INTRINSIC OPERATORS IN THREE-SPACE

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1. Introduction. In Euclidean three-space there are three important classical intrinsic operators, namely the intrinsic curl, the intrinsic divergence, and the intrinsic (or generalized) Laplacian. Usually they are given in terms of differential operators, but the occasion arises sometimes when they cannot be so defined. In particular if u is the Newtonian potential due to a continuous distribution, then in general u is only a function in class C^1, and consequently the usual Laplacian of u, the usual curl of grad u, and the usual divergence of grad u cannot be defined. Nevertheless, as it is easy to show, the intrinsic curl of grad u is equal to zero, the intrinsic (or generalized) Laplacian of u equals the intrinsic divergence of grad u, and furthermore Poisson's equation holds. The question arises whether the converse is true. The answer to questions of this nature is the subject matter of this paper. In particular we shall establish the following result (with the precise definitions given in the next section):

THEOREM 1. *Let D be a domain in Euclidean three-space and let v be a continuous vector field defined in D. Then a necessary and sufficient condition that v be locally in D the gradient of a potential of a distribution with continuous density is that the intrinsic curl of v be zero in D and the intrinsic divergence of v be continuous in D.*

2. Definitions and notation. We shall use the following vectorial notation: $x = (x_1, x_2, x_3)$, $\alpha x + \beta y = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)$, $(x, y) =$ the usual scalar product, $x \times y =$ the usual cross product, and $|x| = (x, x)^{1/2}$.

Let $v(x) = [v_1(x), v_2(x), v_3(x)]$ be a continuous vector field defined in the neighborhood of the point x_0. Then we define the upper intrinsic curl of v at x_0 to be the vector, curl ${}^*v(x_0) = [w_1^*(x_0), w_2^*(x_0), w_3^*(x_0)]$ where $w_j^*(x_0) = \limsup_{r\to 0}(\pi r^2)^{-1}\int_{C_j(x_0,r)}(v, dx)$, $j = 1, 2, 3$, with $C_j(x_0, r)$ the circumference of the circle of radius r and center x_0 in the plane through x_0 normal to the x_j-axis where $C_j(x_0, r)$ is oriented in the counterclockwise direction when seen from the side in which the x_j-axis points. In a similar manner using lim inf, we define the lower intrinsic curl of v at x_0, $\operatorname{curl}_* v(x_0)$. If $\operatorname{curl}^* v(x_0) = \operatorname{curl}_* v(x_0)$ is finite, we call this

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common value the intrinsic curl of v at x_0 and designate it by $\operatorname{curl} v(x_0)$. This definition is essentially the intrinsic definition of the curl as given in [4, p. 71].

Next, we define the intrinsic divergence. Let $v(x)$ be a continuous vector field defined in a neighborhood of the point x_0. Then with $S(x_0, r)$ the spherical surface with center x_0 and radius r, we define the upper intrinsic divergence of v at x_0 as follows

$$\operatorname{div}^* v(x_0) = \limsup_{r\to 0} 3(4\pi r^3)^{-1}\int_{S(x_0,r)}(v, n)dS$$

where n is the outward pointing unit normal on $S(x_0, r)$ and dS is the natural surface area element on $S(x_0, r)$. Similarly we define the lower intrinsic divergence, $\operatorname{div}_* v(x_0)$, using lim inf. If $\operatorname{div}_* v(x_0) = \operatorname{div}^* v(x_0)$ is finite, we call this common value the intrinsic divergence of v at x_0 and designate it by $\operatorname{div} v(x_0)$ (see [9]).

If $u(x)$ is a continuous function defined in a neighborhood of the point x_0, then the upper intrinsic (or generalized) Laplacian of u at the point x_0, $\operatorname{Lap} u(x_0)$, is usually defined as

$$\operatorname{Lap}^* u(x_0) = \limsup_{r\to 0}\left[(4\pi r^2)^{-1}\int_{S(x_0,r)} udS - u(x_0)\right]6r^{-2} .$$

Similarly we define $\operatorname{Lap}_* u(x_0)$ using lin inf. If $\operatorname{Lap}^* u(x_0) = \operatorname{Lap}_* u(x_0)$ is finite, we call this common value the intrinsic (or generalized) Laplacian of u at x_0 and designate it by $\operatorname{Lap} u(x_0)$.

