SETS OF RADIAL CONTINUITY OF ANALYTIC FUNCTIONS

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1. Introduction. A point set E on the unit circle C ($|z| = 1$) will be called a *set of radial continuity* provided there exists a function $f(z)$, regular in the interior of C, with the property that $\lim_{r \to 1} f(re^{i\theta})$ exists if and only if $e^{i\theta}$ is a point of E. From Cauchy's criterion it follows that the set E of radial continuity of a function $f(z)$ is given by the formula

$$E = \prod_{k=1}^{\infty} \sum_{n=1}^{\infty} \prod \mathop{\mathrm{E}}_{e^{i\theta}} \left\{ |f(r_1 e^{i\theta}) - f(r_2 e^{i\theta})| \le \frac{1}{k} \right\},$$

where the inner intersection on the right is taken over all pairs of real values r_1, r_2 with $1 - 1/n \le r_1 < r_2 < 1$. From the continuity of analytic functions it thus follows that every set of radial continuity is a set of type $F_{\sigma\delta}$. The main purpose of the present note is to prove the following result.

THEOREM 1. *If E is a set of type F_σ on C, it is a set of radial continuity.*

The theorem will be proved by means of a refinement of a construction which was used by the authors in an earlier paper [**2**] to show that every set of type F_σ on C is the set of convergence of some Taylor series.

2. A special function. That the set consisting of all points of C is a set of radial continuity is trivial. In proving Theorem 1, it may therefore be assumed that the complement of E is not empty. In order to surmount difficulties one at a time, we begin with a new proof of the well-known fact that the empty set is a set of radial continuity (see [**1**, vol. 2, pp. 152-155]).

Let

$$f(z) \equiv \sum_{n=N}^{\infty} C_n(z),$$

where

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$$C_n(z) \equiv \frac{z^{k_n}}{n^2} \Big\{ 1 + z/\omega_{n1} + (z/\omega_{n1})^2 + \cdots + (z/\omega_{n1})^{n^2-1}$$

$$(1) \qquad + z^{n^2}[1 + z/\omega_{n2} + (z/\omega_{n2})^2 + \cdots + (z/\omega_{n2})^{n^2-1}]$$

$$+ \cdots$$

$$+ z^{(n-1)n^2}[1 + z/\omega_{nn} + (z/\omega_{nn})^2 + \cdots + (z/\omega_{nn})^{n^2-1}] \Big\};$$

here

$$\omega_{nj} = e^{2\pi ij/n},$$

and $\{k_n\}$ is a sequence of nonnegative integers which increases rapidly enough so that no two of the polynomials $C_n(z)$ contain terms of like powers of z, and so that a certain other requirement is met; the positive integer N, which is the lower limit of the foregoing series, will be determined later.

If z is one of the points ω_{nj}, then $|C_n(z)| = 1$. On the other hand, let z lie on the unit circle, and let $\Gamma_n(z)$ be any sum of consecutive terms from (1). If z is different from each of the roots of unity ω_{nj} that enter into $\Gamma_n(z)$, and δ denotes the (positive) angular distance between z and the nearest of these ω_{nj}, then

$$(2) \qquad |\Gamma_n(z)| < \frac{A_1}{\delta n^2},$$

where A_1 is a universal constant (see [**2**, Lemma A]). Now, if

$$(3) \qquad z = e^{i\theta}\omega_{nj}, \quad |\theta| < \frac{\pi}{n^2},$$

and $R_{nj}(z)$ denotes the sum of the terms in the jth row of (1) (including the factor z^{k_n}/n^2), then

$$(4) \qquad |R_{nj}(z)| = \frac{\sin(n^2\theta/2)}{n^2 \sin(\theta/2)} > A_2,$$

where A_2 is again a positive universal constant. But if the angular distance between z and ω_{nj} is less than π/n^2, the angular distances between z and the remaining nth roots of unity are all greater than $1/n$, and therefore (3) implies that, for sufficiently large n, by (2) and (4),

$$|C_n(z)| > A_2 - 2A_1/n > 5A_3,$$

where $A_3 = A_2/6$. We now choose N so large that the second of these inequalities holds whenever $n \geq N$.

