A SUBSTITUTION THEOREM FOR THE LAPLACE TRANSFORMATION AND ITS GENERALIZATION TO TRANSFORMATIONS WITH SYMMETRIC KERNEL

R. G. BUSCHMAN

In the problem of the derivation of images of functions under the Laplace transformation, the question arises as to the type of image produced if t is replaced by $g(t)$ in the original. Specific examples have been given by Erdélyi [**3**, vol. I §§ 4.1, 5.1, 6.1], Doetsch [**1**, 75–80], McLachlan, Humbert, and Poli [**6**, pp. 11–13] and [**7**, pp. 15–20], and Labin [**5**, p. 41] and a general formula is also listed by Doetsch [**1**, 75–80].

The Laplace transformation will be taken as

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt$$

in which the integral is taken in the Lebesgue sense and which, as suggested by Doetsch [**2**, vol. I, p. 44], will be denoted by

$$F(t) \underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}} f(s)\,.$$

(The symbol will be read " $F(t)$ has a Laplace transform $f(s)$ ".)

THEOREM 1. *If*

(i) *k, g, and the inverse function $h = g^{-1}$ are single-valued analytic functins, real on $(0, \infty)$, and such that $g(0) = 0$ and $g(\infty) = \infty$ (or $g(0) = \infty$ and $g(\infty) = 0$)*;

(ii) $F(t) \underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}} f(s)$ *and this Laplace integral converges for $0 < \Re s$*;

(iii) *there exists a function $\Phi(s, u)$, $\Phi(s, u) \underset{u\quad p}{\overset{\mathscr{L}}{\circ\!-\!\bullet}} \phi(s, p)$ and this Laplace integral converges for $0 < \Re p$, and $\phi(s, p) = e^{-sh(p)} k[h(p)]\, |h'(p)|$; and*

(iv) $\int_0^\infty \left[\int_0^\infty |e^{-up} \Phi(s, u) F(p)|\, du\right] dp$ *converges for $a < \Re s$*;

then

$$k(t)\, F[g(t)] \underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}} \int_0^\infty \Phi(s, u) f(u)\, du$$

Received June 25, 1957. This paper is a portion of a thesis submitted to the Graduate School of the University of Colorado.

and this Laplace integral converges for $a < \Re s$.

Proof. From (iii) and (iv) it follows that

$$\int_0^\infty e^{-sh(p)}k[h(p)]\,|h'(p)|\,F(p)dp$$

is absolutely convergent for $a < \Re s$. There are two cases to be considered. Since from (i) both g and h are single valued, h is monotonic.

Case 1. If $g(0)=0$ and $g(\infty)=\infty$, then $0 \leqq h'(p)$.

Case 2. If $g(0)=\infty$ and $g(\infty)=0$, then $h'(p) \leqq 0$.
In either case, therefore, if the substitution $t=h(p)$ is made in the integral

$$\int_0^\infty e^{-st}k(t)F[g(t)]dt\,,$$

then

$$k(t)\,F[g(t)] \underset{t\quad s}{\overset{\mathscr{L}}{\circ\!\!-\!\!\bullet}} \int_0^\infty e^{-sh(p)}k[h(p)]\,|h'(p)|\,F(p)\,dp\,.$$

From (iii) $\Phi(s, u)$ can be introduced and by (iv) the order of integration changed so that

$$k(t)\,F[g(t)] \underset{t\quad s}{\overset{\mathscr{L}}{\circ\!\!-\!\!\bullet}} \int_0^\infty \left[\int_0^\infty e^{-up}F(p)\,dp\right]\Phi(s, u)\,du\,.$$

Finally, from (ii)

$$k(t)\,F[g(t)] \underset{t\quad s}{\overset{\mathscr{L}}{\circ\!\!-\!\!\bullet}} \int_0^\infty \Phi(s, u)f(u)\,du\,.$$

To show that there are functions $\phi(s, p)$ as assumed in (iii), let, for example, $g(t)=t^2$ and $k(t)=1$ so that

$$\Phi(s, p)=(4p)^{-1/2}e^{-sp^{1/2}}$$

and

$$\phi(s, u)=(4\pi u)^{-1/2}e^{-s^2/4u}.$$

From this the known relation

$$F(t^2) \underset{t\quad s}{\overset{\mathscr{L}}{\circ\!\!-\!\!\bullet}} \int_0^\infty (4\pi u)^{-1/2}e^{-s^2/4u}f(u)du$$

is obtained.

