AN ULTRASPHERICAL GENERATING FUNCTION

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1. **Introduction.** Let $P_n^{(\alpha,\alpha)}(v)$ denote ultraspherical polynomials and let

$$w=2(v-t)(1-2vt+t^2)^{-1/2}\,, \tag{1}$$

$$g=1-2vt+t^2\,,$$

$$y=-tu(1-2vt+t^2)^{-1/2}\,,$$

$$r=(1-2yw+y^2)^{1/2}\,,$$

with the roots to be those assuming the value 1 for $t=0$. Then this note will prove that

$$g^{-\alpha-1/2}\,{}_2F_1\left[\begin{matrix}c,\ 1+2\alpha-c; \\ 1+\alpha\quad ;\end{matrix}\ \frac{1-y-r}{2}\right]{}_2F_1\left[\begin{matrix}c,\ 1+2\alpha-c; \\ 1+\alpha\quad ;\end{matrix}\ \frac{1+y-r}{2}\right] \tag{2}$$

$$=\sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{(1+\alpha)_n}\,{}_3F_2\left[\begin{matrix}-n,\ c,\ 1+2\alpha-c; \\ 1+\alpha,\ 1+2\alpha\quad ;\end{matrix}\ u\right]P_n^{(\alpha,\alpha)}(v)t^n\,,$$

valid for t sufficiently small. In (2), c is an arbitrary parameter. Equation (2) is a direct generalization of Rice's result given in [8, equ. 2.14], to which it reduces for $\alpha=0$. (A different generalization of Rice's result is given in [3].) For c the non-positive integer $-k$, the left side of (2) reduces to a product of ultraspherical polynomials:

$$g^{-\alpha-1/2}\frac{k!k!}{(1+\alpha)_k(1+\alpha)_k}P_k^{(\alpha,\alpha)}(r+y)P_k^{(\alpha,\alpha)}(r-y) \tag{3}$$

$$=\sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{(1+\alpha)_n}\,{}_3F_2\left[\begin{matrix}-n,\ -k,\ 1+2\alpha+k; \\ 1+\alpha,\ 1+2\alpha\quad ;\end{matrix}\ u\right]P_n^{(\alpha,\alpha)}(v)t^n\,.$$

In addition, this note will show other results on ultraspherical polynomials. Further, it will provide a new way of deriving some results of Weisner. These will be shown later.

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2. **A preliminary result.** It will be established in this section that

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$$(4)\quad \sum_{n=0}^{\infty}\frac{(b)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n, & a; \\ b & ;\end{matrix}\; x\right]{}_{p+1}F_q\left[\begin{matrix}-n, c_1, c_2, \cdots, c_p; \\ d_1, d_2, \cdots, d_q \;;\end{matrix}\; u\right]t^n$$

$$=(1-t)^{a-b}(1-t+xt)^{-a}\sum_{n=0}^{\infty}\frac{(c_1)_n\cdots(c_p)_n(-tu)^n(b)_n}{(d_1)_n\cdots(d_q)_n(1-t)^n n!}\,{}_2F_1\left[\begin{matrix}-n, & a; \\ b & ;\end{matrix}\; \frac{x}{xt+1-t}\right],$$

for

$$|t|<1\,,\ |tu/(1-t)|<1\,,\ xt+1-t\neq 0\,,\ p\leqq q\,.$$

Start with

$$(5)\quad (1-t)^{a-b-k}(xt+1-t)^{-a}\,{}_2F_1\left[\begin{matrix}-k, & a; \\ b & ;\end{matrix}\; \frac{x}{1-t+xt}\right]$$

$$=(1-t)^{-b-k}(1-x)^{-a}\,{}_2F_1\left[\begin{matrix}b+k, & a; \\ b & ;\end{matrix}\; \frac{x}{(x-1)(1-t)}\right]$$

$$=(1-x)^{-a}\sum_{n=0}^{\infty}\frac{(b+k)_n t^n}{n!}\,{}_2F_1\left[\begin{matrix}b+n+k, & a; \\ b & ;\end{matrix}\; \frac{x}{x-1}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(b+k)_n t^n}{n!}\,{}_2F_1\left[\begin{matrix}-n-k, & a; \\ b & ;\end{matrix}\; x\right].$$

Multiply the first and last lines of (5) by

$$(6)\quad \frac{(b)_k(c_1)_k(c_2)_k\cdots(c_p)_k(-tu)^k}{(d_1)_k(d_2)_k\cdots(d_q)_k k!}$$

and sum on k from 0 to ∞. A shift of indices will then give equation (4). The restrictions given insure the absolute convergence of the various series which are multiplied together.

It should be here noted that (4) includes two results by Weisner as special cases. See [7, equ's. 4.3 and 4.6]. The first follows from (4) by taking

$$(7)\quad p=1,\ q=1,\ c_1=d,\ d_1=b$$

and summing the result by Chaundy's equation 25 in [4].