Let $k_N = 0$; let r_N be a number $(0 < r_N < 1)$ such that

$$|C_N(re^{i\theta}) - C_N(e^{i\theta})| < \frac{A_3}{N!}$$

for $r_N \leq r \leq 1$ and all θ. Next, let k_{N+1} be large enough so that

$$|C_{N+1}(r_N e^{i\theta})| < \frac{A_3}{(N+1)!}$$

for all θ; and let r_{N+1} be greater than r_N, and near enough to 1 so that

$$|C_{N+1}(re^{i\theta}) - C_{N+1}(e^{i\theta})| < \frac{A_3}{(N+1)!}$$

for $r_{N+1} \leq r \leq 1$ and all θ. Let this construction be continued indefinitely.

Now let L be a line segment joining the origin to a point $e^{i\theta}$, and let n be an integer such that $n > N$ and

$$|C_n(e^{i\theta})| > 5A_3. \tag{5}$$

We then write

$$f(r_n e^{i\theta}) - f(r_{n-1}e^{i\theta}) = C_n(e^{i\theta}) + [C_n(r_n e^{i\theta}) - C_n(e^{i\theta})] - C_n(r_{n-1}e^{i\theta})$$

$$+ \sum_{j=N}^{n-1} \{[C_j(r_n e^{i\theta}) - C_j(e^{i\theta})] - [C_j(r_{n-1}e^{i\theta}) - C_j(e^{i\theta})]\}$$

$$+ \sum_{j=n+1}^{\infty} \{C_j(r_n e^{i\theta}) - C_j(r_{n-1}e^{i\theta})\}$$

and obtain from the inequalities above

$$|f(r_n e^{i\theta}) - f(r_{n-1} e^{i\theta})| > A_3\left[5 - \frac{1}{n!} - \frac{1}{n!} - 2\sum_{j=N}^{n-1} \frac{1}{j!} - 2\sum_{j=n+1}^{\infty} \frac{1}{j!}\right]$$

$$\geq A_3[5 - 2(e-1)] > A_3 .$$

It follows that, if there exist infinitely many integers n for which (5) is satisfied $f(z)$ does not approach a finite limit as z approaches $e^{i\theta}$ along the line L. But for each real θ there exist infinitely many integers n with the property that, for some integer j_n,

$$\left|\frac{\theta}{2\pi} - \frac{j_n}{n}\right| < \frac{1}{2n^2}$$

(see [3, p. 48, Theorem 14]), so that each z on C admits infinitely many representations (3). It follows that $\lim_{r\to 1} f(re^{i\theta})$ does not exist for any value θ.

3. Closed sets of radial continuity. Let E be a closed set on C, and let G denote its (nonempty) complement. Again, let $f(z)$ be the function defined in § 2, except for the following modification. In the polynomial $C_n(z)$, let ω_{n1}, $\omega_{n2}, \cdots, \omega_{np_n}$ denote those nth roots of unity which lie in G and have the additional property that the angular distance of each one of them from E is greater than $n^{-1/2}$. The exponent of z in the factor outside of the brackets in the last row of the right member of (1) becomes $(p_n - 1)n^2$. And the p_n nth roots of unity ω_{nj} that occur in $C_n(z)$ must be so labelled that their arguments increase as the index j increases, with $\arg \omega_{n1} > 0$ and $\arg \omega_{np_n} \leq 2\pi$. Then every partial sum $\Gamma_n(z)$ of consecutive terms of $C_n(z)$ satisfies the inequality $|\Gamma_n(z)| < A_1 n^{-3/2}$ for all z belonging to E, and therefore the Taylor series of $f(z)$ converges on E. On the other hand, let the exponents k_n in (1) be chosen in a manner similar to that of § 2, and let L be a line segment joining the origin to a point $e^{i\theta}$ in the (open) set G. Then there exist infinitely many integers n for which (5) is satisfied by our newly constructed polynomials $C_n(z)$, and therefore $\lim_{r\to 1} f(re^{i\theta})$ does not exist.

