CONJUGATE SERIES AND A THEOREM OF PALEY

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1. Introduction. It is known that a trigonometric series

$$\sum_{-\infty}^{\infty} a_n e^{inx} \tag{1}$$

does not have to satisfy condition on the size of its coefficients stronger than the trivial one

$$\sum_{-\infty}^{\infty} | a_n |^2 < \infty$$

in order to be the Fourier series of a continuous function. One theorem which gives precise content to this general statement is the following: *If* $\{w_n\}_{-\infty}^{\infty}$ *is a sequence of non-negative numbers such that*

$$\sum_{-\infty}^{\infty} | a_n | w_n < \infty$$

whenever (1) *is the Fourier series of a continuous function, then*

$$\sum_{-\infty}^{\infty} w_n^2 < \infty \ .$$

The fact that (1) is the Fourier series of a continuous function does not by any means imply the same for

$$\sum_{0}^{\infty} a_n e^{inx} \tag{2}$$

Therefore the following rather neglected theorem of Paley [5] lies deeper than the result just stated.

THEOREM 1 (Paley). *If* $\{w_n\}_0^{\infty}$ *is a sequence of non-negative numbers such that*

$$\sum_{0}^{\infty} | a_n | w_n < \infty \tag{3}$$

whenever (2) *is the Fourier series of a continuous function, then*

$$\sum_{0}^{\infty} w_n^2 < \infty \ . \tag{4}$$

In the next section we offer a new and simple proof of this theorem. The proof depends on the fact that the conjugate series of a Fourier-

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Stieltjes series is summable (by Abel or Cesàro means) almost everywhere.

In several or infinitely many variables the analogue of the first result mentioned is true. The main purpose of this note is to investigate a few of the possible generalizations of Paley's theorem to functions of several variables.

In § 3 we develop the notion of conjugate function in several variables from a point of view somewhat different from that of other writers. We cannot answer the natural questions about the summability of conjugate series, but our result makes it possible (in § 4) to apply a theorem of Zygmund on the summability of multiple power series in order to generalize Paley's Theorem.

Finally, in the last section, we show by a simple example that Paley's theorem cannot be extended to power series in infinitely many variables. We do not know, however, whether power series in *two* variables have Paley's property or not.

2. Proof of Paley's theorem. For each continuous function φ with Fourier series (2) define

$$F[\varphi] = \sum_{0}^{\infty} a_n w_n \ . \tag{5}$$

The series converges on account of the hypothesis (3), and evidently F is a continuous linear functional defined on a closed subspace of the space C of continuous periodic functions. Extend F to a linear functional on all of C. By a well-known theorem of F. Riesz, there is a complex bounded measure μ defined on Borel subsets of the circle such that

$$F[\psi] = \int_0^{2\pi} \psi(-x) d\mu(x) \qquad (\text{all } \psi \in C) \ . \tag{6}$$

In particular, for the functions $\psi(x) = e^{inx}$ $(n = 0, 1, 2, \cdots)$ we have two representations for $F[\psi]$ given by (5) and (6):

$$w_n = F[\psi] = \int_0^{2\pi} e^{-inx} d\mu(x) \ . \tag{7}$$

If we define w_n by (7) for $n < 0$, the Fourier-Stieltjes series of $d\mu$ is

$$\sum_{-\infty}^{\infty} w_n e^{inx} \ .$$

Now every Fourier-Stieltjes series is summable almost everywhere by Abel (or by Cesàro) means [7, p. 59]. The same is true, although more difficult to prove, for the conjugate series of a Fourier-Stieltjes series [7, p. 146]. It follows from these facts together that

$$\sum^{\infty} w_n e^{inx}$$

is summable almost everywhere.

Instead of (5) we could have defined F as

$$F[\varphi] = \sum_0^{\infty} a_n w_n e^{it_n} ,$$

where $t_0, t_1, \cdots$ are any real numbers. The same argument shows that

$$\sum_0^{\infty} w_n e^{it_n} e^{inx}$$

is summable for almost all x, no matter what values the t_n have. But this implies (4) [7, p. 125], as was to be proved.

