

# COMMUTATOR GROUPS OF MONOMIAL GROUPS

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This paper is a study of the commutator groups of certain generalized permutation groups called complete monomial groups. In [2] Ore has shown that every element of the infinite permutation group is itself a commutator of this group. Here it is shown that every element of the infinite complete monomial group is the product of at most two commutators of the infinite complete monomial group. The commutator subgroup of the infinite complete monomial group is itself, as is the case in the infinite symmetric group, [2]. The derived series is determined for a wide class of monomial groups.

Let  $H$  be an arbitrary group, and  $S$  a set of order  $B$ ,  $B \geq d$ ,  $d = \aleph_0$ . Then one obtains a monomial group after the manner described in [1]. A monomial substitution over  $H$  is a linear transformation mapping each element  $x$  of  $S$  in a one-to-one manner onto some element of  $S$  multiplied by an element  $h$  of  $H$ , the multiplication being formal. The element  $h$  is termed a factor of the substitution. If substitution  $u$  maps  $x_i$  into  $h_j x_j$ , while substitution  $v$  maps  $x_j$  into  $h_i x_i$ , then the substitution  $uv$  maps  $x_i$  into  $h_j h_i x_i$ . A substitution all of whose factor are the identity  $e$  of  $H$  is called a permutation and the set of all permutations is a subgroup which is isomorphic to the symmetric group on  $B$  objects. A substitution which maps each element of  $S$  into itself multiplied by an element of  $H$  is called a multiplication. The set of all multiplications form a subgroup which is the strong direct product of groups  $H_\alpha$ , each  $H_\alpha$  isomorphic to  $H$ . Hereafter monomial substitutions which are permutations will be denoted by  $s$ , while those that are multiplications will be denoted by  $v$ . The monomial group whose elements are the monomial substitutions, restricted by the definitions of  $C$  and  $D$  as given below, will be denoted by  $\Sigma(H; B, C, D)$ , where the symbols in the name are to be interpreted as follows,  $H$  the given arbitrary group,  $B$  the order of the given set  $S$ ,  $C$  a cardinal number such that the number of non-identity factors of any substitution of the group is less than  $C$ ,  $D$  a cardinal number such that the number of elements of  $S$  being mapped into elements of  $S$  distinct from themselves by any substitution of the group is less than  $D$ . In the event  $C = D = B^+$ ,  $B^+$  the successor of  $B$ , the resulting monomial group is termed the complete monomial group generated by the given group  $H$  and the given set  $S$ .  $S(B, M)$ ,  $d \leq M \leq D$ , will denote the subgroup of permutations which map fewer than  $M$  elements of  $S$  onto elements of  $S$  distinct from themselves, while  $V(B, N)$ ,

$d \leq N \leq C$ , will denote the subgroup of multiplications which have fewer than  $N$  nonidentity factors. In particular  $S(B, d)$  denotes the subgroup of finite permutations and  $V(B, d)$  the subgroup consisting of those multiplications which have finitely many nonidentity factors. The concept of alternating as associated with permutation groups may be extended in an obvious manner to monomial groups.  $A(B, d)$  will denote the alternating subgroup of the permutation group  $S(B, d)$ , while  $\sum_A(H; B, d, d)$  will denote the alternating subgroup of the monomial group  $\sum(H; B, d, d)$ . Any substitution may be written as the product of a multiplication and a permutation. Hence we may write  $\sum(H; B, C, D) = V(B, C) \cup S(B, D)$ , where  $\cup$  here and throughout will mean group generated by the set.  $G'$  will be used to denote the commutator subgroup of the group  $G$ .

**THEOREM 1.** *The commutator subgroup  $V'(B, C)$ ,  $d \leq C \leq B^+$ , of  $V(B, C)$  is the set of all elements*

$$v' = (h'_1, h'_2, h'_3, \dots), h'_i \in H',$$

where there exists an integer  $N$  such that each  $h'_i$  is the product of  $N$  or fewer commutators of  $H$ .

*Proof.* The theorem follows from the fact that  $V(B, C)$  is the strong direct product, each of whose summands is isomorphic to  $H$ , together with the remark following the lemma page 308 of [2].

**THEOREM 2.** *The commutator subgroup  $S'(B, C)$ ,  $d < C \leq B^+$ , of  $S(B, C)$  is  $S(B, C)$ . The commutator subgroup  $S'(B, d)$  of  $S(B, d)$  is  $A(B, d)$ .*

The proof is contained in [2].

**THEOREM 3.** *The commutator subgroup  $\sum'(H; B, d, d)$  of  $\sum(H; B, d, d)$  is  $A(B, d) \cup V^+(B, d)$  where  $V^+(B, d)$  is the set of all elements of  $V(B, d)$  whose product of factors is a member of  $H'$ .*

*Proof.* By reason of Theorem 2 we have  $\sum'(H; B, d, d) \supset A(B, d)$ , and that  $\sum'(H; B, d, d) \supset V^+(B, d)$

will now be demonstrated.

