A STRONG MAXIMUM PRINCIPLE FOR WEAKLY SUBPARABOLIC FUNCTIONS

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Introduction. It has been proved by E. Hopf [3], over thirty years ago, that solutions of second order elliptic equations satisfy the maximum principle. A similar principle, well known for solutions of the heat equation, has been, relatively recently, extended to second order parabolic equations by Nirenberg [5]. In various problems, such as in solving the Dirichlet problem by the methods of Poincaré and Perron, subsolutions have been introduced and the maximum principle has been extended to such functions. In the elliptic case (see [6]) the subsolutions used are continuous, whereas in the parabolic case, they may have certain discontinuities (see [2]). In the elliptic case, they are called L-subharmonic or subelliptic functions. Likewise, in the parabolic case, we call them L-subcaloric or subparabolic functions; L is the elliptic or the parabolic operator.

Recently, Walter Littman [4] has generalized the concept of L-subharmonic functions to include measurable integrable functions. This generalization is obtained by expressing the condition $Lu \geqq 0$ in an integrated form, namely, $\int uL^*vdx \geqq 0$ for any twice differentiable $v \geqq 0$ with compact support, L^* being the adjoint of L. He then established the maximum principle in the following sense: If an L-subharmonic function assumes its essential supremum at a point of continuity, then it is equal to a constant almost everywhere.

The purpose of this paper is to prove a similar result for measurable L-subcaloric functions. The general outline of the proof is similar to that of Littman's method. However, the crucial step in the proof is the construction of two kernal functions with certain required properties. Our construction is entirely different from that of Littman.

In § 1 we state some definitions and the results of the paper. In § 2 we prove Lemma 2. In § 3 we recall some properties of fundamental solutions. These are used in § 4 to prove Lemma 1. Lemmas 1, 2 immediately yield the maximum principle.

1. Statement of the results.

Consider the differential operators

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x, t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x, t)\frac{\partial u}{\partial x_i} + a(x, t)u - \frac{\partial u}{\partial t}$$

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$$L^*u \equiv \sum_{i,j=1}^{n} b_{ij}(x, t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x, t)\frac{\partial u}{\partial x_i} + b(x, t)u + \frac{\partial u}{\partial t}$$

where L^* is the adjoint of L (thus, $b_{ij} = a_{ij}$, etc.). Throughout this paper it will always be assumed that:

$$a_{ij}, \frac{\partial}{\partial x_k} a_{ij}, \frac{\partial^2}{\partial x_k \partial x_m} a_{ij}, a_j, \frac{\partial}{\partial x_k} a_i, a$$

are Hölder continuous (exponent α) in (x, t) which varies in a bounded domain D, and that

$$a \leqq 0 \text{ in } D,\ \Sigma a_{ij}\xi_i\xi_j \geqq A_0 \Sigma \xi_i^2 \text{ in } D(A_0 > 0)$$

for any real vector ξ.

DEFINITION. A bounded measurable function $u(x, t)$ in D is called *weakly L-subcaloric* (or simply, *weakly subparabolic* when there is no confusion about the L) if for any compact subdomain E of D with piecewise smooth boundary (so that Green's formula holds)

$$\iint_E u(x, t)L^*v(x, t)dxdt \geqq 0 \tag{1}$$

for any function $v(x, t)$ satisfying the following properties:

(i) $v \geqq 0$ in E,

(ii) $v, \partial v/\partial v_i, \partial^2 v/\partial x_i \partial x_j, \partial v/\partial t$ are continuous in E and vanish on the boundary ∂E of E.

We note that, for the establishment of the maximum principle below, it is enough that (1) holds only for some special types of domains, namely, for cylindrical domains and for certain sections of paraboloids.

DEFINITIONS. For any point $P(x^0, t^0)$ in D, we denote by $C(P)$ the set of all points (x^1, t^1) in D such that there exists a differentiable curve connecting (x^0, t^0) to (x^1, t^1) and along which the t-coordinate is non-increasing. A function $u(x, t)$ is said to be *continuous from below* at a point $P = (x^0, t^0)$ if u, as a function in $C(P)$, is continuous at P in the usual sense. By a *neighborhood-from-below* of a point P we mean the intersection of a neighborhood of P with $C(P)$.

Our purpose is to prove the following theorem.

THEOREM. *Let u be a weakly L-subcaloric in D. If u assumes its essential supremum M (in D) at a point $P = (x^0, t^0)$ at which u is continuous from below, and if $M \geqq 0$, then $u = M$ almost everywhere in $C(P)$.*

As in [4], the proof follows immediately once we have established the following lemmas.