3. **Conjugate series.** Denote by T_k the torus group in k dimensions, and by T the compact infinite-dimensional torus. The dual of T_k is the lattice group $\mathscr{L}_k$ in k dimensions. The dual $\mathscr{L}$ of T is the group whose elements are sequences of integers $(n_1, n_2, \cdots)$ which are all zero but a finite number. A summable function f on T_k has Fourier series

$$f(x_1, \cdots, x_k) \sim \sum a(n_1, \cdots, n_k) e^{i(n_1x_1 + \cdots + n_kx_k)} . \tag{8}$$

(The formal summation is extended over all integral values of $n_1, \cdots, n_k$.) Similarly we write for a summable f on T

$$f(x_1, \cdots) \sim \sum a(n_1, \cdots) e^{i(n_1x_1 + \cdots)} , \tag{9}$$

where the sum in the exponent is actually finite at each lattice point $(n_1, \cdots)$.

Let N and X denote generic points of $\mathscr{L}_k$ (or of $\mathscr{L}$) and of T_k (or of T) respectively. In place of (8) and (9) we shall write

$$f(X) \sim \sum a(N) e^{iN\cdot X}$$

In the same way we write the Fourier-Stieltjes series of a bounded complex measure μ on T_k or on T

$$d\mu(X) \sim \sum a(N) e^{iN\cdot X} \tag{10}$$

Let S be a subset of $\mathscr{L}_k$ or of $\mathscr{L}$; we are interested in the operator T_S which carries every series (10) into

$$\sum_S a(N) e^{iN\cdot X} . \tag{11}$$

If T_S carries the Fourier series of one function space into those of another, we shall consider T_S at the same time as an operation on functions. Our aim is to prove, by specializing S, that T_S is continuous in certain topologies.

DEFINITION. A subset S of $\mathscr{L}_k$ or of $\mathscr{L}$ is a *half-space* if

1° The origin is not in S

2° $N \in S$ if and only if $-N \notin S$ (N not the origin)

3° $M + N$ belongs to S with M and N.

Fourier series with coefficients lying in a half-space have many of the properties of analytic Fourier series of one variable [4]. The following theorem of Bochner [1] generalizes a well-known theorem of M. Riesz.

BOCHNER'S THEOREM. *If S is a half-space, then T_S is a bounded operator mapping L^p into itself for each $p > 1$.*

The main result of this section is of the same character.

THEOREM 2. *If S is a half-space and f is a trigonometric polynomial then*

$$\| T_S F \|_p \leqq K_p \| f \|_1 \qquad \text{(all } p<1) , \tag{12}$$

where K_p is a constant depending only on p.

The proof is like that of the corresponding theorem in one variable [7, p. 150], applied not to analytic functions but to elements in an appropriate Banach algebra. The observations which follow are not new.

Denote by C_S the class of continuous function φ on the torus having Fourier series of the form

$$\varphi(X) \sim b + \sum_S b(N) e^{iN \cdot X} . \tag{13}$$

C_S is the closure, in the topology of uniform convergence, of the set of trigonometric polynomials having the form (13). Therefore C_S is an algebra, and in fact is a Banach algebra in the uniform topology. To each maximal ideal M in C_S is associated a multiplicative linear functional of norm one on C_S. The value of this functional on φ is denoted by $\hat{\varphi}(M)$. The *spectrum* of φ is the closed set of complex numbers $\hat{\varphi}(M)$, where M ranges over the space of maximal ideals. If F is an analytic function defined on a region of the plane containing the spectrum of φ then there is an element ψ in C_S such that

$$\hat{\psi}(M) = F[\hat{\varphi}(M)] \qquad \text{(all } M) ;$$

we may write simply $\psi = F(\psi)$.

Suppose φ is in C_S and $\Re\, \varphi(X) \geqq \varepsilon > 0$ for all X. We assert that $\Re \hat{\varphi}(M) \geqq \varepsilon$ for all M. (The second statement contains the first, because to each point X there is a maximal ideal M_X with $\hat{\varphi}(M_X) = \varphi(X)$ for all φ.) Indeed, the linear functional associated with any M can be extended to the space of all continuous functions without increasing its bound, and so has a representation

$$\hat{\varphi}(M) = \int \varphi(X) d\mu_M(X) \qquad (\text{all } \varphi \in C_S) \tag{14}$$

with

$$\int | d\mu_M(X) | = 1 . \tag{15}$$

The function φ_0 constantly equal to one is in C_S and its value in every maximal ideal is one. From (14) we obtain

$$1 = \hat{\varphi}_0(M) = \int d\mu_M(X) . \tag{16}$$

It follows from (15) and (16) together that μ_M is a non-negative measure. Therefore in (14) we can separate real and imaginary parts to obtain

$$\Re\hat{\varphi}(M) = \int \Re\varphi(X) d\mu_M(X) \geqq \varepsilon$$

as was to be proved.