If  $h_i$  is the only nonidentity factor of the multiplication  $v_i$ , then the commutator  $v_i s v_i^{-1} s$ , where  $s = (x_i, x_j)$ , is a multiplication whose only nonidentity factors are  $h_i$  and  $h_i^{-1}$ . It then follows that any multiplication  $v$  of  $V^+(B, d)$  with  $n$  nonidentity factors can be written as the product of  $n + 1$  multiplications,  $n$  of which are of the type of the commutator

described above, and the remaining member having as its only nonidentity factor the product of the factors of  $v$ . But the first  $n$  members of the product belong to  $\Sigma'(H; B, d, d)$ , while the other member of the product is an element of  $V'(B, d)$ , by reason of Theorem 1, and hence

$$\Sigma'(H; B, d, d) \supset V^+(B, d), \text{ since } V'(B, d) \subset \Sigma'(H; B, d, d).$$

Then

$$\Sigma'(H; B, d, d) \supset V^+(B, d) \cup A(B, d).$$

Since  $G/G'$  is abelian for any group  $G$ , and  $G'$  is the smallest group for which this is true, to demonstrate that

$$\Sigma(H; B, d, d)/V^+(B, d) \cup A(B, d)$$

is abelian will imply that

$$\Sigma'(H; B, d, d) \subset V^+(B, d) \cup A(B, d),$$

and the conclusion of the theorem will follow.

That  $V^+(B, d) \supset V'(B, d)$  follows from the definition of  $V^+(B, d)$ , and hence  $V(B, d)/V^+(B, d)$  is abelian. Therefore any two multiplications commute mod  $V^+(B, d) \cup A(B, d)$ . Since  $A(B, d)$  consists of all even permutations there are but two cosets of  $A(B, d)$  in  $S(B, d)$ , namely,  $A(B, d)$  and  $(x_1, x_2)A(B, d)$ . Thus any element of the factor group  $\Sigma(H; B, d, d)/V^+(B, d) \cup A(B, d)$  has one of the forms

$$v[V^+(B, d) \cup A(B, d)]$$

or

$$v(x_1, x_2)[V^+(B, d) \cup A(B, d)], \quad v \in V(B, d).$$

But  $v(x_1, x_2)v^{-1}(x_1, x_2)$  is the commutator  $(h_1h_2^{-1}, h_2h_1^{-1}, e, \dots)$  which belongs to  $V^+(B, d)$ . That is,  $(x_1, x_2)$  and  $v$  commute mod  $[V^+(B, d) \cup A(B, d)]$ , and hence  $\Sigma(H; B, d, d)/V^+(B, d) \cup A(B, d)$  is abelian, which implies  $\Sigma'(H; B, d, d) \subset V^+(B, d) \cup A(B, d)$ , and we have

$$\Sigma'(H; B, d, d) = V^+(B, d) \cup A(B, d).$$

The following theorem asserts that the derived series for  $\Sigma(H; B, d, d)$  consists of but two distinct terms.

**THEOREM 4.** *The commutator subgroup  $\Sigma''(H; B, d, d)$  of  $\Sigma'(H; B, d, d)$  is  $\Sigma'(H; B, d, d)$ .*

*Proof.*  $A(B, d) = A'(B, d)$ , as was demonstrated in Theorem 7 of [2], and hence  $\Sigma''(H; B, d, d)$  contains  $A(B, d)$ .

Consider elements  $v_1$  and  $v_2$  of  $\Sigma'(H; B, d, d)$ , where the factors of  $v_1$  are all  $e$  except the first two and they are inverses of one another, and the factors of  $v_2$  are all  $e$  except the first and third and they are inverses of one another. The commutator  $v_1 v_2 v_1^{-1} v_2^{-1}$ , which is an element of  $\Sigma''(H; B, d, d)$ , has as its first factor a commutator of  $H$  and all other factors  $e$ . It then follows that any element of  $V'(B, d)$  is the product of elements of  $\Sigma''(H; B, d, d)$  and hence is an element of  $\Sigma''(H; B, d, d)$ . That is  $\Sigma''(H; B, d, d) \supset V'(B, d)$ . Then one can in the manner described in the first part of Theorem 3 write any element  $v$  of  $V^+(B, d)$  as the product of  $n + 1$  elements, each member of the product being an element of  $\Sigma''(H; B, d, d)$ . That is  $\Sigma''(H; B, d, d)$  contains  $V^+(B, d)$ , and hence  $\Sigma''(H; B, d, d)$  contains  $V^+(B, d) \cup A(B, d) = \Sigma'(H; B, d, d)$ .

**THEOREM 5.** *The commutator subgroup*

$$\Sigma'_A(H; B, d, d) \text{ of } \Sigma_A(H; B, d, d) \text{ is } V^+(B, d) \cup A(B, d) .$$

This theorem together with Theorem 3 states that  $\Sigma(H; B, d, d)$  has for its commutator subgroup  $\Sigma'_A(H; B, d, d)$ . This is the analogue for monomial groups of the result Ore obtains for permutation groups in [2], and as stated in the second part of Theorem 2.

