ON INVARIANT PROBABILITY MEASURES II

J. R. BLUM AND D. L. HANSON

1. **Summary.** We continue the work begun in [1]. In this paper we investigate convergence properties of sequences of probability measures which are asympototically invariant.

2. **Introduction.** Let Ω be a set, $\mathscr{A}$ be a σ-algebra of subsets of Ω, and T be a mapping of Ω onto Ω which is one-to-one and bimeasurable. A set $A \in \mathscr{A}$ is said to be invariant if $A = TA$, a probability measure Q defined on $\mathscr{A}$ is invariant if $Q(A) = Q(TA)$ for all $A \in \mathscr{A}$, and an invariant probability measure P is said to be ergodic if every invariant set A is trivial for P, i.e., if $P(A) = 0$ or $P(A) = 1$. Alternately an invariant probability measure P is ergodic if whenever $P(A) > 0$ we have

$$P\Big(\bigcup_{n=-\infty}^{\infty} T^n A\Big) = 1 .$$

Let $\{Q_n\}$ be a sequence of probability measures defined on $\mathscr{A}$. We shall say that the sequence is asymptotically invariant if $\lim_n [Q_n(A) - Q_n(TA)] = 0$ for every $A \in \mathscr{A}$. In § 3 we give a simple condition which yields convergence of such a sequence to a given ergodic measure. In § 4 an example is given which shows that a reasonable conjecture is in fact false, and further conditions are given which insure uniform convergence of a sequence of asymptotically invariant measures. In the last section we investigate convergence properties of certain sequences of probability density functions.

Throughout the paper we shall have occasion to refer to the following theorem, proved in [1]. We state it here as:

THEOREM 1. *If P and Q are invariant measures which agree on the invariant sets then $P = Q$.*

3. **A convergence theorem.** Let P be an ergodic measure (we shall assume throughout that every measure considered is a probability measure) and let Q be a measure absolutely continuous with respect to P. Define the sequence $\{Q_n\}$ for $n = 1, 2, \cdots$ by the formula

$$Q_n(A) = \frac{1}{n}\sum_{i=0}^{n-1} Q(T^i A)\,, \quad A \in \mathscr{A}$$

Then it is an immediate consequence of the individual ergodic theorem that $\lim_n Q_n(A) = P(A)$ for every $A \in \mathscr{A}$. Clearly the sequence $\{Q_n\}$ is

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asymptotically invariant. It is equally clear that the sequence $\{Q_n\}$ is uniformly absolutely continuous with respect to P. It is the object of this section to show that in fact these properties alone are sufficient to insure convergence to P, and that the averaging is only incidental in this case.
More precisely we have

THEOREM 2. *Let P be an ergodic measure and $\{Q_n\}$ a sequence of measures satisfying*

(i) $\lim_n [Q_n(A) - Q_n(TA)] = 0$ *for every* $A \in \mathscr{A}$.

(ii) *For every $\alpha > 0$ there exists $\delta > 0$ and for every $A \in \mathscr{A}$ an integer $N_{A,\alpha,\delta}$* such that if *$P(A) \leqq \delta$ and $n \geqq N_{A,\alpha,\delta}$ then $Q_n(A) \leqq \alpha$.*
Then $\lim_n Q(A) = P(A)$ *for every* $A \in \mathscr{A}$.

Proof. If the conclusion is false there exists $\alpha_0 > 0$, a set $A \in \mathscr{A}$ and a subsequence $\{Q_n\}$ (to avoid multiple subscripting we shall index subsequences in the same way as the original sequence) such that

$$|Q_n(A) - P(A)| \geqq \alpha_0, \text{ all } n . \tag{3.1}$$

Now let Σ be the class of sets $\{\phi, \Omega, T^nA, T^nA^c, n = 0, \pm 1, \cdots\}$, let $\mathscr{F}$ be the smallest field of sets containing Σ, and let $\mathscr{A}'$ be the smallest σ-algebra containing $\mathscr{F}$. We have $\Sigma \subset \mathscr{F} \subset \mathscr{A}' \subset \mathscr{A}$. Note that if $\beta \in \Sigma$ then $TB \in \Sigma$ and $T^{-1}B \in \Sigma$. Now $\mathscr{F}$ consists of finite intersections of finite unions of sets in Σ and it follows from the properties of T that $\mathscr{F}$ has the same property, i.e., T is bimeasurable with respect to $\mathscr{F}$. Let

$$\mathscr{B} = \{A \mid A \in \mathscr{A}', TA \in \mathscr{A}', T^{-1}A \in \mathscr{A}'\} ,$$

Then $\mathscr{F} \subset \mathscr{B} \subset \mathscr{A}'$. Suppose $A \in \mathscr{B}$. Then $TA^c = (TA)^c$ and $T^{-1}A^c = (T^{-1}A)^c$ and it follows that $A^c \in \mathscr{B}$. Similarly let $\{A_n\}$ be a sequence of elements of $\mathscr{B}$. Then $T \bigcup_n A_n = \bigcup_n TA_n$ and $T^{-1} \bigcup_n A_n = \bigcup_n T^{-1}A_n$. It follows that $\mathscr{B}$ is σ-algebra and consequently $\mathscr{B} = \mathscr{A}'$. Thus T is bimeasurable with respect to $\mathscr{A}'$.

