

A FINITE ALGORITHM FOR THE SOLUTION OF CONSISTENT LINEAR EQUATIONS AND INEQUALITIES AND FOR THE TCHEBYCHEFF APPROXIMATION OF INCONSISTENT LINEAR EQUATIONS

ALLEN A. GOLDSTEIN AND WARD CHENEY

I. Introduction. The three problems of the title are treated here from a unified geometric standpoint, and an algorithm is presented for their solution. Algorithms for these problems already exist in [8], [4], [5], and [6]. In the present algorithm, each problem is reinterpreted as one of finding the lowest points (if any exist) of a polytope in an Euclidean space; the techniques of steepest descent and elimination of variables are then combined to work downward from vertex to vertex. Professor T. S. Motzkin kindly called our attention to references [8] and [3] in which a similar viewpoint is exploited. Our thanks are also due to the referee for helpful suggestions, and to Norman Levine, who has coded the algorithm on the "704" automatic digital computer.

Section II which follows provides a description of the problems and an outline of the algorithm. Section III contains a detailed statement of the algorithm. Section IV concerns the special case of n variables and $n+1$ equations or inequalities. Section V is devoted to a proof of finiteness of the algorithm. In §VI, relationships among the three above problems and linear programming are discussed. In particular, it is shown that a simple modification of the algorithm will permit its application to linear programming.

II. The problems defined. Let $\{A^1, \dots, A^m\}$ be a subset of E_n with $m \geq n$, and let b denote a fixed point of E_m . We emphasize that throughout the paper the matrix A composed of the rows A^1, \dots, A^m is assumed to be of rank n . The notation (u, v) will be used for $\sum u_i v_i$, and a vector $\bar{x} = (x_1, \dots, x_{n+1})$ will be said to have *abscissa* $x = (x_1, \dots, x_n)$ and *ordinate* x_{n+1} . The word *polytope* denotes the intersection of a finite number of half-spaces. Define the polytope

$$\mathcal{P} = \left\{ \bar{x} : x_{n+1} \geq \max_{1 \leq i \leq m} [(A^i, x) - b_i] \right\}.$$

If \mathcal{P} intersects the half-space $\{\bar{x} : x_{n+1} \leq 0\}$, then the system

Received August 1, 1957, and in revised form March 11, 1958. Parts of this paper were presented to the American Mathematical Society as follows: June 1958 abstract 589, August 1957 abstract 748. The paper was presented by title to the Conference on Matrix Computation, Wayne State University, September 1957.

$$(A^i, x) \leq b_i \quad (1 \leq i \leq m)$$

is consistent. Any point \bar{x} of \mathcal{S} lying in the aforementioned half-space has an abscissa which satisfies this system. In particular, if \mathcal{S} possesses lowest points (i.e., points of minimum ordinate), then these points have this property, and their abscissae minimize the function

$$F(x) = \max_{1 \leq i \leq m} [(A^i, x) - b_i] .$$

Consider next the system of linear equations

$$(A^i, x) = b_i \quad (1 \leq i \leq m) .$$

The Tchebycheff problem for this system is to obtain x in E_n , called a *minimax solution*, which minimizes the function

$$F^*(x) = \max_{1 \leq i \leq m} |(A^i, x) - b_i| .$$

If one defines $A^{m+i} = -A^i$ and $b_{m+i} = -b_i$ when $1 \leq i \leq m$, then

$$F^*(x) = \max_{1 \leq i \leq 2m} [(A^i, x) - b_i] .$$

In this case another polytope may be defined, and any lowest point \bar{x} will have a minimax solution x for abscissa and $F^*(x)$ for ordinate. Note that the case of consistent linear equations is encompassed by the Tchebycheff problem, and that in the Tchebycheff problem the polytope must have a lowest point. The all-inclusive problem, then, to which attention is now turned, is that of locating the lowest points of \mathcal{S} or, equivalently, of minimizing the function F . This unified geometric viewpoint is due to Fourier, [3].

In the simplest case, the algorithm is as follows. Define *residual functions*

$$R^i(x) = (A^i, x) - b_i \quad (1 \leq i \leq m) .$$

Then $F(x) = \max_{1 \leq i \leq m} R^i(x)$. An arbitrary initial vector x^0 will generally provide one maximum residual; suppose $F(x^0) = R^1(x^0)$ for convenience. On the ray $\{x^0 - tA^1 : t \geq 0\}$ let x^1 be the first point at which another residual equals R^1 ; suppose $F(x^1) = R^1(x^1) = R^2(x^1)$. The equation $R^1(x) = R^2(x)$ serves to eliminate one variable from the original system, and the algorithm may begin anew in this reduced system. After at most $n-1$ eliminations and descents a point is obtained which is the abscissa of a vertex of \mathcal{S} . A direction for the next descent is obtained by solving a system of inequalities according to a simple prescription. The precise description of the algorithm in §III is more complicated because of certain singular cases that may arise.

