

THE GEOMETRIC INTERPRETATION OF A SPECIAL CONNECTION

I. M. SINGER

1. Introduction. An esthetic problem remaining in the foundations of Riemannian geometry is this. To each Riemannian metric on a manifold one associates a connection on its bundle of bases. This connection is uniquely characterized by its having zero torsion and its parallel translation preserving the metric. By very definition at each tangent space of the bundle the connection assigns a complement to the vertical. Yet the various existence proofs of the Riemannian connection are computational and do not exhibit this complementary space (the horizontal space) directly. The problem is how can one do so directly from the metric, given (say) by the reduction of the bundle to the bundle of frames.

Under rather special circumstances, we exhibit a geometric and coordinate free construction of a connection in §2. This includes the class of complex analytic bundles and gives a geometric construction of a connection of type $(1, 0)$ with curvature of type $(1, 1)$, useful in algebraic geometry (§4). See [2] and [5]. For the case of complex vector bundles Nakano computes this connection in terms of coordinates [6]. For Kähler manifolds it turns out (§2) that this connection is the Riemannian connection restricted to the unitary subbundle. Hence we obtain a solution to the problem posed in the preceding paragraph for Kähler manifolds.

We assume the reader is familiar with the bundle approach to connections, say as in [1]. We use the notation of that paper.

2. The connection. Let $\mathfrak{B} = (B, M, G, G, \pi, \phi)$ be a principal bundle. Throughout this discussion we shall suppose that G is the complexification of a compact¹ group K (such is the case for example if G is complex semisimple) and that B is an almost complex manifold. If $b \in B$, let J_b denote the linear transformation on B_b (the tangent space of B at b) giving the almost complex structure on B , so that $J_b^2 = -I$. We assume the following two conditions on this almost complex structure:

(i) the structure is invariant under right translation by G , that is $R_g J_b = J_{b_g} R_g$

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¹ The results of this paper are valid whenever the Lie algebra \mathfrak{g} of G is the complexification of the Lie algebra \mathfrak{k} of K . The compactness of K is assumed only to guarantee that the reduction to K is possible.

(ii) J_b leaves the vertical subspace $V_b = \pi^{-1}(0)$ of B_b invariant and therefore J_b restricted to V_b induces an almost complex structure on the fibre containing b . This almost complex structure should agree with the two sided invariant complex structure of G , when the fibre is identified with G .

One knows [7] that the structural group of \mathfrak{B} can be reduced to the maximal compact subgroup K of G . This fact implies the existence of a submanifold B_k of B passing through a fixed point $b \in B$, where B_k is the underlying space of a subbundle $\mathfrak{B}_k = (B_k, M, K, K, \pi, \Phi')$ of \mathfrak{B} . The main observation is this

THEOREM. *For every reduction of \mathfrak{B} to a subbundle \mathfrak{B}_k , there exists a unique connection H on \mathfrak{B} satisfying the two conditions:*

(α) *If $c \in B$, $J_c(H_c) = H_c$, that is, the connection H is invariant under the almost complex structure.*

(β) *If $c \in B_k$, $H_c \subset (B_k)_c$, the subspace of B_c tangent to B_k . This condition is equivalent to the statement that H restricted to B_k is a connection on the bundle \mathfrak{B}_k .*

Proof. Since G is the complexification of K , its Lie algebra \mathfrak{g} is the complexification of the subalgebra \mathfrak{k} , the Lie algebra of K . Let c be a point of B_k , and identify the fibre containing c with G , so that c becomes the identity of G . Then V_c is identified with \mathfrak{g} and under this identification $V_c \cap (B_k)_c$ becomes \mathfrak{k} . Since $\mathfrak{g} = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{k}$, and since (ii) holds, we have $V_c = V_c \cap (B_k)_c \oplus J_c(V_c \cap (B_k)_c)$. To simplify the notation, let $W = (B_k)_c$ and $W' = J_c((B_k)_c)$, so that $V_c = V_c \cap W \oplus V_c \cap W'$ because $J_c(V_c) = V_c$.