It is clear that if $v(x)$ is in class C^1 and $u(x)$ is in class C^2, then $\operatorname{curl} v(x)$, $\operatorname{div} v(x)$, and $\operatorname{Lap} u(x)$ exist and equal the usual curl, divergence, and Laplacian respectively, defined in terms of the partial derivatives.

If $f(x)$ is a function defined in a neighborhood of the point x_0 and if $f(x)$ is in L^1 in $S_1(x_0, r)$ for some $r > 0$ where $S_1(x_0, r)$ is the open solid sphere with center x_0 and radius r, we shall designate by $A^*f(x_0)$ the following upper limit:

$$A^*f(x_0) = \limsup_{r\to 0}(4\pi r^3)^{-1}3\int_{S_1(x_0,r)} f(x)dx .$$

Similarly, we shall designate by $A_*f(x_0)$ the corresponding value obtained by using lim inf. As is well-known, for almost all x in $S_1(x_0, r)$, $A_*f(x) = A^*f(x)$.

Given $v(x)$ a continuous vector field defined in a domain D, we shall say that $v(x)$ is locally in D the gradient of a potential of a distribution with bounded density if for each point x_0 in D there exists an $S_1(x_0, r)$ contained in D and two functions $f(x)$ and $h(x)$ defined in $S_1(x_0, r)$ with $f(x)$ bounded in $S_1(x_0, r)$ and $h(x)$ harmonic in $S_1(x_0, r)$ such that

$$(1) \quad u(x) = -(4\pi)^{-1}\int_{S_1(x_0,r)} f(y)|x - y|^{-1}dy + h(x) \text{ for } x \text{ in } S_1(x_0, r) ,$$

and $v(x) = \operatorname{grad} u(x)$ for x in $S_1(x_0, r)$. It is understood that $f(x)$ is bounded in $S_1(x_0, r)$ but need not be bounded in D.

It is well-known that if $u(x)$ is defined by (1), then $u(x)$ is in class C^1 in $S_1(x_0, t)$, and furthermore $\operatorname{Lap} u(x) = f(x)$ (see [7]) at every point where $A^*f(x) = A_*f(x)$. We shall show that $\operatorname{curl} \operatorname{grad} u(x) = 0$, $\operatorname{div}^* \operatorname{grad} u(x) = A^*f(x)$, and $\operatorname{div}_* \operatorname{grad} u(x) = A_*f(x)$.

$\bar{E}$ will designate the closure of the set E.

3. **Statement of main results.** We shall prove the theorems stated below.

THEOREM 2. *Let D be a bounded domain in Euclidean three-space, and let $v(x)$ be a continuous vector field defined in D. Then a necessary and sufficient condition that $v(x)$ be locally in D the gradient of a potential of a distribution with bounded density is that*

(i) $\operatorname{curl}_* v(x)$ *and* $\operatorname{curl}^* v(x)$ *be finite-valued in D.*

(ii) $\operatorname{curl}_* v(x) = \operatorname{curl}^* v(x) = 0$ *almost everywhere in D.*

(iii) $\operatorname{div}_* v(x)$ *and* $\operatorname{div}^* v(x)$ *be locally bounded in D.*

In the next theorem, the definitions of regular curves and regular surfaces are those given in [4, Chapter 4].

THEOREM 3. *Let D be a bounded domain in Euclidean three-space, and let $v(x)$ be a continuous vector field defined in D. Suppose that*

(i) $\operatorname{curl}^* v(x)$ *and* $\operatorname{curl}_* v(x)$ *are finite valued in D.*

(ii) *there exists a continuous vector-field $w(x)$ such that* $w(x) = \operatorname{curl}_* v(x) = \operatorname{curl}^* v(x)$ *almost everywhere in D.*

Then $\operatorname{curl} v(x)$ *exists everywhere in D and is equal to $w(x)$. Furthermore Stokes' theorem with respect to v and* $\operatorname{curl} v$ *holds for every open two-sided regular surface contained in the interior of D, that is*

$$\int_C (v, dx) = \int_S (\operatorname{curl} v, n) dS \tag{2}$$

where C is the regular curve which is the boundary of S oriented in the counter-clockwise sense when seen from the side of S towards which n points.

