APPLICATIONS OF THE SUBORDINATION PRINCIPLE TO UNIVALENT FUNCTIONS

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1. Introduction. Let

$$(1.1) \qquad f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

be regular and univalent in $|z| < 1$ and map $|z| < 1$ onto a simply-connected domain D. Let

$$(1.2) \qquad \phi(z) = b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots$$

also be regular in $|z| < 1$. $\phi(z)$ is said to be subordinate to $f(z)$ if for each z of the unit circle $|z| < 1$ the corresponding point $w = \phi(z)$ lies in the domain D. In this case [2] there exists an analytic function $\omega(z)$ regular in $|z| < 1$ for which $\omega(0) = 0$, $|\omega(z)| \leq |z| < 1$ and $\phi(z) \equiv f\{\omega(z)\}$.

It is the purpose of this paper to establish the following basic Theorems A and B which concern analytic functions $F(z, t)$ and $\omega(z, t)$, depending upon a real parameter t, and then to use them to obtain results in the theory of univalent functions. Some of the results are well known and others are new, but the method of attack seems to be novel, simple and of sufficient generality to be of interest in itself. The functions $F(z, t)$ and $\omega(z, t)$ will be related to the univalent function $f(z)$ of (1.1) by means of the subordination concept.

An interesting biproduct of Theorem B is the following statement. A sufficient condition that $f(z)$, regular and univalent in $|z| < 1$, be convex in $|z| < 1$ is that the de la Vallée Poussin means $V_n(z)$ of (1.1) be subordinate to $f(z)$ in $|z| < 1$ for $n = 1, 2, \cdots$. Recently [3] G. Pólya and I. J. Schoenberg showed that this condition for convexity is also necessary.

THEOREM A. *Let*

$$(1.3) \qquad \omega(z, t) = \sum_1^\infty b_n(t) z^n$$

be regular in $|z| < 1$ for $0 \leq t \leq 1$. Let

$$|\omega(z, t)| < 1 \text{ for } |z| < 1, 0 \leq t \leq 1, \omega(z, 0) \equiv z .$$

Let ρ be a positive real number for which

$$(1.4) \qquad \omega(z) = \lim_{t \to 0+}\left\{\frac{\omega(z, t) - z}{z t^\rho}\right\}$$

exists. Then

(1.5) $\mathscr{R}\omega(z) \leqq 0 \; for \; |z| < 1$.

If $\omega(z)$ is also analytic in $|z| < 1$ and $\mathscr{R}\omega(0) \neq 0$, then

$$\mathscr{R}\omega(z) < 0 \; for \; |z| < 1 .$$

Proof. By Schwarz' lemma we have for $|z| < 1 \, |\omega(z, t)| \leqq |z|$ with equality only if $\omega(z, t) = z \exp i\theta(t)$, then the function

(1.6) $$\mu(z, t) = \frac{\omega(z, t) - z}{\omega(z, t) + z}$$

is regular and $\mathscr{R}\mu(z, t) < 0$ for $|z| < 1$. But when $\omega(z, t) = z \exp i\theta(t)$, $\mu(z, t) = i \tan(1/2\theta(t))$ is purely imaginary. Thus $\mu(z, t)$ is regular and $\mathscr{R}\mu(z, t) \leqq 0$ in $|z| < 1$ with equality occurring only if $\omega(z, t) = z \exp i\theta(t)$.
For $t > 0, |z| < 1$ we may write

(1.7) $$\mathscr{R}\left\{\frac{\omega(z, t) - z}{zt^\rho} \cdot \frac{2z}{\omega(z, t) + z}\right\} = \mathscr{R}\left\{\frac{2\mu(z, t)}{t^\rho}\right\} \leqq 0 .$$

(1.4) implies that $\lim_{t \to 0+}\omega(z, t) = z = \omega(z, 0)$. Therefore, on letting $t \to 0$ in (1.7) we obtain $\mathscr{R}\omega(z) \leqq 0$ for $|z| < 1$. When $\omega(z)$ is also analytic in $|z| < 1$ and $\mathscr{R}\omega(0) \neq 0$ we have further that $\mathscr{R}\omega(z) < 0$ in $|z| < 1$. This follows since the maximum, in this case zero, of a non-constant harmonic function cannot occur at an interior point.
As an illustration of Theorem A, the following example is useful. Let

(1.8) $$\omega(z, t) = \frac{(1 - 2t)z + z^2}{1 + (1 - 2t)z} , \qquad 0 \leqq t \leqq 1 .$$

Then $\omega(z, 0) \equiv z, |\omega(z, t)| \leqq 1$ in $|z| < 1, 0 \leqq t \leqq 1$.

