $N \cap H$ is nilpotent, since we have assumed that the orders of N and G/N are relatively prime, it follows by the theorem of Zassenhaus that C and D are conjugate in H. Otherwise, there exists a prime p such that the normalizer in H of a p-Sylow subgroup of $N \cap H$ is a proper subgroup of H. By (c) there exist p-Sylow subgroups P and Q of $N \cap H$ which are normalized by C and D respectively. There exists an element x in H such that $P = Q^x$, and the complement $E = D^x$ of N in G normalizes P. Thus, if we let T denote the normalizer of P in H, $\{C, E\} \leq T < H$. Now it follows by the induction hypothesis that there exists an element y in $\{C, E\}$ such that $C = E^y = D^{xy}$. Since both x and y are in $H = \{C, D\}$, so is xy.

Added in proof. The very interesting fact, that the hypothesis that the extension be abelian is indeed necessary for Gaschütz's theorem (i), as stated in the introduction of the present note, is shown by the following example communicated to the author by Professor Zassenhaus:

Let G be the group with generators A_{ik}, B_i, C_i $(i, k = 1, 2)$ and the defining relations

$$A_{11}^2 = A_{12}^2 = (A_{11}A_{12})^2 = A_{21}^2 = A_{22}^2 = (A_{21}A_{22})^2,\ A_{1i}A_{2k} = A_{2k}A_{1i};$$

$$B_i^3 = 1,\ B_iA_{i1}B_i^{-1} = A_{i2},\ B_iA_{i2}B_i^{-1} = A_{i1}A_{i2},\ B_1A_{2k} = A_{2k}B_1,\ B_2A_{1k} = A_{1k}B_2,$$

$$B_1B_2 = B_2B_1;\ C_i^2 = (C_iA_{i1})^2 = (C_iA_{i1}A_{i2})^2 = A_{i1},\ C_iB_iC_i^{-1} = B_i^{-1},$$

$C_1 A_{2k} C_1$, $C_2 A_{1k} = A_{1k} C_2$, $C_1 B_2 = B_2 C_1$, $C_2 B_1 = B_1 C_2$, $C_1 C_2 = C_2 C_1$ $(i, k = 1, 2)$.

The subgroup N generated by the four elements A_{ik} is the direct product of two quaternion groups with identified centers, thus N is of order 32. The group N is normal in G and the subgroup G_1 of G generated by N, B_1 and B_2 is normal too such that G_1/N is abelian of type (3,3). The factor group G/G_1 is of type (2,2). Thus G is of order 1152.

The group G does not split over its normal subgroup N. But the factor group G_1/N is the 3-Sylow subgroup of the factor group G/N such that G_1 splits over N with the subgroup generated by B_1 and B_2 as representative subgroup. Moreover the factor group generated by N, C_1 and C_2 over N is a 2-Sylow subgroup of G/N such that the subgroup generated by $C_1 A_{21}$ and $C_2 A_{12}$ is a representative subgroup of order 4.

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