III. The algorithm. A point x of E_n is sought which minimizes the function

$$F(x) = \max_{1 \leq i \leq m} R^i(x) = \max_{1 \leq i \leq m} [(A^i, x) - b_i].$$

The solution is obtained by repeating a cycle of steps, each of which determines a vertex of the polytope \mathcal{P} at a lower level than the vertex of the preceding cycle. Since the vertices of \mathcal{P} are finite in number, only one cycle need be described here. In each cycle (except the first) a point x is on hand from the preceding cycle. (In the first cycle, this point is arbitrary.) It is necessary to consider certain auxiliary systems

$$(r) \begin{cases} {}^rF(x) = \max_i {}^rR^i(x) \\ {}^rR^i(x) = ({}^rA^i, x) - {}^rb_i \end{cases}$$

in which $x \in E_n$, ${}^rA^i \in E_n$, ${}^rb \in E_m$, ${}^1F = F$, ${}^1R^i = R^i$, ${}^1A^i = A^i$, ${}^1b = b$, $1 \leq r \leq n$, and $1 \leq i \leq m$. A system of index $(r+1)$ is obtained from a system of index (r) by the elimination of a variable via an equation of the form ${}^rR^i(x) = {}^rR^j(x)$.

JUNCTURE I. Assume that system (r) is provided and that a point y in E_n is specified. By reindexing, obtain for suitable k ,

$${}^rF(y) = {}^rR^1(y) = \dots = {}^rR^k(y) > {}^rR^{k+i}(y) \quad (1 \leq i \leq m-k)$$

$$({}^rA^1, {}^rA^1) \geq ({}^rA^i, {}^rA^i) \quad (1 \leq i \leq k)$$

If ${}^rA^1 = 0$, then $r = 1$ and y is a solution. (See proof of Theorem J.) Otherwise, there are three cases to distinguish:

Case 1 Not all of ${}^rA^1, \dots, {}^rA^k$ are multiples of ${}^rA^1$. By rearranging, assume ${}^rA^2 \neq c {}^rA^1$, and set ${}^rR^1(x) = {}^rR^2(x)$. By means of this equation, some variable, say x_j , may be eliminated from system (r) . The new system that arises in this way is, by definition, system $(r+1)$. The vector taken now at Juncture I (with $r+1$ replacing r) is identical with y , except that the j th component is set equal to zero.

Case 2 ${}^rA^1, \dots, {}^rA^k$ are all positive multiples of ${}^rA^1$. (This includes the case $k=1$.) Let t be the least positive scalar for which the point $y - t {}^rA^1$ induces $k+1$ or more equal maximum residuals. Define $z = y - t {}^rA^1$, and return to Juncture I with z in place of y . If no such t exists then \mathcal{P} has no lowest point. This can happen only in the inequality problem; any $z (= y - t {}^rA^1)$ for which ${}^rF(z) \leq 0$ will have for its image in the original system a solution of the inequalities.

Case 3 All of ${}^rA^1, \dots, {}^rA^k$ are multiples of ${}^rA^1$, but not all positive multiples. (This includes the case of some vanishing ${}^rA^i$.) Proceed to Juncture II.

JUNCTURE II. Restore those $r-1$ components of y which were set equal to zero during occurrences of Case 1, obtaining thereby a vector x in E_n for which

$$F(x) = R^1(x) = \cdots = R^k(x) > R^{k+i}(x) \quad (1 \leq i \leq m-k).$$

If further descent is possible, then the system of inequalities

$$(1) \quad (A^i, d) \geq 1 \quad (1 \leq i \leq k)$$

is consistent. By Theorem H, there exists (and is discoverable by trial) a set of r indices $\{i_1, \dots, i_r\} \subset \{1, \dots, k\}$ such that the set $\{A^{i_1}, \dots, A^{i_r}\}$ is linearly independent and such that the solution d of $(A^{i_j}, d) = 1$ ($1 \leq j \leq r$) satisfies (1). In the case of usual occurrence, $k = r + 1$, and d may be constructed as in the proof of Theorem F. The point v at which the minimum of F occurs on the ray $\{x - td : t \geq 0\}$ is taken for the initial point of the next cycle. If no such point exists, any t for which $F(x - td) \leq 0$ solves the inequalities.