We define the horizontal space of our connection, H_c , to be $W \cap W'$. We must show that H_c is a linear complement to V_c in B_c . $V_c \cap H_c = V_c \cap W \cap W' = (V_c \cap W) \cap (V_c \cap W') = 0$ because $V_c = V_c \cap W \oplus V_c \cap W'$. This shows that H_c is a complement to V_c . To show that it is a full complement, we need only check that the dimension of H_c equals the dimension of M . Let $\dim(M) = m$. Let $\dim(K) = n$, so that the real $\dim(G) = 2n$, $\dim(B) = 2n + m$, and $\dim W = \dim W' = \dim B_k = n + m$. Since $\dim(W) + \dim(W') = \dim(W \cap W') + \dim(W \cup W')$, to prove $\dim(H_c) = m$ it suffices to show that $\dim(W \cup W') = 2n + m$. In fact, we show that $W \cup W' = B_c$. If $t \in B_c$, then $\exists w \in W \ni \pi(w) = \pi(t)$ for B_k is a bundle over M . Since $\pi(w - t) = 0$, $w - t = v \in V_c$ and therefore $w = t + v \in W \cup V_c$; that is

$$B_c \subset W \cup V_c \subset W \cup (V_c \cap W \oplus V_c \cap W') \subset W \cup W'.$$

As c varies over B_k , the choice H_c gives a connection on \mathfrak{B}_k . For $H_c \subset W$ and we have already shown that H_c is a full complement to V_c

in B_c and hence to $V_c \cap W$ (the vertical subspace of \mathfrak{B}_k) in W . Moreover H_c is invariant under right translation by $k \in K$ by (i), that is

$$\begin{aligned} R_k(H_c) &= R_k(W \cap W') \\ &= R_k(W) \cap R_k(J_c(W)) = R_k(W) \cap J_{ck}(R_k(W)) = H_{ck} . \end{aligned}$$

Finally, because the J_c are chosen ‘‘smoothly’’ as c varies over B , one can show the H_c give a ‘‘smooth’’ choice.

It is easy to see that this connection on \mathfrak{B}_k can be extended uniquely to a connection on \mathfrak{B} by the use of right translation. Since $J_c(H_c) \subset H_c$ if $c \in \mathfrak{B}_k$ and since (i) holds, condition (α) is satisfied also on \mathfrak{B} . (β) is satisfied because of the way the connection was constructed.

In order to prove the uniqueness of the connection, suppose H' is a connection on \mathfrak{B} satisfying (α) and (β) . If $c \in B_k$, (β) implies that $H'_c \subset W$, and therefore $J(H'_c) \subset J(W) = W$. But (α) implies that $J(H'_c) = H'_c$ so that $H'_c \subset W \cap W' = H_c$. Since $\dim(H'_c) = m = \dim(H_c)$, $H'_c = H_c$. Thus the connection H' restricted to \mathfrak{B}_k equals H restricted to B_k . Since the extension to \mathfrak{B} is unique, $H' = H$.

3. An example. Let M be a complex analytic manifold of complex dimension r and let \mathfrak{B} be the bundle of complex bases over M , so that $B = (p; e_1, \dots, e_n)$ where $p \in M$ and e_1, \dots, e_n is a complex basis of M_p . In this case, \mathfrak{B} is a principal bundle with structural group G the complex linear group. G is the complexification of the unitary group K . B is a complex analytic manifold. A complex coordinate neighborhood $U(\varepsilon)$ of the point $(p; e_1, \dots, e_r)$ is obtained by first choosing complex coordinates z_1, \dots, z_r about p ; then

$$e_i = \sum_{j=1}^r a_{ij} \frac{\partial}{\partial z_j} , \det(a_{ij}) \neq 0 ,$$

and $(q, f_1, \dots, f_r) \in U(\varepsilon)$ if $|z_i(q)| < \varepsilon$ and $|b_{ij} - a_{ij}| < \varepsilon$ where

$$f_i = \sum_{j=1}^r b_{ij} \frac{\partial}{\partial z_j} .$$

The coordinates of points in $U(\varepsilon)$ are $\tilde{z}_1, \dots, \tilde{z}_r, \tilde{z}_{ij}$ where $z_i(q, f_1, \dots, f_i) = z_i(q)$ and $z_{ij}(q, f_1, \dots, f_r) = b_{ij}$. It is easy to check that the operation of G on B preserves the complex structure of B so that (i) is true. Moreover the z_{ij} 's restricted to a fiber give the complex structure of G , i.e., (ii) also holds. The reduction of G to the unitary group K is equivalent to making M into a Hermitian space [7]; in this case $B_k = (p, e_i, \dots, e_r)$ where e_i, \dots, e_r are complex orthonormal relative to the hermitian inner product on M_p .