*Proof.* We have

$$\Sigma'(H; B, d, d) \subset \Sigma'_A(H; B, d, d) \subset \Sigma(H; B, d, d) ,$$

hence,

$$\Sigma''(H; B, d, d) \subset \Sigma'_A(H; B, d, d) \subset \Sigma'(H; B, d, d) .$$

Then by reason of Theorem 4,

$$\Sigma'(H; B, d, d) = \Sigma''(H; B, d, d) = V^+(B, d) \cup A(B, d) .$$

Hence  $\Sigma'_A(H; B, d, d) = V^+(B, d) \cup A(B, d)$ .

**THEOREM 6.** *The commutator subgroup  $\Sigma'(H; B, C, D)$ ,  $d < C \leq D \leq B^+$ , of  $\Sigma(H; B, C, D)$  is  $\Sigma(H; B, C, D)$ .*

This theorem is also an analogue of a result Ore obtains in [2] for permutation groups as stated in the first part of Theorem 2.

*Proof.* It is shown in [2] that the commutator subgroup  $S'(B, D)$  of  $S(B, D)$  is  $S(B, D)$ . Hence  $\Sigma'(H; B, C, D)$  contains  $S(B, D)$ . The conclusion of the theorem will then follow if it can be demonstrated that  $\Sigma'(H; B, C, D) \supset V(B, C)$ . Let

$$s = (\dots, x_{-1}, x_0, x_1, \dots)$$

and

$$v = (\dots, h_{-1}, h_0, h_1, \dots)$$

be elements of  $\Sigma(H; B, C, D)$ . Then the commutator  $svs^{-1}v^{-1}$  an element of  $\Sigma'(H; B, C, D)$  has the form

$$(\dots, h_0h_{-1}^{-1}, h_1h_0^{-1}, h_2h_1^{-1}, \dots) .$$

Let

$$v_c = (\dots, c_{-1}, c_0, c_1, \dots)$$

be an arbitrary element of  $V(B, C)$ , and consider the following set of equations.

$$\dots, h_0h_{-1}^{-1} = c_{-1}, h_1h_0^{-1} = c_0, h_2h_1^{-1} = c_1, \dots$$

This set of equations has solutions,

$$h_0 = c_{-1}, h_{-1} = e, h_n = c_{n-1}h_{n-1}, h_{-n} = \left[ \prod_{i=2}^n c_{-i} \right]^{-1} .$$

Then if the factors of  $v$  be represented in terms of the factors of  $v_c$  as indicated above, we see that

$$svs^{-1}v^{-1} = v_c \in \Sigma'(H; B, C, D) ,$$

and hence  $\Sigma'(H; B, C, D)$  contains  $V(B, C)$ , and therefore

$$\Sigma(H; B, C, D) = \Sigma'(H; B, C, D) .$$

**COROLLARY 1.** *Any element  $u$  of  $\Sigma(H; B, C, D)$ ,  $d < C \leq D \leq B^+$ , is the product of at most two commutators.*

*Proof.* Every element of  $S(B, D)$  is a commutator of  $S(B, D)$ , as was shown in [2]. Every element of  $V(B, C)$  is a commutator of  $\Sigma(H; B, C, D)$ , as was shown in Theorem 6. Therefore any element of  $\Sigma(H; B, C, D)$  which is either a multiplication or a permutation is a commutator. But every element of  $\Sigma(H; B, C, D)$  maybe written as the product of a multiplication and a permutation and consequently may be written as the product of two commutators.

To see that the assertion that every element of  $\Sigma(H; B, C, D)$  is the product of at most two commutators is the strongest possible, suppose every element of  $\Sigma(H; B, C, D)$  is a commutator. Let

$$u \in \Sigma(H; B, d, d) \subset \Sigma(H; B, C, D) .$$

Then  $u = u_1u_2u_1^{-1}u_2^{-1}$ ,  $u_1$  and  $u_2$  elements of  $\Sigma(H; B, C, D)$ . But since  $u$  belongs to  $\Sigma(H; B, d, d)$  we can choose a  $u_1^*$  and  $u_2^*$  in  $\Sigma(H; B, d, d)$  by

causing  $u_1$  and  $u_2$  to become the map of  $x_i$  into  $ex_i$  except for those maps which yield the permutation and nonidentity factors of  $u$ . It then follows that  $u$  is an element of  $\Sigma'(H; B, d, d)$ , and hence  $\Sigma(H; B, d, d) = \Sigma'(H; B, d, d)$ . But this is a contradiction to Theorem 3.

#### REFERENCES

1. Ore, Oystein, *Theory of monomial groups*, Trans, Amer. Math. Soc. **51** (1942).
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