LEMMA 1. *Under the assumptions of the theorem, there exists a neighborhood-from-below N of P such that $u = M$ almost everywhere in N.*

LEMMA 2. *Let u be a weakly L-subcaloric function in D. If $u = M$ almost everywhere in a neighborhood-from-below of some point P of D, and $M \geqq 0$, then $u = M$ almost everywhere in $C(P)$.*

2. Proof of Lemma 2. We shall prove that, given a compact subset E of D, we can construct, for each point $Q = (y, \tau)$ in E a domain

$$\Omega = \Omega_{\delta\varepsilon} : -\delta < t - \tau < 0, \varepsilon | x - y |^2 < | t - \tau | (\varepsilon > 0, \delta > 0)$$

and a function $w(x, t) = w^{y,\tau}(x, t)$ having the following properties:

(a) $w > 0$ in Ω.

(b) $w, \partial w/\partial x_i, \partial^2 w/\partial x_i \partial x_j, \partial w/\partial t$ are continuous in $\bar{\Omega} - \{(y, \tau)\}$ and vanish on the boundary $\partial\Omega - \{(y, \tau)\}$.

(c) $L^* w > 0$ in Ω.

Furthermore, ε may be any number between 0 and 1 and δ may be taken to be dependent *only* on L, ε and E, but not on the particular point $Q = (y, \tau)$. Finally, as $\varepsilon \to 0$, the radius of the base (or δ/ε) can be taken to be bounded away from zero.

Once w has been constructed, a simple argument of [4] can easily be extended to complete the proof of the lemma. For the sake of completeness we reproduce it here.

Let S be the set of points (x, t) in $C(P)$ having the property that $u = M$ almost everywhere in an open-from-below set containing (x, t). By assumption S is nonempty. Clearly S is open from below. If we show that S is also closed, then S coincides with $C(P)$. To prove it, we take any sequence $Q_m \to R$, Q_m in S, R in D, and use the above construction with $E = \{R, Q_1, Q_2, \cdots\}$. If we show that $u = M$ almost everywhere in each domain Ω_i corresponding to Q_i, then it would follow that R also belongs to S. (Note that in the construction of the Ω below, the radius of the base of Ω can be made bounded away from zero as $\varepsilon \to 0$.)

For simplicity we denote Ω_i by Ω and the corresponding w_i by w. We now modify the definition of $w(x, t)$ in the intersection of Ω with a neighborhood-from-below N of Q_i where $u = M$ almost everywhere. The modified function is denoted by W, and is taken to satisfy the conditions imposed on the function v in the definition of subcaloricity (in § 1) with E replaced by Ω. Denote $A = N \cap \Omega$, $B = \Omega - A$. Using the definition of weakly L-subcaloric functions, we get

$$(2) \qquad \iint_\Omega u L^* W dx dt \geqq 0 ,$$

Now,

$$\iint_\Omega L^* W dxdt = \iint_\Omega WL1 dxdt \leqq 0\ ;$$

hence

$$\iint_B L^* W dxdt \leqq - \iint_A L^* W dxdt\ . \tag{3}$$

On the other hand, by (2),

$$\iint_B uL^* W dxdt \geqq \iint_A uL^* W dxdt = -M \iint_A L^* W dxdt\ .$$

Using (3) we obtain

$$\iint_B (u - M) L^* W dxdt \geqq 0\ .$$

Since $L^* W = L^* w > 0$ in B, $u - M$ must vanish in B almost everywhere.

To complete the proof of Lemma 2 we have to construct a function w which the required properties (a) — (c). For simplicity we shall do it in the special case is $\tau = 0$, $y = 0$; the general case is immediately obtained by translation.