(1.9) $$\omega(z) = \lim_{t \to 0}\left\{\frac{\omega(z, t) - \omega(z, 0)}{zt}\right\} = \frac{1}{z}\left[\frac{\partial}{\partial t}\omega(z, t)\right]_{t=0} = 2\frac{z - 1}{z + 1} ,$$

(1.10) $$\mathscr{R}\omega(z) = 2\mathscr{R}\left(\frac{z - 1}{z + 1}\right) < 0 , \quad |z| < 1 .$$

Theorem A is a special case of Theorem B to follow. However, the proof of Theorem B depends upon Theorem A.

THEOREM B. *Let*

(1.11) $$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

be regular and univalent in $|z| < 1$. For $0 \leqq t \leqq 1$ let $F(z, t)$ be regular in $|z| < 1$. Let $F(z, 0) \equiv f(z)$ and $F(0, t) \equiv 0$: Let ρ be a positive

real number for which

$$(1.12) \qquad F(z) = \lim_{t \to 0+} \left\{ \frac{F(z, t) - F(z, 0)}{zt^\rho} \right\}$$

exists. *Let* $F(z, t)$ *be subordinate to* $f(z)$ *in* $|z| < 1$ *for* $0 \leqq t \leqq 1$. *Then*

$$(1.13) \qquad \mathscr{R}\left\{ \frac{F(z)}{f'(z)} \right\} \leqq 0 , \quad |z| < 1 .$$

If in addition $F(z)$ *is also analytic in* $|z| < 1$ *and* $\mathscr{R}F(0) \neq 0$, *then*

$$\mathscr{R}\left\{ \frac{f'(z)}{F(z)} \right\} < 0 , \quad |z| < 1 .$$

Proof. Since $F(z, t)$ is subordinate to $f(z)$ in $|z| < 1$ we have

$$F(z, t) = f\{\omega(z, t)\} , \quad |z| < 1 , \quad 0 \leqq t \leqq 1 ,$$

where $\omega(z, t)$ is regular and bounded $|\omega(z, t)| \leqq 1$ in $|z| < 1, 0 \leqq t \leqq 1$. Since $F(z, 0) \equiv f(z)$ and since $f(z)$ is univalent in $|z| < 1$ we have $\omega(z, 0) \equiv z$. Also since $f(0) = 0$, $F(0, t) = 0$ and since $f(z)$ is univalent we have $\omega(0, t) = 0$. We now write

$$(1.14) \quad \frac{F(z, t) - F(z, 0)}{zt^\rho} = \left[\frac{f(\omega(z, t)) - f(\omega(z, 0))}{\omega(z, t) - \omega(z, 0)} \right] \left[\frac{\omega(z, t) - \omega(z, 0)}{zt^\rho} \right] .$$

(1.12) implies that $F(z, t)$ is continuous from the right at $t = 0$ and a similar statement holds for $\omega(z, t)$ because of the subordination. Let $t \to 0+$ in (1.14). The left side of equation (1.14) has for a limit $F(z)$ by (1.12). On the right side of (1.14) the square bracket has a limit $f'(z) \neq 0$. Thus

$$(1.15) \qquad \omega(z) = \lim_{t \to 0+} \left[\frac{\omega(z, t) - \omega(z, 0)}{zt^\rho} \right]$$

exists and equals $F(z)/f'(z)$. Furthermore $\mathscr{R}\omega(0) = \mathscr{R}F(0)$. If $F(z)$ is analytic so is $\omega(z)$. Since the conditions of Theorem A are fulfilled by $\omega(z, t)$ we have

$$(1.16) \qquad \mathscr{R}\frac{F(z)}{f'(z)} = \mathscr{R}\omega(z) \leqq 0 , \quad |z| < 1 .$$