The sufficiency conditions of Theorems 1 and 2 follow as corollaries of Theorem 5 to be stated in § 5. As a further corollary of Theorem 5, we obtain the following extension of a theorem of Beckenbach's [**1**, Theorem 1] (i.e. we remove the uniformity conditions stated in his theorem).

THEOREM 4. *Let $v(x)$ be a continuous vector field defined in a bounded domain D of Euclidean three-space. Then a sufficient condition that $v(x)$ be a Newtonian vector field in D is that*

(i) $\operatorname{curl} v(x) = 0$ *in* D

(ii) $\operatorname{div} v(x) = 0$ *in* D.

The curl of a vector field which is only assumed continuous in a domain can be defined in a different manner than that given above, namely by using spherical surfaces and the cross product. We shall consider this definition and the analogues of Theorem 1, 2, 3, and 4 in the concluding section of this paper.

4. Proof of Theorem 3. Since we need the result of Theorem 3 in order to establish Theorems 1, 2, and 4, we shall prove the former theorem first. In order to do this, we need the following lemma:

LEMMA 1. *Let $v(x) = [v_1(x), v_2(x), v_3(x)]$ be a continuous vector field defined and continuous in a neighborhood of the point x_0, and let $\lambda(x)$ be a non-negative function in class C^1 in a neighborhood of the point x_0. Let $v'(x) = \lambda(x)v(x)$, that is $v'_j(x) = \lambda(x)v_j(x)$, $j = 1, 2, 3$. Then*

(a) $\operatorname{curl}^* v'(x_0) = \lambda(x_0)\operatorname{curl}^* v(x_0) + \operatorname{grad}\lambda(x_0) \times v(x_0)$

(b) $\operatorname{curl}_* v'(x_0) = \lambda(x_0)\operatorname{curl}_* v(x_0) + \operatorname{grad}\lambda(x_0) \times v(x_0)$

where $\lambda(x_0)\operatorname{curl}^* v(x_0) = \lambda(x_0)\operatorname{curl}_* v(x_0) = 0$ in case $\lambda(x_0) = 0$.

To prove the lemma, it is sufficient to prove (a) for (b) will follow on considering $-v(x)$. To prove (a), we have to show with $w^*(x_0) = \operatorname{curl}^* v(x_0)$ that

$$\begin{aligned}&\lambda(x_0)w_k^*(x_0) + v_j(x_0)\lambda_{x_i}(x_0) - v_i(x_0)\lambda_{x_j}(x_0)\\&\quad = \limsup_{r\to 0}(\pi r^2)^{-1}\int_{\sigma_k(x_0, r)} \lambda(x)v_i(x)dx_i + \lambda(x)v_j(x)dx_j\end{aligned}$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$ and $\lambda(x_0)w_k^*(x_0) = 0$ in case $\lambda(x_0) = 0$. But this follows immediately from [9, Lemma 8].

To prove Theorem 3, it is sufficient to establish

$$\int_{\sigma}(v, dx) = \int_{S}(w, n)dS \tag{3}$$

for every open two-sided regular surface S contained in the interior of D. For once (3) is established, it holds in the particular case when S is a disc. Consequently the assumed continuity of w in D and (3) implies that

$$\lim_{r\to 0}(\pi r^2)^{-1}\int_{\sigma_j(x_0, r)}(v, dx) = w_j(x_0) \qquad j = 1, 2, 3\,.$$

Therefore $\operatorname{curl} v$ exists everywhere in D and is equal to w, and consequently (3) is equivalent (2).