DEFINITION OF w:

$$w = (\delta + t)^2 (-t - \varepsilon r^2)^3 \tilde{r}^{-k} \tag{4}$$

where

$$r = |x|,\ \tilde{r}^2 = r^2 - k^{1/2} t,$$

where k is a positive integer to be determined later. Clearly, w satisfies (a), (b). It remains to prove that $L^* w > 0$ in Ω. We have

$$\frac{\partial w}{\partial x_i} = -6\varepsilon x_i (\delta + t)^2 (-t - \varepsilon r^2)^2 \tilde{r}^{-k} - k x_i (\delta + t)^2 (-t - \varepsilon r^2)^3 \tilde{r}^{-k-2}\ ,$$

$$\begin{aligned}\frac{\partial^2 w}{\partial x_i \partial x_j} &= 24\varepsilon^2 x_i x_j (\delta + t)^2 (-t - \varepsilon r^2) \tilde{r}^{-k} - 6\varepsilon \delta_{ij} (\delta + t)^2 (-t - \varepsilon r^2)^2 \tilde{r}^{-k} \\ &\quad + 12 k \varepsilon x_i x_j (\delta + t)^2 (-t - \varepsilon r^2)^2 \tilde{r}^{-k-2} - k \delta_{ij} (\delta + t)^2 (-t - \varepsilon r^2)^3 \tilde{r}^{-k-2} \\ &\quad + k(k+2) x_i x_j (\delta + t)^2 (-t - \varepsilon r^2)^3 \tilde{r}^{-k-4}\ ,\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial t} &= 2(\delta + t)(-t - \varepsilon r^2)^3 \tilde{r}^{-k} - 3(\delta + t)^2 (-t - \varepsilon r^2)^2 \tilde{r}^{-k} \\ &\quad + \frac{1}{2} k^{3/2} (\delta + t)^2 (-t - \varepsilon r^2)^3 \tilde{r}^{-k-2}\ .\end{aligned}$$

We now form $L^* w$, and restrict δ to be sufficiently small and restrict $|x|$ to be sufficiently small (depending only on L^*), say $|x| \leqq \rho$. Then, the contribution to $L^* w$ made by the terms of $\Sigma b_i \partial w / \partial x_i + bw$ is small compared with the corresponding last two terms in $\partial w / \partial t$. Also, the negative contribution in $\Sigma b_{ij} \partial^2 w / \partial x_i \partial x_j$ corresponding to the fourth term in $\partial^2 w/\partial x_i \partial x_j$ (calculated above) can be neglected as compared to the third term in $\partial w/\partial t$ (provided k is sufficiently large, depending on b_{ij}). Discarding (as we may) the positive contribution corresponding to the first and the last terms in $\partial^2 w/\partial x_i \partial x_j$, we conclude that in order to prove that $L^* w > 0$, it is sufficient to prove that

$$k\varepsilon \frac{r^2}{\tilde{r}^2} + k^{3/2}(-t - \varepsilon r^2)\frac{1}{\tilde{r}^2} \geqq \lambda > 0 \tag{5}$$

where λ is a constant depending only on L and ρ ($|x| \leqq \rho$ in Ω).

To prove (5) we take $k > 1/\varepsilon^2$, which imples that, in Ω (where $\varepsilon r^2 < |t|$),

$$k^{1/2}|t| \leqq \tilde{r}^2 = r^2 + k^{1/2}|t| \leqq 2k^{1/2}|t| .$$

Hence (5) is a consequence of

$$k^{1/2}\varepsilon r^2 + k(-t - \varepsilon r^2) \geqq 2\lambda |t|$$

which is clearly true if $k^{1/2} \geqq 2\lambda$, $k \geqq 1$.

3. **Properties of fundamental solutions.** Assume that the closure of a cylinder $C: |x|^2 < \beta, -\delta < t < 0$ with base B is contained in D. By our assumptions on L, there exists (by Pogorzelski [7]) in C a fundamental solution $\Gamma(x, t; \xi, \tau)(t < \tau)$ of L^* with pole (ξ, τ); $L^*\Gamma = 0$ as a function of (x, t), and Γ can be constructed as follows:

Let (B_{ij}) be the matrix inverse to (b_{ij}) and define

$$\sigma(x, t; \xi, \tau) = \Sigma B_{ij}(\xi, \tau)(x_i - \xi_i)(x_j - \xi_j)$$
$$Z(x, t; \xi, \tau) = (\tau - t)^{-n/2} \exp\left\{-\frac{\sigma(x, t; \xi, \tau)}{4(\tau - t)}\right\}$$

$$\Gamma(x, t; \xi, \tau) = Z(x, t; \xi, \tau) + \int_t^\tau \int_B Z(x, x; \eta, s)\Phi(\eta, s; \xi, \tau) d\eta ds$$

where Φ is the solution of the integral equation

$$L^*_{(x,t)} Z(x, t; \xi, \tau) - \rho(x, t)\Phi(x, t; \xi, \tau) + \int_\tau^t \int_B [L^*_{(x,t)} Z(x, t; \eta, s)]\Phi(\eta, s; \xi, \tau) d\eta ds = 0 .$$