When $F(z)$ is analytic in $|z| < 1$ and $\mathscr{R}F(0) \neq 0$ we also have

$$(1.17) \qquad \mathscr{R}\left\{ \frac{f'(z)}{F(z)} \right\} < 0 , \quad |z| < 1 .$$

2. Applications to univalent functions. The properties of univalent

functions $W = f(z)$, given by (1.1), which are also star-like with respect to the origin in $|z| < 1$ are well-known [2]. If $W = f(z)$ maps $|z| < 1$ onto a star-like domain D of the W-plane, then by definition the line segment joining the origin to the point $W = f(z)$ lines entirely within D for each z in $|z| < 1$. One then shows that it is necessary that

$$(2.1) \qquad \mathscr{R}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \text{ in } |z| < 1 .$$

In establishing (2.1) one is obliged to show first that if $f(z)$ is star-like with respect to the unit circle it is also star-like with respect to each smaller circle $|z| = r < 1$. At this stage one then appeals to an alternative definition of a star-like domain. This requires that the radius vector, joining the origin to the point $f(z)$, turns always in one direction as the argument of z advances.

A much simpler proof of the necessity of (2.1) follows immediately from Theorem B. Since $(1 - t)f(z)$ is subordinate to $f(z)$ for $0 \leqq t \leqq 1$, we have

$$(2.2) \qquad (1 - t)f(z) = f\{\omega(z, t)\}$$

where $\omega(z, t)$ satisfies the conditions of Theorem A. Taking $\rho = 1$ and letting

$$(2.3) \qquad F(z, t) = (1 - t)f(z)$$

in Theorem B we obtain at once $F(z) = -f(z)/z \neq 0$, so that (2.1) follows from (1.17) very simply.

More generally we have the following theorem.

THEOREM 1. *Let.*

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

be regular and univalent in $|z| < 1$ and such that $(1 - te^{i\alpha})f(z)$ is subordinate to $f(z)$ in $|z| < 1$ for an interval $0 \leqq t \leqq t_0$, α a real constant $|\alpha| < \pi/2$, then

$$(2.4) \qquad \mathscr{R}\left\{e^{-i\alpha}\frac{zf'(z)}{f(z)}\right\} > 0 , \quad |z| < 1 .$$

For the proof of Theorem 1 we take

$$(2.5) \qquad F(z, t) = (1 - te^{i\alpha})f(z)$$

in Theorem B and (1.13) becomes (2.4) in this case. The condition (2.4) is the one given for spiral-like functions by L. Špaček [7].

The following theorem from an intuitive point of view appears to be almost self-evident. Our new technique, however, furnishes an easy

and precise proof.

THEOREM 2. *Let $f(z)$ of (1.1) be regular and univalent in $|z| < 1$. For an interval $0 \leqq t \leqq t_0$ let the function*

$$(2.6) \qquad \frac{1}{2}[f(e^{it}z) + f(e^{-it}z)]$$

be subordinate to $f(z)$ in $|z| < 1$. Then.

$$(2.7) \qquad \mathscr{R}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad |z| < 1,$$

and $f(z)$ is convex in $|z| < 1$.

Proof. In Theorem B we choose $\rho = 2$ and $F(z, t)$ to be the function (2.6). Then

$$(2.8) \qquad F(z) = \lim_{t \to 0} \frac{F(z, t) - F(z, 0)}{zt^2} = \lim_{t \to 0} \frac{1}{2zt} \frac{\partial F(z, t)}{\partial t}$$

$$F(z) = \lim_{t \to 0} \frac{1}{2z} \frac{\partial^2 F(z, t)}{\partial t^2} = -zf''(z) - f'(z) .$$

Since $f'(0) = 1$, it follows that $F(0) = -1$ so that $\mathscr{R}F(0) \neq 0$. Thus (1.17) of Theorem B is equivalent to

$$(2.9) \qquad \mathscr{R}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} > 0, \quad |z| < 1 .$$

It is well known [2] that (2.9) implies that $f(z)$ is convex in $|z| < 1$.