4. The general case. Suppose finally that E is a set of type F_σ on C. Then the complement G of E is of type G_δ; that is, it can be represented as the intersection of open sets $G_1, G_2, \cdots$, with $G_k \supset G_{k+1}$ for all k. In turn, we can represent G_1 as the union of closed intervals I_{1h} in such a way that no two distinct intervals I_{1h} and $I_{1h'}$ contain common interior points, and in such a way that no point of G_1 is a limit point of end points of intervals I_{1h}. Similarly, each set G_k can be represented as the union of closed intervals I_{kh} satisfying similar restrictions.

Let n_0 be any positive integer. Since the denumerable set of all open arcs

$$z = e^{i\theta}, \ |\theta - 2\pi j/n| < \pi/n^2 \qquad (j = 1, 2, \cdots, n, \ n > n_0)$$

covers the entire unit circle, there exists a set of finitely many such arcs covering the unit circle. It follows that we can choose a finite number of terms $C_n(z)$ (see (1)), modified as in § 3, such that their sum $f_1(z)$ has the following properties:

i) for each θ in I_{11}, there exist two values ρ' and ρ'', $0 < \rho' < \rho'' < 1$, such that $|f_1(\rho' e^{i\theta}) - f_1(\rho'' e^{i\theta})| > A_3$;

ii) for each point $e^{i\theta}$ outside of I_{11} and outside of the two neighboring intervals I_{1h} and $I_{1h'}$, and for each n for which $C_n(z)$ occurs in $f_1(z)$, the modulus of any sum of consecutive terms of $C_n(e^{i\theta})$ is less than $A_1 n^{-3/2}$.

Next we accord a similar treatment to I_{12}, then to I_{21}, I_{13}, I_{22}, I_{31}, I_{14}, and so forth. The sum $f(z)$ of the polynomials $f_1(z)$, $f_2(z)$, $\cdots$ thus constructed has the following properties: if $e^{i\theta}$ lies in E, that is, lies in only finitely many of the intervals I_{kh}, the Taylor series of $f(z)$ converges at $z = e^{i\theta}$; if $e^{i\theta}$ lies in G, there exist pairs of values ρ' and ρ'' arbitrarily near to 1 and such that

$$|f(\rho' e^{i\theta}) - f(\rho'' e^{i\theta})| > A_3 .$$

It follows that E is the set of radial continuity of $f(z)$, and the proof of Theorem 1 is complete.

5. Sets of uniform radial continuity. The following theorem is analogous to Theorem 2 of [2].

THEOREM 2. *If E is a closed set on C, then there exists a function $f(z)$, regular in $|z| < 1$, such that $\lim_{r \to 1} f(re^{i\theta})$ exists uniformly with respect to all $e^{i\theta}$ in E and does not exist for any $e^{i\theta}$ not in E.*

For the proof of Theorem 2, we refer to the function $f(z)$, constructed in § 3. Note that $|\Gamma_n(z)| < A_1 n^{-3/2}$ for all z in E. Hence the Taylor series of $f(z)$ converges uniformly in E. It then follows easily, by the use of Abel's summation, that the convergence

$$\lim_{r \to 1} f(re^{i\theta}) = f(e^{i\theta})$$

is also uniform in E.

6. An unsolved problem. The converse of Theorem 1 is false, since a set of radial continuity can be the complement of a denumerable set which is dense on C. We do not know whether there exist sets of type $F_{\sigma\delta}$ that are not sets of radial continuity.

References

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