# ON THE RADIUS OF UNIVALENCE OF THE FUNCTION $\exp z^2 \int_0^z \exp(-t^2)dt$

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**1. Introduction.** We shall determine the radius of univalence $\rho_u$ of the function

$$E(z) = e^{z^2}\int_0^z e^{-t^2}dt \ . \tag{1.1}$$

We shall write $E(z) = w = u(x, y) + iv(x, y)$. On the imaginary axis we have $u = 0$ and $v$, regarded as a function of $y$, has a single maximum at the solution $y = \rho$ of

$$2yv(0, y) = 1 \ .$$

The value of $\rho$ to eight decimal places has been determined by Lash Miller and Gordon [1] and is

$$\rho = 0.92413887 \ . \tag{1.2}$$

It is evident that $\rho_u \leqq \rho$. We shall prove the following theorem.

THEOREM. *The number $\rho$ is the radius of univalence of $E(z)$.*

Recently, the radius of univalence of the error function

$$erf(z) = \int_0^z e^{-t^2}dt$$

was determined [2]. It is interesting to note that when proceeding from $erf(z)$ to $E(z)$ we meet an entirely different situation. In the case of $erf(z)$, points $z_1, z_2$ closest to the origin and such that $erf(z_1) = erf(z_2)$ are conjugate complex and lie far apart from each other. In the case of $E(z)$ points of that nature can be found in an arbitrarily small neigborhood of the point $z = i\rho$.

The actual situation is made clear by the diagram and tables given below. In Fig. 1 we show the curves $R = |E| =$ constant and $\gamma = arg\, E =$ constant in the square $0 \leqq x \leqq 1.5$, $0 \leqq y \leqq 1.5$ of the $z$-plane. The table shows the values of $E$ for $z$ on the curve $C$ (defined below). The values given were obtained by summing an adequate number of terms of the power series on the Datatron 205 at the California Institute of Technology ; some were checked by comparison with the tables of Karpov [4, 5] from which values of $E(z)$ can be obtained.

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**2. Idea of proof.** Since

$$E(z)=\sum_{n=0}^{\infty}\frac{2^n}{1.3.5\cdots(2n+1)}z^{2n+1}\,,\qquad |z|<\infty\,,\tag{2.1}$$

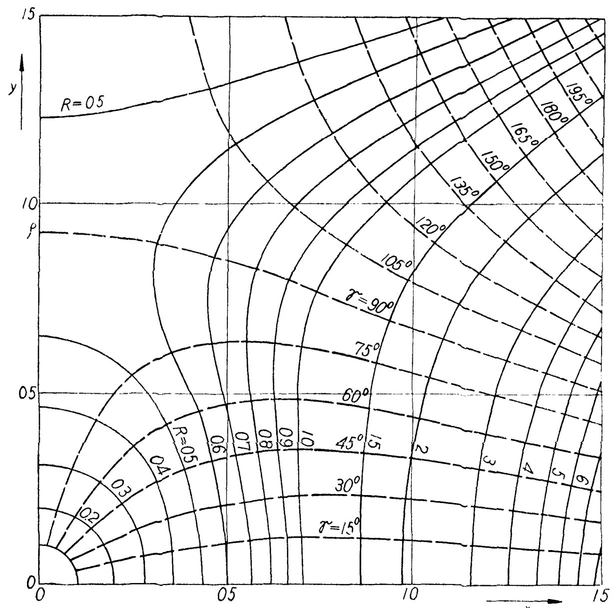


Fig. 1. Curves $R=|E|=$ const. and $\gamma=arg\,E=$ const. in the $z$-plane.