For odd functions and an appropriate choice of $F(z, t)$ we obtain a result perhaps not so intuitively obvious as Theorem 2. It is the following theorem.

THEOREM 3. *Let*

$$(2.10) \qquad f(z) = z + \sum_{2}^{\infty} a_{2n-1} z^{2n-1} ,$$

be an odd function, regular and univalent in $|z| < 1$. For all real α and for an interval $0 \leqq t \leqq t_0$ let the function

$$(2.11) \qquad \frac{1}{2}\left[f\left(\frac{z + x}{1 + \bar{x}z}\right) + f\left(\frac{z - x}{1 - \bar{x}z}\right)\right], \quad x = te^{i\alpha} ,$$

be subordinate to $f(z)$ in $|z| < 1$. Then $f(z)$ is convex in $|z| < 1$.

For the proof of Theorem 3 we take $F(z, t)$ of Theorem B to be the function (2.11) and select $\rho = 2$. A calculation of $F(z)$ in (1.12),

together with (1.16), leads to the inequality

$$(2.12) \qquad \mathscr{R}\left[(1 - e^{-2i\alpha}z^2)^2 e^{2i\alpha}\frac{f''(z)}{zf'(z)} - 2(1 - e^{-2i\alpha}z^2)\right] \leqq 0, \quad |z| < 1.$$

Choose $\alpha = \operatorname{amp} z$. Let $|z| = r < 1$. Then (2.12) becomes

$$(2.13) \qquad \mathscr{R}\left[(1 - r^2)^2 \cdot \frac{zf''(z)}{r^2 f'(z)} - 2(1 - r^2)\right] \leqq 0,$$

$$(2.14) \qquad \mathscr{R}\left[1 + \frac{zf''(z)}{f'(z)}\right] \leqq \frac{1 + r^2}{1 - r^2}.$$

Similarly, for $\alpha = \pi/2 + \operatorname{amp} z$, we obtain

$$(2.15) \qquad \mathscr{R}\left[1 + \frac{zf''(z)}{f'(z)}\right] \geqq \frac{1 - r^2}{1 + r^2} > 0.$$

It follows from (2.15) that $f(z)$ is convex for $|z| < 1$. It is to be noticed that equality occurs in (2.13) for the convex function

$$f(z) = \frac{1}{2}\nu \cdot \log\left(\frac{1 + z}{1 - z}\right).$$

when $\alpha = 0$. In this case $F(z) \equiv 0$.

For another application of Theorem B we turn now to a class of function which need not be convex but which form a subclass of the class of close-to-convex functions introduced by W. Kaplan [1].

It is well known that if

$$(2.16) \qquad f(z) = z + \sum_{2}^{\infty} a_n z^n$$

is univalent and convex in $|z| < 1$, then $|a_n| \leqq 1$ [2]. The author [5, 6] has shown that if the coefficients are all real and if $f(z)$ is univalent and convex only in the direction of the imaginary axis for $|z| < 1$, then again $|a_n| \leqq 1$, but that if the coefficients are complex the results $|a_n| \leqq n$ is sharp. For the class of functions $f(z)$ which are close-to-convex in $|z| < 1$, the inequalities $|a_n| \leqq n$ again hold [4]. We now consider another class of functions, which are also close-to-convex in $|z| < 1$, but not necessarily convex, for which $|a_n| \leqq 1$. This class contains the odd star-like functions as a sub-class. The result is stated in the following theorem.

THEOREM 4. *Let the function*

$$(2.17) \qquad (1 - t)f(z) + tf(-z)$$

be subordinate to the univalent, regular function

$$(2.18) \qquad f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

in $|z| < 1$ *for an interval* $0 \le t \le t_0$. *Then*

$$(2.19) \qquad \mathscr{R}\left\{ \frac{z f'(z)}{f(z) - f(-z)} \right\} > 0 , \quad |z| < 1 ,$$

and the vector $\{ f(z) - f(-z) \}$ *turns continuously in one direction as* z *traverses each circle* $|z| = r < 1$. $f(z)$ *is close-to-convex in* $|z| < 1$.