REMARK. In the event that $k > r + 1$ at Juncture II, the selection of the set of r indices may be avoided by a perturbation. Of the first k components of the vector b , $k - r - 1$ may be increased by arbitrarily small quantities, and the algorithm continued with this perturbed system. This situation arises only a finite number of times, and a solution of a perturbed system is eventually obtained. It may be shown that if the final perturbed vector b^* satisfies $\max |b_i - b_i^*| < \frac{1}{2}\delta$, where δ is the vertical separation between the two lowest vertex-bearing levels of \mathcal{P} , then starting with the solution of the perturbed system, the algorithm will produce in one cycle a solution of the original system. The perturbation method is well-adapted to machine computation.

IV. The case $m = n + 1$. The case $m = n + 1$ of the Tchebycheff problem was considered in [6], where it was shown how the solution of an $m \times n$ system may be obtained as the solution of a certain $(n+1) \times n$ subsystem discoverable by trial. Algorithms which systematize the search for this subsystem appear in [4]. Techniques developed in this section are applicable also in the algorithm of §III whenever there is obtained an "ordinary" vertex of the polytope \mathcal{P} (that is, a vertex involving precisely $n+1$ planes).

The norms, N , which are employed in this section belong to a large class of *monotone* norms; i.e., they have the property that $N(u) \leq N(v)$ whenever $|u_i| \leq |v_i|$ for all i . Examples of such norms are these:

$$\begin{aligned} N_p(x) &= (\sum |x_i|^p)^{1/p} & (p \geq 1) \\ N_\infty(x) &= \max_i |x_i| \end{aligned}$$

$$N_w(x) = (\sum x_i^2 |w_i|)^{1/2} \quad (w_i \neq 0 \text{ all } i).$$

In these terms, a well-known inequality states $(x, y) \leq N_1(x) \cdot N_\infty(y)$. The word *orthant* is used for the n -dimensional analogue of *octant*. Thus, two vectors x and y lie in the same orthant if $\text{sgn } x_i = \text{sgn } y_i$ for all i . It is understood that $\text{sgn } 0 = 0$.

THEOREM A. *Let H denote a hyperplane in E_m . Let N and N' denote monotone norms. The points of H which minimize N or N' may be taken to lie in the same orthant.*

Proof. The theorem is trivial when H contains the origin. In the other case, let u minimize N on H and let w minimize N_2 on H . Then $H = \{v : (v, w) = (w, w)\}$. By setting $u'_i = u_i \text{sgn } w_i$, one obtains a vector u' in the same orthant as w ; by the monotone property of N , $N(u') \leq N(u)$. Since the signs of u'_i agree with those of w_i , $(u', w) \geq (u, w) = (w, w) > 0$, and the number $k = (w, w)/(u', w)$ is therefore well-defined. The inequality just written shows that $0 < k \leq 1$, whence $N(ku') \leq N(u')$. Since $(ku', w) = (w, w)$, ku' lies on H . Thus ku' minimizes N on H . The argument may now be repeated with N' and N_2 to conclude the proof. It should be observed that the theorem would not be valid if H denoted a translate of an arbitrary subspace, as for example a line in E_3 . See [7].

THEOREM B. *Let H denote a hyperplane in E_m not through the origin.*

(1) *A vector u of H which minimizes N_∞ is unique if and only if the vector w of H which minimizes N_2 has no zero components.*

(2) *If the vector w exhibits zero components, then there is a unique vector u whose corresponding components vanish.*

(3) *Statements (1) and (2) remain true if w is replaced by the vector v which minimizes N on H , N being any strictly convex monotone norm.*

Proof. Define $u_i = (w, w) \text{sgn } (w_i)/N_1(w)$. Then $(u, w) = (w, w)$, whence u lies on H . If $N_\infty(u') < N_\infty(u)$ then

$$(u', w) \leq N_\infty(u') \cdot N_1(w) < N_\infty(u) \cdot N_1(w) = (w, w)$$

so that u' does not lie on H . Hence the vector u defined above minimizes N_∞ on H . If all components of w are non-zero, then it is easy to see that any alteration of the components of u will either increase $N_\infty(u)$ or decrease (u, w) , so that u is unique. If w has zero components, the corresponding components of u are zero by the above definition. The remaining components of u are uniquely determined by the same argument. Statement (3) depends on the observation that v is unique, and on Theorem A.

