SOME INEQUALITIES BETWEEN LATENT ROOTS AND MINIMAX (MAXIMIN) ELEMENTS OF REAL MATRICES

LOUIS GUTTMAN

1. Introduction. Because of the usual tediousness of computing latent roots, any quick information about them is often welcome and useful. We develop here some lower bounds to the absolute value of the major latent root (the one largest in absolute value) of any real symmetric matrix that depend only on a simple inspection of its elements. Also, lower bounds are developed for the largest latent root of a Gramian matrix of the form AA' that require inspection only of the elements of A. The latter case is especially important in linear regression theory of statistics, in factor analysis theories of psychology, and elsewhere.

The original motivation for our inequalities was to study the relationship between latent roots and the von Neumann value of a two-person zero-sum game matrix. We actually use the von Neumann theory to establish our bounds to latent roots, and in return we show how latent roots can be used to bound the game value of a matrix. The latter kind of bound will be useful whenever it is easier to get at the appropriate latent root than at the desired game value.

The bounds to latent roots are first exhibited in §§ 2–3, and then proved in § 4. How to reverse their emphasis to provide bounds for game values is shown in § 5.

2. Lower bounds to the major latent root. Let A be any real matrix of order $m \times n$. Let a_{ij} be the typical element of A ($i=1, 2, \cdots, m$; $j=1, 2, \cdots, n$), and let p_i and q_j be defined respectively as

$$p_i = \min_j a_{ij}, \qquad q_j = \max_i a_{ij} \qquad \begin{pmatrix} i=1, 2, \cdots, m \\ j=1, 2, \cdots, n \end{pmatrix}. \tag{1}$$

Furthermore, let p and q be defined respectively as

$$p = \max_i p_i, \qquad q = \min_j q_j. \tag{2}$$

From (1), it immediately follows that

$$p_i \leqq a_{ij} \leqq q_j \qquad \begin{pmatrix} i=1, 2, \cdots, m \\ j=1, 2, \cdots, n \end{pmatrix}, \tag{3}$$

Received February 20, 1956.

and in particular that $p \leqq q$.

Let λ^2 be the largest latent root of AA', where A' is the transpose of A. We shall prove in § 4 below that both of the following inequalities hold:

$$|\lambda| \geqq p\sqrt{n} \tag{4}$$

$$|\lambda| \geqq -q\sqrt{m}\ . \tag{5}$$

Inequality (4) is a useful lower bound to $|\lambda|$ if and only if $p > 0$, while (5) is useful if and only if $q < 0$. If $p \leqq 0 \leqq q$, we obtain no information about $|\lambda|$.

One interesting feature of (4) and (5) is that they show that λ^2 is generally at least of the order of m or of n, depending on whether $q < 0$ or $p > 0$.

Corresponding inequalities can be developed by considering A' in place of A. Let p'_j and q'_i be defined respectively as

$$p'_j = \min_i a_{ij}\ , \qquad q'_i = \max_j a_{ij} \qquad \begin{pmatrix} i=1, 2, \cdots, m \\ j=1, 2, \cdots, n \end{pmatrix}, \tag{6}$$

so that

$$p'_j \leqq a_{ij} \leqq q'_i \qquad \begin{pmatrix} i=1, 2, \cdots, m \\ j=1, 2, \cdots, n \end{pmatrix}. \tag{7}$$

Let p' and q' be defined by

$$p' = \max_j p'_j\ . \qquad q' = \min_i q'_i\ , \tag{8}$$

whence, from (7), $p' \leqq q'$.

Now, AA' and $A'A$ have the same nonzero latent roots, which are all positive. So if λ^2 is the largest latent root of AA', it is also the largest latent root of $A'A$. In addition to (4) and (5), we can write

$$|\lambda| \geqq p'\sqrt{m} \tag{9}$$

$$|\lambda| \geqq -q'\sqrt{n}\ . \tag{10}$$

Notice that the roles of m and n in (9) and (10) are reversed from those in (4) and (5). If $p' > 0$, λ^2 is at least of order m, while if $q' < 0$, λ^2 is at least of order n. If either of p or p' is positive, or if either of q or q' is negative, we get some information about $|\lambda|$.