Proof. Let $\rho = 1$ and let $F(z, t)$ be the function in (2.17). Then $F(z)$ of (1.12) reduces to $(1/z)[f(-z) - f(z)]$ and $F(0) \ne 0$. (1.13) then leads to (2.19).

Now let

$$\arg [f(z) - f(-z)] = \phi, \arg z = \theta .$$

$$(2.20) \quad \frac{d\phi}{d\theta} = \mathscr{R} z \left\{ \frac{f'(z) + f'(-z)}{f(z) - f(-z)} \right\}$$

$$= \mathscr{R}\left\{ \frac{z f'(z)}{f(z) - f(-z)} \right\} + \mathscr{R}\left\{ \frac{(-z) f'(-z)}{f(-z) - f(z)} \right\} > 0 , \quad |z| < 1 ,$$

by (2.19).

Since by (2.20) $\{ f(z) - f(-z) \}$ is univalent and star-like in $|z| < 1$, it follows that

$$(2.21) \qquad \psi(z) = \int_0^z \frac{f(t) - f(-t)}{t} \, dt , \quad |z| < 1 ,$$

is convex in $|z| < 1$. Thus (2.19) may be cast in the form

$$(2.22) \qquad \mathscr{R}\left\{ \frac{f'(z)}{\psi'(z)} \right\} > 0 , \quad |z| < 1 , \quad \varphi(z) \text{ convex,}$$

which implies that $f(z)$ is close-to-convex [1] in $|z| < 1$. This completes the proof of Theorem 4.

In a recent paper [3] G. Pólya and I. J. Schoenberg have shown that if $f(z)$ of (1.1) is univalent and convex in $|z| < 1$ then so are the de la Vallée Poussin means $V_n(z)$ of the power series (1.1),

$$(2.23) \qquad V_n(z) = \frac{n}{n+1} z + \frac{n(n-1)}{(n+1)(n+2)} a_2 z^2 + \cdots$$

$$+ \frac{n(n-1) \cdots 1}{(n+1)(n+2) \cdots (2n)} a_n z^n ,$$

and if D and D_n denote the convex domains into which the unit circle is mapped by $f(z)$ and $V_n(z)$, respectively, then $D_n \subset D$. In other words, $V_n(z)$ is necessarily subordinate to $f(z)$ for $n = 1, 2, \cdots$ when $f(z)$ is

univalently convex. By means of Theorem B we can now prove that the condition $D_n \subset D$ for infinitely many values of n is also a sufficient condition that $f(z)$ be convex when $f(z)$ is univalent. The theorem of Pólya and Schoenberg in its extended form is now stated as Theorem 5.

THEOREM 5. *A necessary and sufficient condition that the function*

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots ,$$

regular and univalent in $|z| < 1$, *be convex in* $|z| < 1$ *is that the de la Vallée Poussin means* $V_n(z)$ *in* (2.23) *be subordinate to* $f(z)$ *in* $|z| < 1$ *for* $n = 1, 2, \cdots$.

Proof of sufficiency. In Theorem B we choose $\rho = 1$ and $F(z, t) = V_n(z)$ where $t = (n + 1)^{-1}$. We define $F(z, 0) = \lim_{t \to 0+} F(z, t) = \lim_{n \to \infty} V_n(z) = f(z)$, uniformly in $|z| \leq r < 1$. For $\rho = 1$ we shall show that the limit defining $F(z)$ in (1.12) exists uniformly and is precisely the analytic function $-\{zf''(z) + f'(z)\}$, $F(0) = -1$. When this is done (1.17) will give

$$\mathscr{R}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 , \quad |z| < 1 ,$$

and the convexity of $f(z)$ follows. We need the following lemma.