THEOREM C. *Let A denote an $(n+1) \times n$ matrix of rank n . Let x denote the least-squares solution of $Ax=b$, and let $w=Ax-b$. Define u as in the proof of Theorem B. Then there exists y in E_n such that $Ay=u+b$, and y is a minimax solution of $Ay=b$.*

THEOREM D. *Let A denote an $(n+1) \times n$ matrix of rank n . The following are equivalent when the system $Ax=b$ is inconsistent:*

- (1) *The vector x which minimizes $N_\infty(Ax-b)$ is unique.*
- (2) *If the vector y minimizes $N_2(Ay-b)$ then $Ay-b$ has no zero components.*
- (3) *A has no singular $n \times n$ submatrices.*
- (4) *The range of A contains no coordinate axis.*

Proof. (1) \Leftrightarrow (2) follows from Theorem B. (1) \Leftrightarrow (3) follows from Theorem G below. (2) \Leftrightarrow (4) follows from the observation that $Ay-b$ is normal to the range of A .

THEOREM E. *Let A denote an $(n+1) \times n$ matrix of rank n . Let x denote the least-squares solution of $Ax=b$. Let $w=Ax-b$. Define $N_w(u) = (\sum u_i^2 |w_i|)^{1/2}$. Let y be chosen to minimize $N_w(Ay-b)$ subject to the condition $(A^i, y)=b_i$ when $w_i=0$. Then y is a minimax solution of $Ay=b$.*

Proof. The conditions on y guarantee its uniqueness. From least-squares theory,

$$(1) \quad \sum_i [(A^i, x) - b_i] A_p^i = 0 \quad (1 \leq p \leq n)$$

$$(2) \quad \sum_i [(A^i, y) - b_i] A_p^i |w_i| = 0 \quad (1 \leq p \leq n).$$

Now let z denote a minimax solution of $Az=b$ for which $Az-b$ lies in the same orthant as $Ax-b$ (Theorem A), and set $M=N_\infty(Az-b)$. Equation (2) remains valid when w_i is replaced by w_i/M , and is then equivalent to the following condition on y (since the indices i for which $w_i \neq 0$ yield $(A^i, z)-b_i=M$):

$$(3) \quad \sum'_i [(A^i, y) - b_i] A_p^i [(A^i, x) - b_i] [(A^i, z) - b_i]^{-1} = 0,$$

where p runs from 1 to n , and where \sum' indicates omission of indices i for which $w_i=0$. If w has no zero components, equation (3) is satisfied if and only if $y=z$ because of (1) and the uniqueness of y . If $w_i=0$ for some i , then by Theorem B a minimax solution z exists for which the corresponding components of $Az-b$ are zero. The remainder of the proof is then as before.

THEOREM F. *Let $\{A^i: 1 \leq i \leq n+1\}$ be a subset of E_m of rank n . Let u be chosen in E_{n+1} so that*

$$(1) \quad \sum_{i=1}^{n+1} u_i A^i = 0,$$

$$(2) \quad u \neq 0, \text{ and}$$

$$(3) \quad (u, b) \geq 0. \quad \text{The system of inequalities}$$

$$(4) \quad (A^i, x) \leq b_i \quad (1 \leq i \leq n+1)$$

is consistent if and only if either $(u, b) = 0$ or u exhibits a positive component.

Proof. If (4) is consistent and $(u, b) > 0$, select x such that $y_i \equiv (A^i, x) \leq b_i$. Since $(b - y, u) = (b, u) - (y, u) = (b, u) - \sum_i (A^i, x) u_i = (b, u) - (x, \sum u_i A^i) = (b, u) > 0$, it is clear that u must exhibit a positive component. Next, if $(u, b) = 0$, then there exists an x such that $(A^i, x) = b_i$, $(1 \leq i \leq n+1)$. Finally, if u exhibits a positive component u_j , then define y_i as b_i or $-(\sum_{k \neq j} b_k u_k)/u_j$ according to whether $i \neq j$ or $i = j$. It is quickly verified that $y_i \leq b_i$ for all i , and that $(y, u) = 0$. Hence there exists an x such that $(A^i, x) = y_i$. This completes the proof. It is true that the set $\{A^i : i \neq j\}$ is linearly independent. For, if on the contrary $\sum c_i A^i = 0$ with $c \neq 0$ and $c_j = 0$, then $\sum (u_i + c_i) A^i = 0$, contradicting the fact that u is unique to within a scalar multiple. Theorems of similar nature but greater complexity can be given for the cases of more than $n+1$ inequalities.