Special cases of $k(t)$ will sometimes simplify the image of $\Phi(s, u)$. If $k(t)=|g'(t)|K[g(t)]$, then

$$\Phi(s, u) \underset{u \quad p}{\overset{\mathscr{L}}{\circ\!-\!\bullet}} K(p)\, e^{-s\,h(p)} .$$

If $k(t)=|g'(t)|[g(t)]^c$, then

$$\Phi(s, u) \underset{u \quad p}{\overset{\mathscr{L}}{\circ\!-\!\bullet}} p^c\, e^{-s\,h(p)} .$$

In the proof of Theorem 1 it is noted that the only important property required of the kernel is that it be symmetric. Therefore consider the transformation

$$f(s)=\int_a^b K(s, t)\, F(t)\, dt$$

in which the integral is taken in the Lebesgue sense and in which the interval (a, b) may be unbounded. This transformation will be called the $\mathscr{T}$-transform and denoted by

$$F(t) \underset{t \quad s}{\overset{\mathscr{T}}{\circ\!-\!\bullet}} f(s) ,$$

The following theorem is for this transformation with symmetric kernel.

THEOREM 2. *If*

(i) *k, g, and $h=g^{-1}$ are single-valued analytic functions, real on (a, b), and such that $g(a)=a$ and $g(b)=b$ (or $g(a)=b$ and $g(b)=a$);*

(ii) *$F(t) \underset{t \quad s}{\overset{\mathscr{T}}{\circ\!-\!\bullet}} f(s)$ and this transformation integral converges for $a<s<b$;*

(iii) *there exists a function $\Phi(s, u)$, $\Phi(s, u) \underset{u \quad p}{\overset{\mathscr{T}}{\circ\!-\!\bullet}} \phi(s, p)$, this transformation integral converges for $a<s<b$, and*

$$\phi(s, p)=K[s, h(p)]\, k[h(p)]\, |h'(p)| ;$$

(iv)

$$\int_a^b \left[\int_a^b |K(u, p)\, \Phi(s, u)\, F(p)|\, du \right] dp$$

converges for $s=s_0$; and

(v) *$K(u, p)\equiv K(p, u)$; then $k(t)\, F[g(t)] \underset{t \quad s}{\overset{\mathscr{T}}{\circ\!-\!\bullet}} \int_a^b \Phi(s, u)\, f(u)\, du$ and this* *transformation integral converges for* $s=s_0$.

The proof follows in a manner similar to that of Theorem 1.

Formulas which hold provided $F(t)$ satisfies (ii) or (iv) of the theorem can be obtained for various transforms for specific $k(s)$ and $g(s)$ with the aid of tables [3, formulas 14.1(6), 8.12(10), 5.5(6)].

Formula 1. For the Stieltjes transformation $K(s, t)=(s+t)^{-1}$.

$$t^{b+1}F(at^2) \underset{t\quad s}{\overset{\mathscr{S}}{\circ\!-\!\bullet}} \int_0^\infty \frac{(u/a)^{b/2}}{2\pi a}\left[\frac{(u/a)\cos b\pi/2 - s\sin b\pi/2}{s^2+u/a}\right] f(u)\,du$$

for a positive.

Formula 2. For the Hankel transformation $K(s, t)=J_\nu(st)(st)^{1/2}$

$$t^{-2}F(a/t) \underset{t\quad s}{\overset{\mathscr{H}}{\circ\!-\!\bullet}} a^{-1}\int_0^\infty \sqrt{aus}\,J_{2\nu}(2\sqrt{aus})f(u)\,du$$

for $-1/2<\nu$ and a positive.

The Laplace transformation will be considered in the next two formulas.

Formula 3.

$$(t+b/a)^d F(at^2+2bt)$$

$$\underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}} (1/2\pi)\, e^{bs/a}\int_0^\infty e^{-b^2 s/a}\, e^{-s^2/8au}(\sqrt{2au})^{-d-1} D_d(s/\sqrt{2au})f(u)\,du$$

for a and b positive and in which $D_d(z)$ is the parabolic cylinder function. The range of permissible values of d will depend, according to (iv), on the particular function $F(u)$.

Formula 4.

$$t^{d-1}F(at^{-b}) \underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}} b^{-1}\int_0^\infty (au)^{d/b}\phi[1/b,\ (d+b)/b;\ -s(au)^{1/b}]f(u)\,du$$

for a and b positive and in which $\phi(A, B; Z)$ is Wrights' function [4, vol. 3, § 18.1]. The range of permissible values of d will depend, according to (iv), on the particular function $F(u)$. In the special case $b=1$ the formula becomes

$$t^{d-1}F(a/t) \underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}} \int_0^\infty (\sqrt{au/s})^d\, J_d(2\sqrt{aus})f(u)\,du\ .$$

References

1. Gustav Doetsch, *Tabellen zur Laplace-transformation,* Berlin, 1947.
2. ———, *Handbuch der Laplace-Transformation,* Basel, 1950.
3. Arthur Erdélyi and others, editors, *Tables of integral transforms,* New York, 1954.
4. ———, *Higher transcendental functions,* New York, 1955.
5. Édouard Labin, *Calcul opérationnel,* Paris, 1949.
6. N. W. McLachlan and Pierre Humbert, *Formulaire pour le calcul symbolique,* Paris, 1941.
7. ———, ——— and Louis Poli, *Supplément du formulaire pour le calcul symbolique,* Paris, 1950.

UNIVERSITY OF COLORADO