Here,

$$\rho(x, t) = (4\pi)^{n/2}/(\det (B_{ij}(x, t)))^{1/2} .$$

Note that

$$0 < \text{const.} \leqq \frac{\sigma(x, t; \xi, \tau)}{|x - \xi|^2} \leqq \text{const.} < \infty .$$

In the following we shall be interested in the special case $(\xi, \tau) = 0$. We define

$$g(x, t) = \Gamma(x, t; 0, 0)$$
$$\sigma(x, t) = \sigma(x, t; 0, 0)$$
$$Z(x, t) = Z(x, t; 0, 0) .$$

By simple calculation we get

$$g(x, t) = Z(x, t)(1 + o(1)) \tag{6}$$

where $o(1) \to 0$ as $t \to 0$. Hence, in particular, $g(x, t) > 0$ if the height δ of C is sufficiently small, as we shall assume. We also mention, although this is not used later on, that for any bounded measurable function $\varphi(x, t)$ in C, which is continuous at $(0, 0)$ we have (see [8])

$$\lim_{t\to 0}\int_B g(x, t)\varphi(x, t)dx = \rho(0, 0)\varphi(0, 0) . \tag{7}$$

We conclude this section with estimating the following expression (which will appear in the next section)

$$I \equiv - \sum_{i,j} b_{ij}(x, t)x_i \frac{\partial g(x, t)}{\partial x_j} . \tag{8}$$

Since

$$- \frac{\partial}{\partial x_j} Z(x, t) = \frac{1}{2t}(\sum_k B_{jk}(0, 0)x_k)Z(x, t) ,$$

and since

$$\sum_{i,k} \sum_{j=1}^{n} b_{ij}(x, t)B_{jk}(0, 0)x_i x_k$$
$$= |x|^2 + \sum_{i,j,k}[b_{ij}(x, t) - b_{ij}(0, 0)]B_{jk}(0, 0)x_i x_k = |x|^2(1 + o(1))$$

where $o(1) \to 0$ as $|x| \to 0$, we conclude that

$$I_1 \equiv -\sum_{i,j} b_{ij}(x, t)x_i \frac{\partial Z(x,t)}{\partial x_j} \geqq \frac{|x|^2}{3|t|} Z(x, t) \tag{9}$$

provided $|x|$ is sufficiently small.

To evaluate $I - I_1$, we use the definitions of g and Γ, and proceed to estimate the x_j-derivatives of the integral which appears in the definition of Γ. Noting that

$$\left| \frac{\partial}{\partial x_j} Z(x, t; \eta, s) \right| \leqq \text{const.}\ (s - t)^{-1/2} Z(x, t; \eta, s)$$

and using the estimate of [7] for Φ and Dressel [1; Lemma 2] we find that

$$|I - I_1| \leqq \lambda_0 |x|^\gamma Z(x, t) \qquad (\lambda_0 > 0, 0 < \gamma \leqq 1)\,,$$

where λ_0, γ depend only on L. In what follows we shall only need the weaker inequality

$$I \geqq -\lambda_0 |x|^\gamma Z(x, t)\,. \tag{10}$$

4. Proof of Lemma 1. We may assume, without loss of generality, that the essential supremum M is assumed at the origin. Following the procedure of Littman [4], we claim that it is enough to construct a function $G(x, t)$ in a cylinder C: $|x|^2 < \beta$, $-\delta < t < 0$, with base B, which satisfies the following conditions:

(a) G, $\partial G/\partial x_i$, $\partial^2 G/\partial x_i \partial x_j$, $\partial G/\partial t$ are continuous in $\bar{C} - \{(0, 0)\}$ and vanish on the boundary $\partial C - \{(0, 0)\}$.

(b) $L^*G > 0$ in C.

(c) If $f(x, t)$ is L-subcaloric in a domain which contains C, and if f is continuous from below at the origin and $f(0, 0) = 0$, then

$$0 \leqq \iint_C f L^* G dx dt\,. \tag{11}$$

Once G is constructed, the proof of Lemma 1 follows very easily. Indeed, $u - M$ is L-subcaloric, and using (c) we get

$$\iint_C (u - M) L^* G dx dt \geqq 0\,.$$

Since, by (b), $L^*G > 0$, we conclude that $u = M$ almost everywhere in C.