THEOREM G. Assume the minimum value of $f(x) = \max_{1 \leq i \leq n+1} |(A^i, x) - b_i|$ to be positive. Let y be chosen as in Theorem C. For each i , select u^i non-zero if possible to satisfy $(u^i, A^j) = 0$, $(1 \leq j \leq n+1, j \neq i)$. Then

(1) $|(A^j, y) - b_j|$ is $f(y)$ or zero according as u^j is or is not zero.

(2) $f(x) = f(y)$ if and only if x is of the form $y + \sum' t_i u^i$ where the summation extends over i for which $u^i \neq 0$ and where $|t_i| \leq f(y)/|(A^i, u^i)|$.

(3) The dimension of the set $S = \{x : f(x) = f(y)\}$ is equal to the number of non-zero u^i .

Proof. (1) If $u^j \neq 0$, then from any x in S one may obtain by proceeding in the direction u^j a point x' satisfying $(A^j, x') = b_j$ and $(A^i, x') = (A^i, x)$ when $i \neq j$. Thus whenever $u^j \neq 0$, $(A^j, y) = b_j$. Conversely, if $u^j = 0$, then the set $\{A^i : i \neq j\}$ is of full rank. Consequently y is a continuous function of $f(y)$ via Cramer's rule. If $|(A^j, y) - b_j| < f(y)$ then $f(y)$ may be decreased slightly without elevating $|(A^j, y) - b_j|$ to $f(y)$, thus contradicting the assumption that $f(y)$ be a minimum. Hence $|(A^j, y) - b_j| = f(y)$.

(2) If x is of the quoted form, then a calculation shows $f(x) = f(y)$. Conversely, if $f(x) = f(y)$, then for $u^i \neq 0$, define $t_i = [(A^i, x) - b_i]/(A^i, u^i)$. Set $z = y + \sum' t_i u^i$. When $u^k \neq 0$ then

$$(A^k, z) = (A^k, y) + t_k (A^k, u^k) = b_k + t_k (A^k, u^k) = (A^k, x).$$

When $u^k=0$ then $(A^k, z)=(A^k, y)$, but then Cramer's rule again shows that $(A^k, y)=(A^k, x)$. Thus for all k we have $(A^k, z)=(A^k, x)$, and $x=z$.

(3) This will follow from (2) if it can be shown that the non-zero u^i are linearly independent. Supposing otherwise, we have $\sum' c_i u^i=0$ where the c_i are not all zero. Then

$$0 = (A^k, \sum' c_i u^i) = \sum' c_i (A^k, u^i) = c_k (A^k, u^k),$$

thus showing that all $c_k=0$. This completes the proof.

V. Finiteness of the algorithm. We begin with the theorem [2, p. 104] which may be invoked at Juncture II whenever more than the usual number (which is $n+1$) of planes is encountered at a vertex of \mathcal{P} .

THEOREM H (Fan). *Let the system*

$$(1) \quad f_i(x) \geq \alpha_i \quad (1 \leq i \leq p)$$

be consistent, where f_1, \dots, f_p are given linear functionals of rank r . There exist r linearly independent functionals f_{n_1}, \dots, f_{n_r} among them such that every solution of the system

$$f_{n_k}(x) = \alpha_{n_k} \quad (1 \leq k \leq r)$$

is a solution of (1).

DEFINITION. A point $x \in E_n$ will be called a *quasi-critical point* if for appropriate indices i_1, \dots, i_k ,

- (1) $F(x) = R^{i_1}(x) = \dots = R^{i_k}(x)$;
- (2) the rank, r , of $\{A^{i_1}, \dots, A^{i_k}\}$ is less than k ;
- (3) the system $(A^{i_j}, d)=1$ ($1 \leq j \leq k$) is inconsistent.

When $r=n$, x is termed a *critical point*, and is the abscissa of a vertex of \mathcal{P} .

THEOREM I. *The set of critical points is finite.*

Proof. A critical point x , satisfying (1), (2) and (3) above, satisfies the system of equations

$$\begin{pmatrix} A_{i_1}^{i_1} & \dots & A_n^{i_1} & -1 \\ \dots & \dots & \dots & \dots \\ A_{i_k}^{i_k} & \dots & A_n^{i_k} & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ F(x) \end{pmatrix} = \begin{pmatrix} b_{i_1} \\ \vdots \\ b_{i_k} \end{pmatrix}$$

where the rank of the k by $n+1$ matrix is $n+1$ because of (3). Hence the solution $(x_1, \dots, x_n, F(x))$ is unique. Since there is only a finite number of such systems, the set of critical points is finite.