The second Weisner result follows from (4) by taking

$$(8)\quad p=0\,,\ q=1\,,\ d_1=b\,,$$

and summing by the formula of Rainville as quoted in [5, p. 267, equ. 25].

3. Proof of (2). The use of a quadratic transformation [6, p. 9] on a standard form of the ultraspherical polynomials converts them into

$$P_n^{(\alpha,\alpha)}(x)=\frac{(1+\alpha)_n}{n!}z^{-n}{}_2F_1\left[\begin{matrix}-n,\ \alpha+1/2; \\ 2\alpha+1 \quad ;\end{matrix}\ 1-z^2\right] \tag{9}$$

with $2x=z+1/z$. This is equivalent to a formula by Weisner [7, p. 1038]. Let

$$v=\frac{1}{2}(2-x)(1-x)^{-1/2}\,,\quad a=\alpha+1/2\,,\quad b=2\alpha+1\,, \tag{10}$$

replace t by $t(1-x)^{-1/2}$ in (4), and let w, g, y, r be defined by (1). Then (4) becomes

$$\sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{(1+\alpha)_n}\,{}_{p+1}F_q\left[\begin{matrix}-n,\ c_1,\ \cdots,\ c_p; \\ d_1,\ \cdots,\ d_q \quad ;\end{matrix}\ u\right]P_n^{(\alpha,\alpha)}(v)t^n \tag{11}$$

$$=g^{-\alpha-1/2}\sum_{n=0}^{\infty}\frac{(c_1)_n\cdots(c_p)_n}{(d_1)_n\cdots(d_q)_n}y^n\frac{(1+2\alpha)_n}{(1+\alpha)_n}P_n^{(\alpha,\alpha)}(w)\,.$$

In (11), take

$$p=2,\ q=2,\ d_1=1+\alpha,\ d_2=1+2\alpha,\ c_1=c,\ c_2=1+2\alpha-c \tag{12}$$

and apply the formula given in [2, equ. 17]. Result (2) above follows immediately.

For an additional result from (11), take

$$p=0\,,\quad q=2\,,\quad d_1=1+\alpha\,,\quad d_2=1+2\alpha\,, \tag{13}$$

and use the result from Bateman [1], that

$${}_0F_1\left(-;1+\alpha;\ \frac{y(w-1)}{2}\right){}_0F_1\left(-;1+\alpha;\ \frac{y(w+1)}{2}\right) \tag{14}$$

$$=\sum_{n=0}^{\infty}\frac{P_n^{(\alpha,\alpha)}(w)y^n}{(1+\alpha)_n(1+\alpha)_n}\,.$$

This gives

$$g^{-\alpha-1/2}{}_0F_1\left(-;1+\alpha;\ \frac{y(w-1)}{2}\right){}_0F_1\left(-;1+\alpha;\ \frac{y(w+1)}{2}\right) \tag{15}$$

$$=\sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{(1+\alpha)_n}\,{}_1F_2\left[\begin{matrix}-n \quad ; \\ 1+\alpha,\ 1+2\alpha;\end{matrix}\ u\right]P_n^{(\alpha,\alpha)}(v)t^n\,.$$

Two further results are obtainable from (11) on ultraspherical polynomials. However they are both special cases of the results by Weisner mentioned above, and so are merely presented here for completeness. For the first, take in (11)

$$(16) \qquad p=q=1\,, \quad d_1=1+2\alpha\,, \quad c_1=a\,,$$

and sum the result by [2, equ. (18)] to get

$$(17) \qquad g^{-\alpha-1/2}(1-yw)^{-a}{}_2F_1\left[\begin{matrix} a/2,\ (a+1)/2; & \dfrac{y^2(w^2-1)}{(1-yw)^2} \\ 1+\alpha \quad ; & \end{matrix}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{(1+\alpha)_n}\,{}_2F_1\left[\begin{matrix} -n,\ a; & u \\ 1+2\alpha\ ; & \end{matrix}\right]P_n^{(\alpha,\alpha)}(v)t^n\,.$$

If a is a non-positive integer $-k$ then (17) becomes

$$(18) \qquad g^{-\alpha-1/2}\frac{r^k k!}{(1+\alpha)_k}P_k^{(\alpha,\alpha)}\left(\frac{1-yw}{r}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{(1+\alpha)_n}\,{}_2F_1\left[\begin{matrix} -n,\ -k; & u \\ 1+\alpha\ \ ; & \end{matrix}\right]P_n^{(\alpha,\alpha)}(v)t^n\,.$$

For the other result of Weisner's, in (11) take

$$(19) \qquad p=0\,, \quad q=1\,, \quad d_1=1+2\alpha\,,$$

and sum to get:

$$(20) \qquad g^{-\alpha-1/2}e^{yw}{}_0F_1\left(-\,;1+\alpha;\ \frac{y^2(w^2-1)}{4}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{(1+\alpha)_n}\,{}_1F_1\left[\begin{matrix} -n\ ; & u \\ 1+2\alpha\,; & \end{matrix}\right]P_n^{(\alpha,\alpha)}(v)t^n\,.$$

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