Matrices of the form AA' or $A'A$ are called Gramian, or nonnegative definite symmetric. In statistics, any correlation matrix R is Gramian. A good deal of work in psychology, for example, is aimed at "factoring" an R into the form $R = AA'$. Given such a factoring, our inequalities immediately given lower bounds to the largest latent root of R from the minimax and maximin element of A. The latter are easily ascertainable by inspection.

3. The case of symmetric matrices. If $m \neq n$, A itself has no latent roots defined. However, if A is square, then it does have a characteristic equation and latent roots. A particularly important case is where A is symmetric, or $A = A'$. Then the latent roots of A are all real, and their squares are the latent roots of $AA' = A^2$. If λ^2 is the largest latent root of AA', then λ must be a root of A largest in absolute value, and conversely. In this symmetric case, we have not only $m = n$, but also $p = p'$, $q = q'$. So (9) and (4) are redundant, as are also (10) and (5). The inequalities can now be interpreted as referring to the major latent root of A itself, and not merely to a root of AA'.

When A is symmetric, we can usually improve on (4) and (5).

Let I be the unit matrix of order n, c be an arbitrary constant, and A^* be defined as

$$A^* = A - cI . \tag{11}$$

If λ is a latent root of A, then $\lambda - c$ is a latent root of A^*, and conversely. Let p^* and q^* be the maximin and minimax of elements of A^* respectively, or, if δ_{ij} is Kronecker's delta,

$$p^* = \max_i \min_j (a_{ij} - c\delta_{ij}) , \qquad q^* = \min_i \max_j (a_{ij} - c\delta_{ij}) . \tag{12}$$

Then in place of (4) and (5), we can write

$$|\lambda - c| \geqq p^* \sqrt{n} , \qquad |\lambda - c| \geqq -q^* \sqrt{n} \qquad (A = A') , \tag{13}$$

where $\lambda - c$ is the major latent root of A^*. In special cases, a judicious choice of c may be apparent that will make maximum $|\lambda - c|$ correspond to a λ which is either the most positive or the most negative latent root of A, and with a better bound than given by (4)-(5).

An especially important symmetric case is where A is a correlation matrix R, with all diagonal elements equal to unity. In such a case, the largest latent root of R cannot be less than 1, for the trace of R is n and all n latent roots are nonnegative. For this case, if $p > 0$, then choose $c = 1 - p$. This implies that the main diagonal elements of R^* are all equal to p. Then, clearly $p = p^*$; and since $\lambda \geqq 1$ for any R, $|\lambda - 1 + p| = \lambda - 1 + p$ when $p > 0$, and (13) becomes

$$\lambda \geqq 1 + p(\sqrt{n} - 1) \qquad (p \geqq 0, \ A = R) . \tag{14}$$

Similarly, if $q < 0$, by choosing $c = 1 - q$ in (13) we get

$$\lambda \geqq 1 - q(\sqrt{n} - 1) \qquad (q \leqq 0,\ A = R)\ . \tag{15}$$

4. Proof of the inequalities. Let P_k denote the space of all k-dimensional probability row vectors. That is $z \in P_k$ if and only if z is a row of k nonnegative numbers whose sum equals unity. Let z' denote the column vector that is the transpose of z. Then zz' is the sum of squares of the components of z, and it is easily established and well-known that

$$\frac{1}{k} \leqq zz' \leqq 1 \qquad (z \in P_k)\ . \tag{16}$$

The equality on the left of (16) is always attained by letting $z = z_1$, where z_1 is a vector whose components all equal $1/k$ (and hence $z_1 \in P_k$).

von Neumann [1] has shown how each real matrix A has associated with it a unique real number v with certain important minimax properties. Since his theorem was developed in the context of his theory of games, we shall call v the *game value* of A. Our present interest of course is to regard von Neumann's theorem as a general theorem on real matrices, without necessary reference to the theory of games.

von Neumann's theorem is as follows. *If A is a real matrix of order $m \times n$, then there exist an x_0 and a y_0, where $x_0 \in P_m$ and $y_0 \in P_n$, and a unique real number v, such that*

$$xAy_0' \leqq v \leqq x_0Ay' \qquad \textit{for all} \quad x \in P_m\ , \quad y \in P_n\ . \tag{17}$$

Furthermore,

$$p \leqq v \leqq q\ , \tag{18}$$

where p and q are as defined in (2).