We shall now proceed to establish (3). In order to do this, we first notice that with no loss of generality (since we are going to use Fourier series to prove (3)) we can assume that the closure of our domain D is contained in the interior of the three-dimensional torus $T_3 = \{x, -\pi < x_j \leqq \pi, j = 1, 2, 3\}$. Now let S be a given open two-sided regular surface contained in the interior of D. Since S itself is a closed point set, between S and D we can put two domains D' and D'' with the following property:

$$S \subset D' \subset \overline{D}' \subset D'' \subset \overline{D}'' \subset D \subset \overline{D} \subset T_3 .$$

Letting $\lambda(x)$ be a localizing function which is non-negative and in class C^∞ on T_3 and which takes the value one on D' and the value zero on $T_3 - \overline{D}''$, we set $v'(x) = \lambda(x)v(x)$ and $w'(x) = \lambda(x)w(x) + \operatorname{grad}\lambda(x) \times v(x)$ for x in D and $v'(x) = w'(x) = 0$ for x in $T_3 - D$. Since $v'(x) = v(x)$ and $w'(x) = w(x)$ for x on S, (3) will be established once we can show that

$$\int_C (v', dx) = \int_S (w', n) dS . \tag{4}$$

In order to establish (4), we first observe from Lemma 1 and (i) and (ii) of Theorem 3 that

(5) $\operatorname{curl}^* v'(x)$ and $\operatorname{curl}_* v'(x)$ are finite-valued in T_3

(6) $\operatorname{curl}^* v'(x) = \operatorname{curl}_* v'(x) = w'(x)$ almost everywhere in T_3.

Next we designate the multiple Fourier series of v'_j and w'_j respectively by

$$\begin{aligned} v'_j(x) &\sim \sum_m a^j_m e^{i(m,x)} \\ w'_j(x) &\sim \sum_m b^j_m e^{i(m,x)} \end{aligned} \qquad j = 1, 2, 3 \tag{7}$$

where m represents an integral lattice point in three-space.

The essential step in proving (4) is to show that

$$b^\alpha_m = i(m_\beta a^\gamma_m - m_\gamma a^\beta_m) \tag{8}$$

where (α, β, γ) is a cyclic permutation of (1, 2, 3).

In order to do this we fix x_α and observe that

$$v'_j(x) \sim \sum_{m_\beta} \sum_{m_\gamma} a^j_{m_\beta m_\gamma}(x_\alpha) e^{i(m_\beta x_\beta + m_\gamma x_\gamma)} \text{ for } j = \beta, \gamma \tag{9}$$

where

$$a^j_{m_\beta m_\gamma}(x_\alpha) = (4\pi^2)^{-1} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-i(m_\beta x_\beta + m_\gamma x_\gamma)} v'_j(x) dx_\beta dx_\gamma . \tag{10}$$

Now by (5),

$$\text{(11)} \qquad \limsup_{r\to 0}(\pi r^2)^{-1}\int_{C_\alpha(x,r)} v'_\beta(x_\alpha, y_\beta, y_\gamma)dy_\beta + v'_\gamma(x_\alpha, y_\beta, y_\gamma)dy_\gamma$$

is finite-valued in T_3 with a similar statement holding for lim inf, and by (6),

$$\text{(12)} \qquad \begin{aligned}\lim_{r\to 0}(\pi r^2)^{-1}\int_{C_\alpha(x,r)} & v'_\beta(x_\alpha, y_\beta, y)dy_\beta + v'_\gamma(x_\alpha, y_\beta, y_\gamma)dy_\gamma \\ &= w'_\alpha(x_\alpha, x_\beta, x_\gamma) \text{ for almost every } (x_\beta, x_\gamma) \text{ if } x_\alpha \text{ lies in} \\ &\quad (-\pi, \pi] - E_\alpha \text{ where } E_\alpha \text{ is a set of linear measure zero.}\end{aligned}$$