DEFINITION OF $G(x, t)$:

$$G(x, t) = (t + \delta)^2 (\beta - r^2)^3 g(x, t) \tag{12}$$

where $g(x, t)$ is defined in § 3. Clearly (a) is satisfied. We proceed to establish (b), (c).

Proof of (b).

$$\frac{\partial G}{\partial x_i} = -6x_i (t + \delta)^2 (\beta - r^2)^2 g + (t + \delta)^2 (\beta - r^2)^3 \frac{\partial g}{\partial x_i}\,,$$

$$\begin{aligned}\frac{\partial^2 G}{\partial x_i \partial x_j} = {} & -6\delta_{ij} (t + \delta)^2 (\beta - r^2)^2 g + 24 x_i x_j (t + \delta)^2 (\beta - r^2) g \\ & - 6x_i (t + \delta)^2 (\beta - r^2)^2 \frac{\partial g}{\partial x_j} - 6x_j (t + \delta)^2 (\beta - r^2)^2 \frac{\partial g}{\partial x_i} \\ & + (t + \delta)^2 (\beta - r^2)^3 \frac{\partial^2 g}{\partial x_i \partial x_j}\,,\end{aligned}$$

$$\frac{\partial G}{\partial t} = 2(t+\delta)(\beta - r^2)^3 g + (t+\delta)^2(\beta - r^2)^3 \frac{\partial g}{\partial t} .$$

Recalling that $L^*g = 0$ we obtain

$$\begin{aligned} L^*G = & -6(t+\delta)^2(\beta - r^2)^2 g \Sigma x_i b_i - 6(\Sigma b_{ii})(t+\delta)^2(\beta - r^2)^2 g \\ & + 24(\Sigma b_{ij} x_i x_j)(t+\delta)^2(\beta - r^2) g - 12(t+\delta)^2(\beta - r^2)^2 \Sigma b_{ij} x_i \frac{\partial g}{\partial x_j} \\ & + 2(t+\delta)(\beta - r^2)^3 g . \end{aligned}$$

Now the first term in L^*G is small compared with the second one, if $|x|$ (or β) is small. Using (8), (10), (6) to estimate the fourth term, we conclude that, if it is negative, then its absolute value is smaller than that of the second term. Hence, if we prove that

$$(13) \qquad (t+\delta)r^2 + (\beta - r^2)^2 > \mu(t+\delta)(\beta - r^2)$$

for sufficiently large μ depending only on L (provided β is smaller than an appropriate constant), then $L^*G > 0$.

To prove (13) we note that if $\mu(\beta - r^2) < r^2$ then (13) clearly holds. Hence it remains to consider the case where

$$\mu(\beta - r^2) \geqq r^2 .$$

However, in this case

$$(\beta - r^2)^2 \geqq \frac{\beta^2}{(1+\mu)^2} > \mu(t+\delta)\beta$$

for sufficiently small δ (i.e., if $(\mu+1)^2\mu\delta < \beta$), from which (13) follows.

Proof of (c). We modify G as follows: Let

$$\sigma_\varepsilon(x,t) = \begin{cases} \sigma(x,t) \text{ if } -\delta < t < -\varepsilon \\ \sigma(x,t) + (t+\varepsilon) \text{ if } -\varepsilon \leqq t \leqq 0 . \end{cases}$$

Clearly $\sigma_\varepsilon(x,t)$ has second continuous x-derivatives and a first continuous t-derivative in $\bar{C}$. We next define

$$Z_\varepsilon(x,t) = \frac{1}{(-t)^{n/2}} \exp\left\{\frac{\sigma_\varepsilon(x,t)}{4t}\right\} .$$

$$g_\varepsilon(x,t) = Z_\varepsilon(x,t) + \int_t^0\int_B Z(x,t;\eta,s)\Phi(\eta,s;0,0)d\eta ds ,$$

$$G_\varepsilon(x,t) = (t+\delta)^2(\beta - r^2)^3 g_\varepsilon(x,t) .$$

G_ε is differentiable also at the origin where it vanishes. We now proceed to prove (c).

By the definition of L-subcaloricity (see (1)) we have,

$$\iint_{C} f(x, t) L^* G_\varepsilon(x, t) dxdt \geqq 0 . \tag{14}$$

If we prove that

$$\lim_{\varepsilon \to 0} \iint_{C_\varepsilon} f x, t) L^* G(x, t) dxdt = 0 , \tag{15}$$

$$\lim_{\varepsilon \to 0} \iint_{C_\varepsilon} f(x, t) L^* G_\varepsilon(x, t) dxdt = 0 , \tag{16}$$

where $C_\varepsilon = C \cap \{-\varepsilon < t < 0\}$, then (c) follows from (14).