LEMMA. *If* n *and* k *are positive integers*, $k \leq n$, *then*

$$(2.24) \qquad (n + 1)\left[1 - \frac{n(n - 1) \cdots (n - k + 1)}{(n + 1)(n + 2) \cdots (n + k)}\right] \leq k^2 .$$

We establish the lemma by mathematical induction. Let n be an assigned positive integer. It is readily seen that (2.24) holds for $k = 1$. Assuming that (2.24) is true for a value $k < n$ we prove that (2.24) also holds when k is replaced by $(k + 1)$. Indeed, we have

$$(2.25) \quad (n + 1)\left[1 - \frac{n(n - 1) \cdots (n - k + 1)(n - k)}{(n + 1)(n + 2) \cdots (n + k)(n + k + 1)}\right]$$

$$= (n + 1)\left[1 - \frac{n(n - 1) \cdots (n - k + 1)}{(n + 1)(n + 2) \cdots (n + k)}\left(1 - \frac{2k + 1}{n + k + 1}\right)\right]$$

$$= (n + 1)\left[1 - \frac{n(n - 1) \cdots (n - k + 1)}{(n + 1)(n + 2) \cdots (n + k)}\right]$$

$$\qquad + (2k + 1)\frac{(n + 1)}{n + k + 1}\left[\frac{n(n - 1) \cdots (n - k + 1)}{(n + 1)(n + 2) \cdots (n + k)}\right]$$

$$\leq k^2 + (2k + 1) = (k + 1)^2 .$$

Turning to the calculation of $F(z)$ we have

(2.26)
$$F(z) = \lim_{t \to 0+}\left[\frac{F(z, t) - F(z, 0)}{tz}\right] = \lim_{n \to \infty}\frac{n + 1}{z}(V_n(z) - f(z))$$

$$= -\lim_{n \to \infty}(n + 1)\sum_{k=1}^{n}\left\{1 - \frac{n(n - 1)\cdots(n - k + 1)}{(n + 1)(n + 2)\cdots(n + k)}\right\}a_k z^{k-1}, a_1 = 1,$$

$$- \lim_{n \to \infty}(n + 1)z^n\sum_{\nu=0}^{\infty}a_{\nu+n+1}z^\nu .$$

Let $|z| \leqq r < 1$. Since $f(z)$ is univalent we have $|a_{\nu+n+1}| < e(\nu + n + 1)$. Consequently for large n

(2.27)
$$(n + 1)z^n\sum_{\nu=0}^{\infty}a_{\nu+n+1}z^\nu = 0\left\{\frac{n^2 r^n}{(1 - r)^2}\right\} = p_n ,$$

(2.28)
$$\lim p_n = 0$$

uniformly in $|z| \leqq r$.

Let N be a positive integer. Then

(2.29)
$$\lim_{n \to \infty}(n + 1)\sum_{k=1}^{N}\left\{1 - \frac{n(n - 1)\cdots(n - k + 1)}{(n + 1)(n + 2)\cdots(n + k)}\right\}a_k z^{k-1}$$

$$= \sum_{k=1}^{N}k^2 a_k z^{k-1} .$$

For $n > N, |z| \leqq r$, by the lemma we have

(2.30)
$$\left|(n + 1)\sum_{k=N+1}^{n}\left\{1 - \frac{n(n - 1)\cdots(n - k + 1)}{(n + 1)(n + 2)\cdots(n + k)}\right\}a_k z^{k-1}\right|$$

$$\leqq \sum_{k=N+1}^{n}k^2|a_k|r^{k-1} < e\sum_{N+1}^{\infty}k^3 r^{k-1} .$$

Given $\varepsilon > 0$, we now choose $N_0(\varepsilon, r)$ so that for $N > N_0$

(2.31)
$$e\sum_{N+1}^{\infty}k^3 r^{k-1} < \varepsilon .$$

From (2.26), (2.28), (2.29), (2.30) and (2.31) it follows that the limit in (2.26) exists uniformly in $|z| \leqq r < 1$ and is the analytic function

(2.32)
$$F(z) = -\sum_{k=1}^{\infty}k^2 a_k z^{k-1}, a_1 = 1, |z| < 1,$$

$$= -zf''(z) - f'(z) .$$

This completes the proof of the sufficiency part of Theorem 5. The necessity part was shown in [3]. In (2.26) since n is a positive integer we have let $t \to 0$ through a discrete set of values of t. This, however, in no way affects the validity of Theorem B.

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