Let p be an arbitrary fixed positive number. Suppose φ is in C_S and $\Re\varphi(X) \geqq \varepsilon > 0$. By what has just been proved, the function z^p is analytic on the spectrum of φ, so we can form φ^p in C_S satisfying

$$\hat{\varphi}^p(M) = \hat{\varphi}(M)^p \qquad (\text{all } M) . \tag{17}$$

Two cases of (17) are important. First, if M_X is the maximal ideal determined by a point X of the torus then (17) becomes

$$\varphi^p(X) = \varphi(X)^p \qquad (\text{all } X) . \tag{18}$$

Second, there is a distinguished maximal ideal M_0 such that[1]

$$\hat{\varphi}(M_0) = \int \varphi(X) d\sigma(X) .$$

We conclude from (17) that

$$\int \varphi^p(X) d\sigma(X) = \left(\int \varphi(X) d\sigma(X)\right)^p . \tag{19}$$

The second property of half-spaces has not been used up to this point. We appeal to it now in order to observe that every real trigonometric polynomial is the real part of a trigonometric polynomial in C_S.

We are ready to give the proof of Theorem 2. Suppose first that f is a *positive* trigonometric polynomial:

$$f(X) = \sum b(N) e^{in \cdot X} \geqq \varepsilon > 0$$

[1] $d\sigma(X)$ denotes the element of Haar measure on the torus normalized to have unit total mass.

and set

$$\varphi(X) = b(0) + 2\sum_S b(N)e^{iN\cdot X} .$$

Then φ belongs to C_S, and we can write $\varphi = f + ig$ where g is a real trigonometric polynomial having mean value zero. The expression for φ in polar form is

$$\varphi = w(\cos\alpha + i\sin\alpha) ,$$

where $w(x) = |\varphi(X)|$ and $-\pi/2 < \alpha(X) < \pi/2$.

Making use of (18) and (19) and the notation just introduced we have

$$\left(\int f d\sigma\right)^p = \left(\int \varphi d\sigma\right)^p = \int \varphi^p d\sigma = \int w^p(\cos p\alpha + i\sin p\alpha)d\sigma$$
$$= \int w^p \cos p\alpha d\sigma \geqq \cos\frac{p\pi}{2}\int (f^2+g^2)^{p/2}d\sigma \geqq \cos\frac{p\pi}{2}\int |g|^p d\sigma .$$

That is, with $A = \left(\cos\frac{p\pi}{2}\right)^{-1/p}$,

$$\left(\int |g|^p d\sigma\right)^{1/p} \leqq A\int f d\sigma .$$

From this inequality it is simple to prove (12) with a certain constant B in place of K_p. By continuity the result holds for trigonometric polynomials which are non-negative but not necessarily bounded from zero.

The passage to trigonometric polynomials of arbitrary sign is not quite trivial. If f is *real* and has the form

$$f = g - h$$

where g and h are non-negative trigonometric polynomials, then by what we have proved

$$\|T_S f\|_p^p \leqq \|T_S g\|_p^p + \|T_S h\|_p^p \leqq B^p[\|g\|_1^p + \|h\|_1^p] . \tag{20}$$

Decompose f into its positive and negative parts:

$$f = f_+ - f_- ;\ f_+, f_- \geqq 0,\ f_+ \cdot f_- \equiv 0 .$$

If f_+ and f_- were trigonometric polynomials we could choose them for g and h and obtain from (20)

$$\|T_S f\|_p^p \leqq B^p[2\|f\|_1^p] . \tag{21}$$

Of course they are never trigonometric polynomials unless one of them vanishes, but they are non-negative continuous functions, and so can be approximated uniformly by non-negative trigonometric polynomials, say by g_n and h_n respectively. Set $f_n = g_n - h_n$. Writing (20) with f_n, g_n, and h_n and passing to the limit we obtain

$$\lim \| T_S f_n \|_p^p \leqq 2B^p \|f\|_1^p .$$

But the sequence $T_S f_n$ obviously tends to $T_S f$ in the norm of L^2, and so in L^p for each $p < 2$, so we obtain (21) after all.