THEOREM J. *At Juncture II the rank of $\{A^1, \dots, A^k\}$ is r , and $r < k$.*

Proof. The elimination of one component (say x_s) from $x = (x_1, \dots, x_n)$ by means of an equation such as $R^k(x) = R^h(x)$ can be effected only if A^k and A^h are linearly independent, and leads to a new system for which the coefficient of x_j in the i th row is $A_j^i + B_{hksj}A_s^i$ where

$$B_{hksj} = (A_j^h - A_j^k)/(A_s^k - A_s^h).$$

Thus the new matrix is obtained from the old by a sequence of elementary column operations followed by the setting of one column equal to zero. The rank of this set of rows is thus one less in the new matrix than in the old. When Juncture II arises, $r-1$ such eliminations have been made using say the residuals $R^1 = R^2 = \dots = R^r$. Thus the rank of A^1, \dots, A^r must be r . Now Juncture II occurs because ${}^rA^1, \dots, {}^rA^k$ are multiples of ${}^rA^1$, but not all positive multiples. Not all of these are zero moreover, because if they were, the rank of the ${}^{r-1}A^1, \dots, {}^{r-1}A^k$ would be unity and an elimination would not have been possible. Thus the rank of A^1, \dots, A^k is at most $k-1$.

THEOREM K. *If for some set I of indices there exists a d in E_n such that $(A^i, d) = 1$ for i in I , then $({}^rA^i, d) = 1$ ($1 \leq r \leq n-1$) whenever i is in I .*

Proof. We consider the case $r=2$. Repetitions of the argument will prove the general case. Assume $(A^i, d) = 1$ for $i \in I$. Then for distinct h and k in I we have $\sum_{j=1}^n d_j(A_j^h - A_j^k) = 0$. Thus $d_s = \sum_{\substack{j \neq s \\ j=1}}^n d_j B_{hksj}$, with B_{hksj} as in the proof of Theorem J. Now for $i \in I$ we have

$$\begin{aligned} 1 &= \sum_{j=1}^n A_j^i d_j = \sum_{\substack{j \neq s \\ j=1}}^n A_j^i d_j + A_s^i \sum_{\substack{j \neq s \\ j=1}}^n d_j B_{hksj} \\ &= \sum_{\substack{j \neq s \\ j=1}}^n (A_j^i + A_s^i B_{hksj}) d_j = ({}^2A^i, d). \end{aligned}$$

THEOREM L. *In each cycle of the algorithm, the point x appearing at Juncture II is a quasi-critical point.*

Proof. By Theorem J, the point x satisfies criteria (1) and (2) for a quasi-critical point. Criterion (3) follows because of Theorem K. For suppose that at Juncture II the system $({}^1A^i, d) = 1$ were consistent; then each reduced system $({}^rA^i, d) = 1$ would be consistent and Case 3 would not have arisen.

THEOREM M. *To each quasi-critical point y there corresponds a critical point x such that $F(x) = F(y)$.*

Proof. Assume that y satisfies (1), (2), and (3) in the definition of a quasi-critical point. If $r=n$, the theorem is trivial. Otherwise, select a non-zero vector z orthogonal to A^{i_1}, \dots, A^{i_k} . Along the ray $\{y+tz : t \geq 0\}$ the residuals R^{i_1}, \dots, R^{i_k} remain constant, and one may select the first point y' at which a new residual joins the maximum ones. This process may be repeated until a point x is obtained such that for appropriate indices i_{k+1}, \dots, i_h we have

$$F(x) = F(y) = R^{i_1}(x) = \dots = R^{i_h}(x),$$

where $h \geq n+1$ and the rank of $\{A^{i_1}, \dots, A^{i_h}\}$ is n . The system $(A^{i_j}, d)=1$ ($1 \leq j \leq h$) is inconsistent because it has the inconsistent subsystem $(A^{i_j}, d)=1$ ($1 \leq j \leq k$).

The following theorem includes a classical result of [1] to the effect that when each n by n submatrix of A is non-singular then the minimax solution of $Ax=b$ possesses $n+1$ equal dominating residuals.