| $x$ | $E(x)$ |
|---|---|
| 0 | 0 |
| 0.1 | 0.1007 |
| 0.2 | 0.2054 |
| 0.3 | 0.3187 |
| 0.4 | 0.4455 |
| 0.5 | 0.5923 |
| 0.6 | 0.7671 |
| 0.7 | 0.9805 |
| 0.8 | 1.2473 |
| 0.9 | 1.5876 |

| $\phi$ | $E(\rho e^{i\phi})$ |
|---|---|
| 0° | 1.6837 |
| 10° | 1.4957+0.6121i |
| 20° | 1.0573+0.9759i |
| 30° | 0.6079+1.0473i |
| 40° | 0.2919+0.9463i |
| 50° | 0.1189+0.8024i |
| 60° | 0.0401+0.6817i |
| 70° | 0.0099+0.6003i |
| 80° | 0.0011+0.5553i |
| 90° | 0.5410i |

| $y$ | $E(iy)$ |
|---|---|
| 0 | 0 |
| 0.1 | 0.0993i |
| 0.2 | 0.1948i |
| 0.3 | 0.2826i |
| 0.4 | 0.3599i |
| 0.5 | 0.4244i |
| 0.6 | 0.4748i |
| 0.7 | 0.5105i |
| 0.8 | 0.5321i |
| 0.9 | 0.5407i |

we have $E(\bar{z})=\overline{E(z)}$ and $E(-z)=-E(z)$ and may restrict our consideration to the first quadrant $x\geqq 0, y\geqq 0$ in the $z$-plane.

In the subsequent section we shall prove the following lemma.

LEMMA.

$$E(z_1)\neq E(z_2)\tag{2.2}$$

*for any two points on the boundary $C$ of the open sector $S$ of the circular disk $|z|<\rho$ in the first quadrant.*

From this it follows, since $E(z)$ is entire and thus regular in $S \cup C$ that $E(z)$ maps $S$ conformally and one-to-one onto the interior of the simple closed curve $C^*$ corresponding to $C$ in the $w$-plane [3, p. 121]. This establishes our theorem.

3. **Proof of the lemma.** Let $z = re^{i\phi}$. The curve $C$ consists of

the segment $S_1$: $y = 0\,, \quad 0 < x < \rho\,,$

the circular arc $K$: $|z| = \rho\,, \quad 0 < \phi < \pi/2\,,$

the segment $S_2$: $x = 0\,, \quad 0 < y < \rho\,.$

and the three common end points of these three arcs.

(A) On $S_1$, $E(z)$ is real and increases steadily with $x$.

(B) On $S_2$, $E(z)$ is imaginary, and $v$ increases steadily with $y$.

(C) $v \neq 0$ on $K$.

(D) On $K$, $|E(z)|$ decreases steadily with increasing $\phi$.

(A) is obvious from (2.1), and (B) follows from the definition of $\rho$.

*Proof of* (C). Integrating along segments parallel to the coordinate axes we have

$$\begin{aligned} v(x, y) = e^{-y^2}\Big[\cos 2xy\int_0^y e^{\tau^2}\cos 2x\tau d\tau \\ + \sin 2xy\Big\{e^{x^2}\int_0^x e^{-t^2}dt + \int_0^y e^{\tau^2}\sin 2x\tau d\tau\Big\}\Big]\,. \end{aligned}$$

In $\{x > 0, y > 0\} \cap \{|z| \leqq \rho\}$ we have $\cos 2xy > 0$, $\sin 2xy > 0$. Therefore $v > 0$ on $K$.

*Proof of* (D). Integrating along a radius $\phi = $ constant from 0 to $\rho$ we have

$$E(z) = e^{i\phi}\int_0^\rho e^{h(r,\phi)}dr$$

where

$$\begin{gathered} h(r, \phi) = a(r, \phi) + ib(r, \phi)\,, \\ a(r, \phi) = (\rho^2 - r^2)\cos 2\phi,\ b(r, \phi) = (\rho^2 - r^2)\sin 2\phi\,. \end{gathered}$$