There is no difficulty in extending the result to arbitrary complex trigonometric polynomials, with a new constant which we call K_p, and so the theorem is proved.

COROLLARY. *If S is a half-space in $\mathscr{L}_k$ or in $\mathscr{L}$, the operation T_S transforms L into L^p for each $p < 1$ in the following sense: every Fourier series is carried by T_S into a series summable in the metric of L^p to a limit function. The summation is effected by every approximate identity consisting of trigonometric polynomials. The transformation carrying f into T_S satisfies* (12).

Let f be an arbitrary summable function on the torus, and let $\{e_1, e_2, \cdots\}$ be an approximate identity consisting of trigonometric polynomials. (That is, each e_j is a non-negative trigonometric polynomial with mean value one and $e_j * f$ tends to f in the norm of L for each f.) Then $\{e_j * f\}$ is a Cauchy sequence in L consisting of trigonometric polynomials. By Theorem 2, $\{T_S(e_j * f\}$ is a Cauchy sequence in L^p for $p < 1$. Consequently $T_S(e_j * f)$ converges in the metric of L^p to a limit function $T_S f$, and (12) clearly holds. This is just the statement of the Corollary.

We do not know whether any method of summation effects pointwise convergence almost everywhere of the series for $T_S f$. However the Corollary shows that $T_S f$ always exists as a limit in mean. If f is a real summable function, its conjugate can be defined as the real function g such that $f + ig = T_S f$. Then g exists as a limit in mean, and has many of the properties one expects of a conjugate function in one variable. Our proofs have made strong use of the fact that S is a half-space ; we do not know whether $T_S f$ exists in any sense whatever if S is, for example, the first quadrant of $\mathscr{L}_2$.

The device used to prove Theorem 2 can be used to extend other classical theorems about Fourier series in one variable.

It is possible to assert the conclusion of Theorem 2 for certain sets S which are not quite half-spaces. For simplicity consider a half-space in $\mathscr{L}_2$. It consists exactly of those lattice-points (m, n) satisfying

$$m\alpha + n\beta > 0 \tag{22}$$

for some *irrationally* related numbers α and β ; or else S consists of the lattice points satisfying (22) for some *rationally* related α and β, together with non-zero points on one ray from the origin of the line

$$m\alpha + n\beta = 0$$

We shall consider a half-space S of the second type. Denote the other ray (including the origin), which is not contained in S, by R. With the use of the theorem on conjugate functions of one variable it is easy to prove that Theorem 2 holds for the set R in place of S. It follows that the augmented half-space H consisting of all (m, n) with $m\alpha + n\beta \geqq 0$ has the same property.

In the same way we can add to a half-space of dimension k in $\mathscr{L}_k$ disjoint half-spaces of lower dimension, obtaining new sets for which Theorem 2 and its corollary hold.

Let R and S be two half-spaces, or more generally, any sets having the property of Theorem 2. Then $T_R - T_S$ is an operator carrying L into L^p for each $p < 1$. For example, let R and S be the modified half-spaces in $\mathscr{L}_2$ defined respectively by $m \geqq 0$ and $n < 0$. $T_R - T_S$ operates on trigonometric series in two variables by multiplying each term by a factor ε_{mn}; this factor is 1 in the first quadrant (including the boundary) and -1 in the third quadrant (excluding the boundary), and vanishes in the second and fourth quadrants.

By Bochner's theorem the operator $T_R - T_S$ carries L^p into itself for $p > 1$. It is interesting to compare these results with theorems of similar nature but different proof by Caldéron and Zygmund [3].