To use this theorem for establishing our own inequalities, apply Schwarz's inequality to (17) to see that

$$-\sqrt{(xx')(y_0A'Ay_0')} \leqq v \leqq \sqrt{(yy')(x_0AA'x_0')} \qquad (x \in P_m,\ y \in P_n)\ . \tag{19}$$

Let λ^2 be the largest latent root of AA' and $A'A$. Then

$$x_0AA'x_0' \leqq \lambda^2 x_0x_0' \leqq \lambda^2\ , \qquad y_0A'Ay_0' \leqq \lambda^2 y_0y_0' \leqq \lambda^2\ , \tag{20}$$

the second inequalities in each part of (20) following from the second inequality in (16). From the first inequality in (16),

$$xx' \geqq \frac{1}{m}\ , \qquad yy' \geqq \frac{1}{n} \qquad (x \in P_m,\ y \in P_n)\ , \tag{21}$$

and we have noted that the equalities in (21) are always attainable, by best possible x_1 and y_1 for this purpose. Using (20) and the equalities of (21) in (19) yield

$$(22) \qquad \frac{-|\lambda|}{\sqrt{m}} \leqq v \leqq \frac{|\lambda|}{\sqrt{n}} .$$

Then (4) and (5) follow from (22) and (18). Inequalities (9) and (10) follow from the restatement of (22) for the game value v' of A':

$$(23) \qquad \frac{-|\lambda|}{\sqrt{n}} \leqq v' \leqq \frac{|\lambda|}{\sqrt{m}} .$$

Inequalities (22) and (23) are of course sharper than those stated in § 2 above. If game values are known, they can be used in place of p, q, p', or q' in the latter inequalities. We have stated our inequalities in the form most practical to use, since p and q can be determined by inspection, whereas v usually cannot, except in the special case where $p=q=v$.

5. Application to game values. Let us now consider the converse problem of bounding game values. If an upper bound to $|\lambda|$ is known, this will serve to bound v and v' via (22) and (23). Thus, useful bounds to v can be set that may sometimes be better than (18) when $p \neq q$. Perhaps more important, (22) and (23) show how the magnitudes of v and v' compare with those of m and n in general, given some notion of the size of $|\lambda|$.

For the purpose of bounding v and v', (22) and (23) can be improved on. Let A_c be the $m \times n$ matrix whose typical element is $a_{ij}-c$, where c is an arbitrary constant. Thus A_c is obtained by subtracting c from *each* element of A (so $A_c \neq A^*$ if $c \neq 0$). It is easily verified that the game value of A_c is $v-c$, and optimal probability vectors x_0 and y_0 for A are optimal also for A_c. Let λ_c^2 be the largest latent root of A_cA_c' (or of $A_c'A_c$). Then we can replace (22) and (23) by the more general inequalities

$$(24) \qquad c-\frac{|\lambda_c|}{\sqrt{m}} \leqq v \leqq c+\frac{|\lambda_c|}{\sqrt{n}}$$

and

$$(25) \qquad c-\frac{|\lambda_c|}{\sqrt{n}} \leqq v' \leqq c+\frac{|\lambda_c|}{\sqrt{m}} .$$

Evidently, the best choice of c is that which will minimize λ_c^2. A practical way to approximate this choice is to minimize instead the *sum of all the latent roots* of A_cA_c', or the trace of A_cA_c'. This requires minimizing

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij}-c)^2 , \tag{26}$$

for which the minimizing value is $c=a$, where

$$\bar{a}=\frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} . \tag{27}$$

Reference

1. J. von Neumann and O. Morgenstern, *Theory of games and economic behavior*, Princeton Univ. Press, 1954.

Center for Advanced Study in the Behavioral Sciences
(on leave from the Israel Institute of Applied Social Research)