Therefore

$$|E|^2 = \int_0^\rho e^h dr\int_0^\rho e^{\bar h}dr\,.$$

Differentiating with respect to $\phi$ and setting

$$\begin{gathered} h^* = a^* + ib^*,\ a^* = a(r^*, \phi),\ b^* = b(r^*, \phi), \\ f = \cos(b^* - b) - i\sin(b^* - b) \end{gathered}$$

we obtain

$$(|E|^2)_\phi = \int_0^\rho e^h h_\phi dr \int_0^\rho e^{\bar{h}^*} dr^* + \int_0^\rho e^{h^*} dr^* \int_0^\rho e^{\bar{h}} \bar{h}_\phi dr$$
$$= \int_0^\rho \int_0^\rho e^{a+a^*} \{f h_\phi + \overline{f h_\phi}\} dr dr^* .$$

Now

$$a_\phi = -2(\rho^2 - r^2) \sin 2\phi, \; b_\phi = 2(\rho^2 - r^2) \cos 2\phi$$

and therefore

$$\begin{aligned} f h_\phi + \overline{f h_\phi} &= 2\Re f h_\phi = 2[\cos(b^* - b) a_\phi + \sin(b^* - b) b_\phi] \\ &= -4(\rho^2 - r^2) \sin(\alpha(\phi)) \end{aligned}$$

where

$$\alpha(\phi) = 2\phi + b - b^* = (r^{*2} - r^2) \sin 2\phi + 2\phi .$$

This yields

$$(3.1) \qquad (|E|^2)_\phi = -4 \int_0^\rho \int_0^\rho e^{a+a^*} (\rho^2 - r^2) \sin(\alpha(\phi)) dr dr^* .$$

Since from (1.2) we have $|r^{*2} - r^2| < 1$, we obtain

$$\alpha'(\phi) = 2 + 2(r^{*2} - r^2) \cos 2\phi > 0 .$$

Hence $\alpha(\phi), 0 \leqq \phi \leqq \pi/2$, has its maximum at $\phi = \pi/2$. Therefore $0 \leqq \alpha(\phi) < \pi$ when $0 \leqq \phi < \pi/2$ and $\sin(\alpha(\phi)) > 0$ when $0 < \phi < \pi/2$. This means that the integrand in (3.1) is positive in the region $0 \leqq r \leqq \rho$, $0 \leqq r^* \leqq \rho$ for all $\phi$ in the interval $0 < \phi < \pi/2$. Thus $(|E|^2)_\phi < 0$ when $0 < \phi < \pi/2$. This proves (D).

We note that (D) remains true if $K$ is replaced by quadrants of circles of radii somewhat larger than $\rho$; this, however, is of no interest here.

For $z_1 \in K$, $z_2 \in S_2$, or $z_1 \in K$, $z_2 \in K$, equation (2.2) holds, as follows from (D). For $z_1 \in K$, $z_2 \in S_1$ the same is true because of (C). In the other cases, $z_1 \in S_1$, $z_2 \in S_1$, etc., the validity of (2.2) is obvious. This proves the lemma.

## References

1. W. Lash Miller and A. R. Gordon, *Numerical evaluation of infinite series and integrals which arise in certain problems of linear heat flow, electrochemical diffusion etc.*, J. Phys. Chem. **35** (1931), 2785-2884. We are indebted to Dr. Walter Gautschi for this reference; Dr. Gautschi has recomputed the value of $\rho$ and communicated the following 10 D value

$$\rho = 0.92413\ 88730.$$

2. Erwin Kreyszig and John Todd, *The radius of univalence of erf z*, Bull. Amer. Math. Soc. **64** (1958), 363-364, and Numerische Mathematik **1** (1959) to appear.

3. J. E. Littlewood, *Lectures on the Theory of Functions*, Oxford, 1944.

4. K. A. Karpov, *Taklicy funkcii* $w(z) = e^{-z^2}\int_0^z e^{x^2}\,dx$ *v kompleksnoi oklasti*, Moscow, 1956.

5. K. A. Karpov *Taklicy funkcii* $F(z) = \int_0^z e^{x^2}dx$ *v kompleksnoi oklasti*, Moscow, 1958.

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