Consequently it follows from (10), (11), (12), a modified version of [9, Lemma 8], and [9, Theorem 2] that for m_β and m_γ any two integers and x_α in $(-\pi, \pi] - E_\alpha$ that

$$\text{(13)} \qquad \begin{aligned} & i[m_\beta a^\gamma_{m_\beta m_\gamma}(x_\alpha) - m_\gamma a^\beta_{m_\beta m_\gamma}(x_\alpha)] \\ &\quad = (4\pi^2)^{-1}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-i(m_\beta x_\beta + m_\gamma x_\gamma)} w'_\alpha(x_\alpha, x_\beta, x_\gamma)dx_\beta dx_\gamma\ .\end{aligned}$$

Letting m_α be any integer, multiplying both sides of (13) by $(2\pi)^{-1}e^{-im_\alpha x_\alpha}$, and then integrating over $(-\pi, \pi]$, we conclude from (10), the fact that E_α is of linear measure zero, and (7) that

$$i(m_\beta a^\gamma_m - m_\gamma a^\beta_m) = b^\beta_m\ ,$$

which is (8).

(4) follows now fairly easily. We introduce for $t > 0$, the vector fields $v'(x, t)$ and $w'(x, t)$ where

$$\text{(14)} \qquad \left.\begin{aligned} v'_j(x, t) &= \Sigma\, a^j_m e^{i(m,x)-|m|t} \\ w'_j(x, t) &= \Sigma\, b^j_m e^{i(m,x)-|m|t}\end{aligned}\right. \qquad j = 1, 2, 3\ .$$

Then, since $v'(x, t)$ and $w'(x, t)$ are vector fields in class C^∞ on T_3 and since we can differentiate under the summation signs in (14), we conclude from (8) that $\operatorname{curl} v'(x, t) = w'(x, t)$. Consequently,

$$\text{(15)} \qquad \int_C (v'(x, t), dx) = \int_S (w'(x, t), n)dS \qquad \text{for } t > 0.$$

But as is well-known [2], $v'(x, t) \to v'(x)$ and $w'(x, t) \to w'(x)$ as $t \to 0$ uniformly for x in T_3. Therefore from the definition of a regular curve, it follows that $\int_C (v'(x, t), dx) \to \int_C (v', dx)$, and from the definition of a regular surface, it follows that $\int_S (w'(x, t), n)dS \to \int_S (w', n)dS$. We conclude from (15) that

$$\int_C (v', dx) = \int_S (w', n)dS$$

which is precisely (4), and the proof of Theorem 3 is complete.

5. **Proof of Theorem 1, 2, and 4.** The necessary conditions of Theorems 1 and 2 follow immediately from the following lemma (for an analogous two-dimensional result, see [3]), which we shall prove:

LEMMA 2. *Let* $u(x) = -(4\pi)^{-1}\int_{S_1(x_0,r_0)} f(y)|x-y|^{-1}dy$ *where* $f(x)$ *is bounded in* $S_1(x_0, r_0)$ *with* $r_0 > 0$. *Then for* x *in* $S_1(x_0, r_0)$

(a) $\operatorname{curl}\operatorname{grad} u(x) = 0$

(b) $A_* f(x) = \operatorname{div}_* \operatorname{grad} u(x)$ *and* $A^* f(x) = \operatorname{div}^* \operatorname{grad} u(x)$

(c) $\operatorname{div}_* \operatorname{grad} u(x) \leqq \operatorname{Lap}_* u(x) \leqq \operatorname{Lap}^* u(x) \leqq \operatorname{div}^* \operatorname{grad} u(x)$

To prove the lemma, it is clearly sufficient to prove it in the case $x = x_0$, and furthermore with no loss of generality, we can assume x_0 is the origin.

Setting $v(x) = \operatorname{grad} u(x)$, we observe that

$$v_j(x) = (4\pi)^{-1}\int_{S_1(0,r_0)} f(y)(x_j - y_j)|x-y|^{-3}dy \qquad j = 1, 2, 3 , \tag{16}$$

and $v_j(x)$ is a continuous function. Observing that

$$\int_{C_j(0,r)} (\operatorname{grad}|x-y|^{-1}, dx) = 0$$

for y not on $C_j(0, r)$ $j = 1, 2, 3,$ we conclude from (16) and Fubini's theorem that $\int_{C_j(0,r)} (v, dx) = 0$ for $j = 1, 2, 3$. Consequently (a) of the lemma is established.