The example just described is the one that arises naturally in the

study of complex manifolds. The reduction of the bundle of complex bases to the unitary group gives rise to a connection usually described in terms of a coordinate system. For example, see [6]. That this connection is the same as the one in §2 can be seen by checking that (α) and (β) hold and using the uniqueness result.

If M is a hermitian manifold, the real part of the hermitian inner product on M_p makes M into a Riemannian manifold. We have the following bundles over M : \mathfrak{B}^1 , the principal bundle of all real bases with G^1 the real linear group in $2r$ dimensions as structural group; B^2 , the principal subbundle of orthogonal bases with G^2 the real orthogonal group as structural group; B^3 , the principal bundle of complex bases with G^3 the complex linear group in r dimensions as structural group; and B^4 the principal bundle of complex orthogonal bases with G^4 , the unitary group as structural group. The mapping

$$(f_1, \dots, f_r) \rightarrow (f_1, \dots, f_r, if_1, \dots, if_r)$$

maps the set of complex bases of a vector space into the set of real bases. Consequently, we can think of B^3 as a subbundle of B^1 and $B^3 \cap B^2 = B^4$.

Let ω_j be the *complex* valued 1-form on B^3 which assigns to $t \in B^3_{(m, f_1, \dots, f_r)}$ the j th component of $\pi(t)$ relative to the complex basis $f_1 \cdots f_r$ at M_m . Then $\omega_j(J_b(t)) = i\omega_j(t)$ so that ω_j is of type $(1, 0)$. In fact, since B^3 is complex analytic, locally $\omega_j = \Sigma = U_{jk} dz_k$ with U_{jk} holomorphic functions so that $d\omega_j$ is of type $(2, 0)$. Let ω_{jk} be the complex valued 2-forms which our connection gives rise to. Condition (α) insures that ω_{jk} is also of type $(1, 0)$ so that the torsion form $d\omega_j - \Sigma \omega_k \wedge \omega_{kj}$ is of type $(2, 0)$. Condition (β) tells us that on \mathfrak{B}^4 $\omega_{kj} = -\bar{\omega}_{jk}$, for our connection restricted to \mathfrak{B}^4 is a connection on \mathfrak{B}^4 . Consequently [3], the vanishing of the torsion of our connection is equivalent to the Kähler condition on the Hermitian metric. The extension of our connection to \mathfrak{B}^3 will then also have zero torsion and must therefore be the Riemannian connection. We have proved the following.

COROLLARY. *If M is a Kähler manifold, then the Riemannian connection can be restricted to the bundle of complex bases and there coincides with the connection constructed in §2.*

4. The curvature. We return to the general case. Let ω denote the one form with values in \mathfrak{g} associated with the connection H on \mathfrak{B} and let A be the curvature form.

THEOREM. *If B is a complex manifold, then for any tangent vectors s and t at $b \in B$, $A(s, t) = A(J_b(s), J_b(t))$.*