In what follows we denote any positive constant (independent of ε) by the same symbol A. To prove (15) we write

$$\iint_{C_\varepsilon} f L^* G dxdt = \int_{-\varepsilon}^{0} \int_{|x|<\eta} f L^* G dxdt + \int_{-\varepsilon}^{0} \int_{\eta<|x|<\beta} f L^* G dxdt , \tag{17}$$

where η is any positive number smaller than β. Since f is continuous from below at (0, 0) and $f(0, 0) = 0$, the first integral on the right side of (17) tends to zero as $\eta \to 0$, independently of ε.

Here we have made use of (see [7])

$$| L^* G(x, t) | \frac{A}{| t |^{(n+\nu+1)/2}} \exp\left\{\frac{Ar^2}{t}\right\} \text{ for some } 0 \leqq \nu < 1 . \tag{18}$$

The second integral on the right side of (17), for any fixed η, also tends to zero as follows by using (18).

Proof of (16). Proceeding similarly to the proof of (15), we find that all that remains to be proved is that

$$\iint_{C_\varepsilon} | L^* G_\varepsilon | dxdt \leqq A < \infty \tag{19}$$

for all $\varepsilon > 0$ (A is independent of ε). Now,

$$\begin{aligned} -L^* g_\varepsilon = L^*(g - g_\varepsilon) = L^*(Z - Z_\varepsilon) &= L^*\left[Z(x, t)\left(1 - \exp\left\{\frac{(t+\varepsilon)^2}{4t}\right\}\right)\right] \\ &= L^* Z\left(1 - \exp\left\{\frac{(t+\varepsilon)^2}{4t}\right\}\right) - Z\left[\exp\left\{\frac{(t+\varepsilon)^2}{4t}\right\}\right]\left[\frac{t+\varepsilon}{2t} - \frac{(t+\varepsilon)^2}{4t^2}\right] . \end{aligned}$$

Since

$$| L^* Z | \leqq \frac{A}{| t |^{(n+1+\nu)/2}} \exp\left\{\frac{Ar^2}{t}\right\}$$

for some $0 \leqq \nu < 1$, we find, denoting $\left|\frac{t+\varepsilon}{2t} - \frac{(t+\varepsilon)^2}{4t^2}\right|$ shortly by $[\cdots]$,

$$\begin{aligned} | L^* G_\varepsilon | \leqq & \frac{A}{| t |^{(n+1+\nu)/2}} \exp\left\{\frac{Ar^2}{t}\right\} \\ & + \frac{A}{| t |^{n/2}} [\cdots] \exp\left\{\frac{Ar^2}{t}\right\} \exp\left\{\frac{(t+\varepsilon)^2}{4t}\right\} \equiv K_1 + K_2 . \end{aligned} \tag{20}$$

The integral of K_1 is easily seen to be bounded. Hence it remains to evaluate

$$J \equiv \iint_{C_\varepsilon} K_2 dxdt .$$

We split J in the following way:

$$J = \int_{-\varepsilon}^{-\varepsilon/2}\int_B K_2 dxdt + \int_{-\varepsilon/2}^{0}\int_B K_2 dxdt \equiv J_1 + J_2 .$$

As for J_1, $[\cdots] \leqq 1$ and hence $J_1 \leqq A$. As for J_2, $[\cdots] \leqq A\varepsilon^2/t^2$ and hence

$$J_2 \leqq A\varepsilon^2 \int_{-\varepsilon/2}^{0}\left(\int_B \frac{1}{|t|^{n/2}} \exp\left\{\frac{Ar^2}{4t}\right\} dr\right)\frac{1}{t^2} \exp\left\{\frac{A\varepsilon^2}{t}\right\} dt .$$

The inner integral is bounded. Substituting $z = \varepsilon^2/|t|$ we get

$$J_2 \leqq A\int_{2\varepsilon}^{\infty} e^{-Az} dz .$$

We have thus proved that $J = J_1 + J_2 \leqq A$, which completes the proof of (19). Hence, the proof of (16) is completed.

REMARK. The maximum principle for subelliptic functions [4] follows from the maximum principle for subparabolic functions proved in this paper. Indeed, as is easily seen, a weak subelliptic function is necessarily a weak subparabolic function.

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