Observing the $-\int_{S(0,r)} (\operatorname{grad}|x-y|^{-1}, n)dS = 4\pi$ if y is in $S_1(0, r)$ and $= 0$ if y is not in $\overline{S}_1(0, r)$, we obtain from (16) and Fubini's theorem that for $0 < r < r_0$.

$$\int_{S(0,r)} (v, n)dS = \int_{S_1(0,r)} f(y)dy . \tag{17}$$

Dividing both sides of (17) by $4\pi r^3/3$ and then taking $\liminf_{r\to 0}$ of both sides and next $\limsup_{r\to 0}$, gives us precisely part (b) of the lemma.

(c) follows from (b), the boundedness of f, and [5].

Theorem 4 and the sufficiency conditions of Theorems 1 and 2 follow from the following more general theorem:

THEOREM 5. *Let* D *be a bounded domain in Euclidean three-space, and let* $v(x)$ *be a continuous vector field defined in* D. *Suppose that*

(i) $\operatorname{curl}_* v(x)$ *and* $\operatorname{curl}^* v(x)$ *are finite-valued in* D

(ii) $\operatorname{curl}_* v(x) = \operatorname{curl}^* v(x) = 0$ *almost everywhere in* D

(iii) $\operatorname{div}_* v(x)$ *and* $\operatorname{div}^* v(x)$ *are finite-valued in* D

(iv) *there exists a function* $f(x)$ *such that* $f(x)$ *is in* L^1 *on every closed subdomain of* D *and such that* $\operatorname{div}_* v(x) \geqq f(x)$ *for* x *in* D.

Then (a) $\operatorname{div} v(x)$ *exists almost everywhere in* D

(b) $\operatorname{div} v(x)$ *is in* L^1 *on every closed subdomain of* D

(c) *for every closed sphere* $\overline{S}_1(x_0, r_0)$ *contained in* D, *there exists a function* $u(x)$ *in class* C^1 *in* $S_1(x_0, r_0)$ *such that for* x *in* $S_1(x_0, r_0)$, $v(x) = \operatorname{grad} u(x)$ *and furthermore*
$u(x) = -(4\pi)^{-1}\int_{S_1(x_0,r_0)} \operatorname{div} v(y)\,|x-y|^{-1}dy + h(x)$ a.e. in $S_1(x_0, r_0)$
where $h(x)$ *is harmonic in* $S_1(x_0, r_0)$.

In order to prove Theorem 5, we first need the following lemma (see [8, p. 381]):

LEMMA 3. *Let* $u(x)$ *be in class* C^1 *in* $S_1(x_0, r_0)$. *Then* $\operatorname{div}_* \operatorname{grad} u(x_0) \leqq \operatorname{Lap}_* u(x_0) \leqq \operatorname{Lap}^* u(x_0) \leqq \operatorname{div}^* \operatorname{grad} u(x_0)$

With no loss in generality, we assume that x_0 is the origin. Then by the mean value theorem

$$\left[(4\pi)^{-1}\int_0^{\pi}\int_0^{2\pi} u(t\sin\theta\cos\varphi, t\sin\theta\sin\varphi, t\operatorname{con}\theta)\sin\theta d\theta d\varphi - u(0)\right]\Big/t^2 6^{-1}$$
$$= (4\pi)^{-1}\int_0^{\pi}\int_0^{2\pi} u_t(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)\ \sin\theta d\theta d\varphi / r 3^{-1}$$

where $0 < r < t$. We conclude that

$$\sup_{0<r<t}\left[(4\pi r^2)^{-1}\int_{S(0,r)} u dS - u(0)\right]\Big/r^2 6^{-1}$$
$$\leqq \sup_{0<r<t} (4\pi r^3)^{-1} 3\int_{S(0,r)} [\operatorname{grad} u, n] dS\ .$$

Consequently from their very definitions, $\operatorname{Lap}^* u(0) \leqq \operatorname{div}^* \operatorname{grad} u(0)$. Similarly we show that $\operatorname{div}_* \operatorname{grad} u(0) \leqq \operatorname{Lap}_* u(0)$, and the proof to the lemma is complete.