Proof. Let \mathfrak{S}_b denote the projection of B_b on the horizontal space H_b . Then we know [1] that $A(s, t) = d\omega(\mathfrak{S}_b s, \mathfrak{S}_b t)$. Let \tilde{s} and \tilde{t} denote horizontal vector fields on B for which $\tilde{s}(b) = \mathfrak{S}_b(s)$ and $\tilde{t}(b) = \mathfrak{S}_b(t)$. Since \tilde{s} and \tilde{t} are horizontal, so are $J(\tilde{s})$ and $J(\tilde{t})$. Since B is a complex manifold $[J\tilde{s}, J\tilde{t}] - [\tilde{s}, \tilde{t}] = J\{[J\tilde{s}, \tilde{t}] + [\tilde{s}, J\tilde{t}]\}$. See [4] for example. Finally, since J_b leaves vertical and horizontal spaces of B_b invariant, $J_b \mathfrak{S}_b = \mathfrak{S}_b J_b$. We can now compute:

$$\begin{aligned} A(J_b(s), J_b(t)) - A(s, t) &= d\omega(\mathfrak{S}_b J_b(s), \mathfrak{S}_b J_b(t)) - d\omega(\mathfrak{S}_b(s), \mathfrak{S}_b(t)) \\ &= d\omega(J_b \mathfrak{S}_b(s), J_b \mathfrak{S}_b(t)) - d\omega(\mathfrak{S}_b(s), \mathfrak{S}_b(t)) \\ &= d\omega(J\tilde{s}(b), J\tilde{t}(b)) - d\omega(\tilde{s}(b), \tilde{t}(b)) \\ &= \frac{1}{2} \left\{ -\omega([J\tilde{s}, J\tilde{t}]_b) + J\tilde{s}(\omega(J\tilde{t}))|_b - J\tilde{t}(\omega(J\tilde{s}))|_b \right. \\ &\quad \left. + \omega([\tilde{s}, \tilde{t}]_b) - \tilde{s}(\omega(\tilde{t}))|_b + \tilde{t}(\omega(\tilde{s}))|_b \right\}. \end{aligned}$$

Since ω annihilates horizontal vectors, this last expression equals

$$-\frac{1}{2} \omega([J\tilde{s}, J\tilde{t}]_b - [\tilde{s}, \tilde{t}]_b) = -\frac{1}{2} \omega(J\{[J\tilde{s}, \tilde{t}]_b + [\tilde{s}, J\tilde{t}]_b\}).$$

However, for any tangent vector u ,

$$\omega(Ju) = \omega(\mathfrak{S}_b Ju + (I - \mathfrak{S}_b)Ju) = \omega((I - \mathfrak{S}_b)Ju) = \omega(J(I - \mathfrak{S}_b)u).$$

But condition (α) implies that this last term equals $i\omega((I - \mathfrak{S}_b)u) = i\omega(u)$. We conclude, therefore, that

$$(A) \quad A(J_b(s), J_b(t)) - A(s, t) = -\frac{1}{2} i \{ \omega([J\tilde{s}, \tilde{t}]_b + [\tilde{s}, J\tilde{t}]_b) \}.$$

Suppose that $b \in B_k$. Since $\tilde{s}, \tilde{t}, J\tilde{s}$, and $J\tilde{t}$ are horizontal, they are tangent to B_k at points of B_k . Therefore, $[J\tilde{s}, \tilde{t}]_b$ and $[\tilde{s}, J\tilde{t}]_b$ are also tangent to B_k . As a consequence of condition (β) the range of ω restricted to $(B_k)_b$ is \mathfrak{k} . Hence the right side of equation (A) lies in $i\mathfrak{k}$. On the other hand, since $b \in B_k$, $A(s, t) = A(\mathfrak{S}_b(s), \mathfrak{S}_b(t)) \in \mathfrak{k}$; similarly, $A(J_b(s), J_b(t)) \in \mathfrak{k}$. Therefore the left side of equation (A) lies in \mathfrak{k} . But $\mathfrak{k} \cap i\mathfrak{k} = 0$ so that both sides of (A) are zero.

Thus we have proved the desired formula for $b \in B_k$. Since A is equi-invariant under right translation by $g \in G$ and since any $c \in B$ is of the form bg for some $g \in G$, the formula holds throughout B .

Since $\omega(J_b u) = i\omega(u)$, w is a connection of type $(1, 0)$. Also

$$\begin{aligned} A(J_b s, t) + A(s, J_b t) &= A(J_b s, t) + A(J_b s, J_b^2 t) \\ &= A(J_b s, t) - A(J_b s, t) = 0 \end{aligned}$$

so that the curvature form is of type $(1, 1)$.

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