THEOREM N. *If F attains a minimum, then it does so at a critical point. Thus there are vertices among the lowest points of \mathcal{P} .*

Proof. Let y minimize F , and suppose that after renumbering we have $F(y)=R^1(y)=\dots=R^k(y)>R^{k+i}(y)$. If the rank of $\{A^1, \dots, A^k\}$ is less than n then the techniques of the proof of Theorem M may be used, repeatedly if necessary, to obtain a point x such that $F(y)=F(x)=R^1(x)=\dots=R^p(x)>R^{p+i}(x)$ and $\{A^i : 1 \leq i \leq p\}$ is of rank n . If $p > n$, then x is a critical point. Otherwise x is a continuous function of $F(x)$ via Cramer's rule, and $F(x)$ may be decreased by so small an amount that none of $R^{p+i}(x)$ ($1 \leq i \leq m-p$) are elevated to $F(x)$ —contradicting the assumption that y was a solution. The proof is concluded by the observation that since x is a solution, the system $(A^i, d)=1$ ($1 \leq i \leq p$) is inconsistent.

THEOREM O. *The algorithm is finite.*

Proof. In view of Theorems I, L, M, and N, it is only necessary to observe that Juncture II arises in each cycle, and that the value of F is decreasing at each step.

VI. Miscellany.

THEOREM P. *If the system of inequalities*

$$(1) \quad (A^i, x) \leq b_i \quad (1 \leq i \leq m)$$

is consistent, then for sufficiently large M , every minimax solution of the system

$$(2) \quad (A^i, x) = b_i - M \quad (1 \leq i \leq m)$$

is a solution of (1).

Proof. Select y satisfying (1). If $M \geq -\min_i [(A^i, y) - b_i]$, then $(A^i, y) - b_i + M \geq 0$ whence $|(A^i, y) - b_i + M| = (A^i, y) - b_i + M$, and

$$\max_i |(A^i, y) - b_i + M| = \max_i [(A^i, y) - b_i + M].$$

Let x minimize $\max_i |(A^i, x) - b_i + M|$. Then

$$\begin{aligned} \max_i [(A^i, x) - b_i + M] &\leq \max_i |(A^i, x) - b_i + M| \\ &\leq \max_i |(A^i, y) - b_i + M| = \max_i [(A^i, y) - b_i + M] \leq M. \end{aligned}$$

Thus x satisfies (1).

REMARKS. One might expect a similar theorem to be valid for the least-squares solution of $(A^i, x) = b_i - M$. A counter-example is provided by the system

$$2x \leq -1, \quad -x \leq 1.$$

The least-squares solution of the system

$$2x = -1 - M, \quad -x = 1 - M$$

is $x(M) = (-3M - 1)/5$; but $-x(M) - 1 > 0$ for large M .

THEOREM Q. *The following three problems are equivalent in the sense that each may be reduced to problems of the other types:*

(1) *The Tchebycheff problem of finding x in E_n which minimizes $\max_{1 \leq i \leq m} |(A^i, x) - b_i|$.*

(2) *The linear inequality problem of finding x in E_n such that $(A^i, x) \leq b_i$, $(1 \leq i \leq m)$.*

(3) *The linear programming problem of finding x in E_n which minimizes (L, x) subject to $(A^i, x) \leq b_i$, $(1 \leq i \leq m)$.*

Proof. That (3) subsumes (2) is trivial as L may be taken arbitrarily. That (2) subsumes (3) was proved by von Neumann; see [1]. To reduce (1) to (3), double the number of A^i as in § II. The problem is that of finding the least positive M for which the system $(A^i, x) - M \leq b_i$ $(1 \leq i \leq 2m)$ is consistent, and of finding x in E_n which satisfies it. This is the same as minimizing the linear form M subject to the inequalities $(A^i, x) - M \leq b_i$ $(1 \leq i \leq 2m)$, and $-M \leq 0$. That (1) subsumes (2) is the content of Theorem P. This concludes the proof.