4. A generalization of Paley's theorem.

THEOREM 3. *Let S be a half-space in $\mathscr{L}_k$. Suppose $\{w(N)\}$ is a set of non-negative numbers defined for N in S, and having the property that*

$$|b| + \sum_S |b(N)|\, w(N) < \infty \tag{23}$$

whenever (13) *is the Fourier series of a function in C_S. Then*

$$\sum_S w(N)^2 < \infty\ . \tag{24}$$

Proof. As for one variable, the hypothesis implies that

$$\sum_S w(N) e^{i\varphi(N)} e^{iN\cdot X} \tag{25}$$

is the image under T_S of some Fourier-Stieltjes series, no matter how the real numbers $\varphi(N)$ are chosen. This assertion remains true if any of the $w(N)$ are replaced by zero, because (23) continues to hold. Suppose now (24) is false. Then at least one of the 2^k congruent cones

$$\pm n_1 \geqq 0,\ \pm n_2 \geqq 0,\ \cdots,\ \pm n_k \geqq 0$$

obtained by choosing the k signs in these inequalities contains points over which the sum (24) diverges. Denote such a cone by C. Replace $w(N)$ by zero for all N in S–C. Finally, by a linear change of variables bring C into coincidence with the preferred cone

$$n_1 \geqq 0,\, n_2 \geqq 0,\, \cdots,\, n_k \geqq 0 \ .$$

(The transformation carries S onto a new half-space.) Now we have a counter-example to the theorem (which is assumed to be false) in which $w(N)$ vanishes for all N not in C.

Extend w to the complement of S so that

$$\sum w(N) e^{i\varphi(N)} e^{iN\cdot X}$$

is the Fourier-Stieltjes series of a measure μ. For $0 < r < 1$ define a function f_r on the torus by the absolutely convergent series

$$\sum w(N) r^{|n_1|+\cdots+|n_k|} e^{i\varphi(N)} e^{iN\cdot X} \ .$$

For each r we have $\| f_r \|_1 \leqq \int | d\mu |$, and so by the corollary to Theorem 2

$$\| T_S f_r \|_p \leqq K_p \| f_r \|_1 \leqq K \qquad (0 < r < 1) \ . \tag{26}$$

Moreover the corollary implies that

$$T_S f_r(X) = \sum_S w(N) r^{|n_1|+\cdots+|n_k|} e^{i\varphi(N)} e^{iN\cdot X} \ .$$

Since $w(N)$ vanishes on S–C this can be written

$$T_S f_r(X) = \sum_C w(N) r^{n_1+\cdots+n_k} e^{i\varphi(N)} e^{iN\cdot X} \ .$$

On account of (26) this series belongs to the space H^p $(p < 1)$, whose elements are analytic functions of k variables. A theorem of Zygmund [**6**, p. 208] asserts that

$$\lim_{r \uparrow 1} T_S f_r(X)$$

exists almost everywhere on the torus. But the numbers $\varphi(N)$ are arbitrary, and so we conclude as in the case of one variable that

$$\sum_C w(N)^2 < \infty \ .$$

This contradiction shows that the theorem is true.

5. A counter-example. Let S be the subset of $\mathscr{L}$ consisting of all lattice points $N = (n_1, \cdots)$ with each $n_j \geqq 0$. (S is not a half-space, but rather the infinite-dimensional analogue of the quadrant in $\mathscr{L}_2$.) Denote by Λ the subset of S on which $\sum n_j = 1$.

THEOREM (Bohr [2, p. 468]). *For each* φ in C_S *with Fourier series* (11) *we have*

$$\sum_{\Lambda} | a(N) | \leqq \| \varphi \|_\infty .$$

This result states exactly that the sequence $\{w(N)\}$ equal to 1 on Λ and 0 on $S-\Lambda$ has the property of Paley's theorem; but $w(N)$ does not even tend to zero, and certainly is not square-summable. So Paley's theorem fails decisively in this setting.

For this sequence the series (25) is

$$\sum_{j=1}^{\infty} e^{it_j} e^{ix_j} , \tag{27}$$

and this is the image under T_S of a Fourier-Stieltjes series no matter how the real numbers t_j are chosen. For appropriate values of $t_1, t_2, \cdots$ (27) is non-summable for almost every point $(x_1, x_2, \cdots)$, and this is true for each Toeplitz method of summation. Hence the theorem on the summability of conjugate series in one variable cannot be extended to this extreme situation. The analogous problem in two variables is open, so far as we know, both for half-spaces and for quadrants.

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