It follows immediately from the three-dimensional analogue of [9, Theorem 2] that (a) and (b) of Theorem 5 hold. To obtain (c) of Theorem 5, we observe that there exists a positive ε such $S_1(x_0, r_0 + \varepsilon) \subset D$. Let x be in $S_1(x_0, r_0 + \varepsilon)$, and let $P(x)$ be the line segment connecting x_0 with x and directed to x. Then we define $u(x) = \int_{P(x)} (v, dy)$, and observe, since by Theorem 3 $\operatorname{curl} v = 0$ everywhere in $S_1(x_0, r_0 + \varepsilon)$ and Stokes' theorem with respect to v and $\operatorname{curl} v$ holds in this domain, that $u(x)$ is in class C^1 in $S_1(x_0, r_0 + \varepsilon)$ and furthermore that $v(x) = \operatorname{grad} u(x)$. Consequently by Lemma 3 and (iii) of the theorem

(18) $\operatorname{Lap}_* u(x)$ and $\operatorname{Lap}^* u(x)$ are finite-valued in $S_1(x_0, r_0 + \varepsilon)$,

and by (a) and (b) of the theorem and Lemma 3

(19) $\quad \operatorname{Lap} u(x) = \operatorname{div} v(x)$ almost everywhere in $S_1(x_0, r_0 + \varepsilon)$.

Therefore by (b) of the theorem, (18), (19), and the three-dimensional analogue of [6, Theorem 1], it follows that for almost all x in $S_1(x_0, r_0)$

$$u(x) = -(4\pi)^{-1}\int_{S_1(x_0,r_0)} \operatorname{div} v(y)\,|x-y|^{-1}dy + h(x)$$

where $h(x)$ is harmonic in $S_1(x_0, r_0)$. But this is precisely (c) of Theorem 5, and the proof to the theorem is complete.

6. The spherical intrinsic curl. Let $v(x)$ be a continuous vector field defined in a neighborhood of the point x_0. Then as mentioned earlier, the upper and lower intrinsic curl of v at x_0 can be defined by means of the cross product and spherical surfaces. In short, we define the upper spherical intrinsic curl to be the component-wise upper limit, $\operatorname{curl}_S^* v(x_0) = \limsup_{r\to 0}(4\pi r^3)^{-1}3\int_{S(x_0,r)} (n \times v)dS$. Similarly we define the lower spherical intrinsic curl, $\operatorname{curl}_{*S} v(x_0)$, using $\liminf_{r\to 0}$. In case $\operatorname{curl}_S^* v(x_0) = \operatorname{curl}_{*S} v(x_0)$ is finite, we say the spherical intrinsic curl of v exists at the point x_0, and we designate this common value by $\operatorname{curl}_S v(x_0)$.

We shall prove the following theorems:

THEOREM 6. *Theorems* 1, 2, 3, 4, *and* 5 *continue to hold if in each of these theorems* $\operatorname{curl}^* v$, $\operatorname{curl}_* v$, *and* curl v *are replaced by* $\operatorname{curl}_S^* v$, $\operatorname{curl}_{*S} v$, *and* $\operatorname{curl}_S v$ *respectively.*

THEOREM 7. *Let D be a bounded domain in Euclidean three-space, and let $v(x)$ be a continuous vector field defined in D. Then*

(a) *if* $\operatorname{curl}_S v(x)$ *exists and is continuous in D, then* $\operatorname{curl} v(x)$ *exists everywhere in D and equals* $\operatorname{curl}_S v(x)$.

(b) *if* $\operatorname{curl} v(x)$ *exists and is continuous in D, then* $\operatorname{curl}_S v(x)$ *exists everywhere in D and equals* $\operatorname{curl} v(x)$.