REMARK. Only small modifications of the algorithm of § III are

required in order to apply it to linear programming. In what follows, let \mathcal{P} denote the polytope $\{x \in E_n : (A^i, x) \geq b_i\}$, and let \mathcal{H} denote the hyperplane $\{x \in E_n : (L, x) = c\}$. If (L, x) attains a minimum for x in \mathcal{P} , then c may be taken *small* enough to ensure that \mathcal{H} and \mathcal{P} are disjoint. (If (L, x) attains a maximum on \mathcal{P} then c is taken *large*.) The point of \mathcal{P} closest to \mathcal{H} gives an extremum for (L, x) . The methods of § III may be applied first to obtain a point x^0 in \mathcal{P} . (Such a point is termed by some a *feasible solution*.) One may select then the largest t for which $x^0 - tL$ still belongs to \mathcal{P} . There will be at least one index i_1 for which $(A^{i_1}, x^1) = b_{i_1}$, where $x^1 = x^0 - tL$. The equation $(A^{i_1}, x) = b_{i_1}$ serves to eliminate one variable from the system, and the above step is repeated in this reduced system. After at most $n-1$ eliminations and descents, a vertex v of \mathcal{P} is ordinarily obtained. In the usual case, $(A^{i_1}, v) - b_{i_1} = \dots = (A^{i_n}, v) - b_{i_n}$. A direction for further descent must have the properties $(A^{i_j}, z) \geq 0$ ($1 \leq j \leq n$) and $(L, z) \leq 0$. Such a system is solvable by Theorem F or H.

REMARK. The algorithm of [8], stated there for the Tchebycheff problem, is also a method which proceeds from vertex on \mathcal{P} . The first vertex is obtained by a process of building-up of equal maximum residuals as follows. Given a point x such that $F(x) = R^1(x) = \dots = R^k(x)$, one obtains first a vector d of the form $\sum_{i=1}^k z_i A^i$ such that $(A^1, d) = \dots = (A^k, d) = 1$. This involves solving a system of $k-1$ linear equations with $k-1$ unknowns. Along the ray $x - td$, the first k residuals remain equal to each other, and the first point at which a new residual joins the maximum ones is taken as a starting point for a repetition of the process. Assuming now that a vertex has been obtained, a test is applied to determine whether it is a lowest point of \mathcal{P} . This test is in essence the same as one of [6], and involves the signs of $n+1$ determinants of order n . If further descent is necessary, one obtains as a by-product of this test an edge issuing downward from the vertex. From this point on, only vertices are encountered, thus avoiding the above building-up process.

The method given in § III for the building-up of equal maximum residuals is more economical than that of [8]. In fact, neglecting the $n-1$ descents which both methods entail, the latter involves (besides certain matrix multiplications) the solving of $n-1$ square systems of linear equations in $2, 3, \dots, n$ variables, while the former involves roughly the labor of solving a single $n \times n$ linear system.

Starting at a vertex, it appears advisable to descend to the minimum of F along whatever ray is chosen, thus presumably by-passing more vertices than when simply moving to an adjacent vertex. This of course entails repeated use of the building-up process. The following alternatives

are offered for determining a direction for descent at a vertex.

(1) Use the method of [8].

(2) In the Tchebycheff problem, use the methods of § IV or [6] to obtain a solution of the $n \times (n+1)$ subsystem determined by those planes which enter the vertex. Move toward this point.

(3) In the inequality problem, use Theorem F to obtain a solution of the $n \times (n+1)$ subproblem. Move toward this point.

REFERENCES

1. George B. Dantzig, *A proof of the equivalence of the programming problem and the game problem*, Activity Analysis of Production and Allocation, ed. by Koopmans, New York, (1951), 330-335.
2. Ky Fan, *On systems of linear inequalities*, Linear Inequalities and Related Systems, ed. by Kuhn and Tucker, Princeton, (1956), 99-156.
3. J. J.-B. Fourier, *Solution d'une question particulière du calcul des inégalités, second extrait*, Histoire de l'Académie des Sciences 1824, p. 48. Also available in CEUVRES DE FOURIER, II, Paris 1890, 325-328.
4. Allen A. Goldstein, James B. Herreshoff and Norman Levine, *On the "best" and "least Q-th" approximation of an overdetermined system of linear equations*, J. Assoc. Comput. Mach. **4** (1957), 341-347.
5. T. S. Motzkin and I. J. Schoenberg, *The relaxation method for linear inequalities*, Canad. J. Math. **6** (1954), 393-404.
6. Ch. J. de la Vallée Poussin, *Sur la méthode de l'approximation minimum*, Ann. Soc. Sci. Bruxelles, Seconde Partie, Mémoires **35** (1911), 1-16.
7. *Problem E-1271*, Amer. Math. Monthly **65** (1958), 122.
8. S. I. Zuhovickii, *An algorithm for the solution of the chebysev approximation problem in the case of a finite system of incompatible linear equations*, Dokl. Akad. Nauk SSSR (N. S.) **79** (1951), 561-564, (Russian). (Translation with commentary to appear in J. Soc. Indust. Appl. Math.)

CONVAIR ASTRONAUTICS
SAN DIEGO, CALIFORNIA