To prove Theorem 6, it follows from the proofs of Theorems 1, 2, 4, and 5 that it is sufficient just to prove Theorem 3 and Lemma 2(a) when $\operatorname{curl}^* v$, $\operatorname{curl}_* v$, and $\operatorname{curl} v$ are replaced respectively by $\operatorname{curl}_S^* v$, $\operatorname{curl}_{*S} v$, and $\operatorname{curl}_S v$.

The analogue of Lemma 2(a) follows immediately from Fubini's theorem and the fact that $\int_{S(x_0,r)} n \times \operatorname{grad}|x-y|^{-1}dS = 0$ if y is not on $S(x_0, r)$.

To prove the new version of Theorem 3, we designate by p^j the unit vector in the direction of the x_j-axis and set $v^j = v \times p^j$ for $j = 1, 2, 3$. Then it follows from the definition of spherical intrinsic curl and intrinsic divergence that the jth component of $\operatorname{curl}_S^* v$ is $\operatorname{div}^* v^j$ with a similar remark holding for $\operatorname{curl}_{*S} v$. Consequently by (i) and (ii) of the new version of Theorem 3 and by the three-dimensional analogue of [9, Theorem 2], we obtain that for $\overline{S}_1(x_0, r)$ contained in D,

$$\int_{S(x_0,r)} (v^j, n)dS = \int_{S_1(x_0,r)} w_j(x)dx \qquad j = 1, 2, 3 . \tag{20}$$

But (20) implies that $\operatorname{curl}_S v(x)$ exists everywhere in D and equals $w(x)$, giving the first part of the theorem.

The last part follows in a manner similar to the original version of Theorem 3, and it suffices to give a sketch of the proof. We first establish the analogue of Lemma 1 for the spherical intrinsic curl. Next with $\overline{D}$ contained in the interior of T_3 and S contained in D, we introduce the periodic vector fields $v'(x) = \lambda(x)v(x)$ and $w'(x) = \lambda(x)w(x) + \operatorname{grad}\lambda(x) \times v(x)$ where $\lambda(x)$ is a non-negative localizing function in class C^∞ which takes the value one in a neighborhood of S and the value zero outside another neighborhood of S for points in T_3. Then with $v'(x, t)$ and $w'(x, t)$ as in Theorem 3, it follows using the three dimensional analogues of the results in [9] that $\operatorname{curl} v'(x, t) = w'(x, t)$. But, as before, this implies that $\int_C (v, dx) = \int_S (w, n)dS$, which fact completes the proof of the theorem.

Theorem 7(a) follows immediately from Theorem 6.

To prove Theorem 7(b), we assume that $\overline{D}$ is contained in the interior of T_3, and we set $w(x) = \operatorname{curl} v(x)$. Then with $\overline{S}_1(x_0, 3r_0)$ contained in D and $\lambda(x)$ a non-negative localizing function of class C^∞ which takes the value one in $S_1(x_0, r_0)$ and the value zero in $T_3 - S_1(x_0, 2r_0)$, we introduce, as before, the periodic vector fields $v'(x) = \lambda(x)v(x)$, $w'(x) = \lambda(x)w(x) + \operatorname{grad}\lambda(x) \times v(x)$, $v'(x, t)$, and $w'(x, t)$. Exactly as in Theorem 3, we obtain that $\operatorname{curl} v'(x, t) = w'(x, t)$. But then on setting $v'^j(x) = v'(x) \times p^j$ and $v'^j(x, t) = v'(x, t) \times p^j$ for $j = 1, 2, 3$, we obtain that

$$\int_{S(x_0,r)} (v'^j(x, t), n)dS = \int_{S_1(x_0,r)} w'_j(x, t)dx \text{ for } r > 0 ,$$

and consequently that

$$\int_{S(x_0,r)} (v'^j(x), n)dS = \int_{S_1(x_0,r)} w'_j(x)dx .$$

This last fact, however, implies that $\operatorname{curl}_S v(x_0) = w(x_0)$, and therefore completes the proof to Theorem 7(b).

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