Now $\mathscr{F}$ is generated by a denumerable collection of sets and is itself denumerable. By the usual diagonalization procedure we may extract a further subsequence $\{Q_n\}$ which converges on every set of $\mathscr{F}$. Define $Q(B) = \lim_n Q_n(B)$ for $B \in \mathscr{F}$. Since each Q_n is a measure on $\mathscr{F}$ it follows that Q is finitely additive and monotone on $\mathscr{F}$. Note that Q satisfies (3.1); i.e., $|Q(A) - P(A)| \geqq \alpha_0$. We proceed to show that Q is a probability measure on $\mathscr{F}$. Clearly $Q(\Omega) = 1$. Let $\{B_n\}$ be a sequence of sets in $\mathscr{F}$ which decrease to the null set. Then $\{Q(B_n)\}$ is a non-increasing sequence of numbers. Suppose $\lim_n Q(B_n) = \rho > 0$. Let $\alpha = \rho/2$ and choose an appropriate $\delta > 0$ according to (ii) of the hypothesis. Since $\lim_n P(B_n) = 0$ we may choose B_k so that $P(B_k) < \delta$. Then for n sufficiently large $Q_n(B_k) \leqq \rho/2$ and hence $Q(B_k) < \rho$ which is a contradiction. Thus $\rho = 0$ and Q is completely additive $\mathscr{F}$.

Since Q is a measure on $\mathscr{F}$ we may employ the usual Caratheodory technique to extend Q uniquely to $\mathscr{A}'$. From the hypothesis it follows that Q is invariant on $\mathscr{F}$ and the method used in extending Q to $\mathscr{A}'$ insures that Q is invariant on $\mathscr{A}'$.

Now let $B \in \mathscr{A}'$ and suppose B is invariant. Then $P(B) = 0$ or $P(B) = 1$. Suppose $P(B) = 0$. It is clear from the hypothesis that in that case $Q(B) = 0$ and similarly $Q(B) = 1$ if $P(B) = 1$. Thus Q agrees with P on the invariant elements of $\mathscr{A}'$, and it follows from Theorem 1 that $Q = P$ on $\mathscr{A}'$. In particular $Q(A) = P(A)$, which is a contradiction. The theorem is proved.

Theorem 2 has an interesting corollary. Consider the condition

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} P(T_i A \cap B) = P(A)P(B) \text{ for all } A, B \in \mathscr{A}. \tag{3.2}$$

It is trivial to verify that if (3.2) holds then P is ergodic. Conversely if P is ergodic one may verify (3.2) by using the individual ergodic theorem. However (3.2) is also an immediate consequence of Theorem 2. It is clearly sufficient to consider the case when $P(B) > 0$. In that case define the sequence $\{Q_n\}$ by

$$Q_n(A) = \frac{1}{P(B)} \frac{1}{n} \sum_{i=0}^{n-1} P(T^i A \cap B) .$$

It follows at once that the hypotheses of Theorem 2 apply and (3.2) holds.

4. On uniform convergence. The converse of Theorem 2 evidently holds. If $\lim_n Q_n(A) = P(A)$ for every $A \in \mathscr{A}$ then (i) and (ii) of Theorem 2 are true. Furthermore if $\lim_n Q_n(A) = P(A)$ uniform for $A \in \mathscr{A}$ then $\lim_n [Q_n(A) - Q_n(TA)] = 0$ uniformly for $A \in \mathscr{A}$. It might therefore be reasonable to except that if hypothesis (i) of Theorem 2 is strengthened to $\lim_n [Q_n(A) - Q_n(TA)] = 0$ uniformly for $A \in \mathscr{A}$ we might obtain uniform convergence of Q_n to P. The following example, which is of some independent interest, shows that this is not the case. Let Ω be the unit interval closed on the left and open on the right, and $\mathscr{A}$ the Borel sets. Define T by $Tx = (x + c) \bmod 1$, where c is an irrational number. Then T is one-to-one, onto, and bimeasurable. Let P be Lebesgue measure. Clearly P is invariant and it can be shown that P is ergodic. For $n = 4, 5, \cdots$ let $A_n = [0, 1/n]$. Since $P(A_n) > 0$ we have

$$P\Big(\bigcup_{i=-\infty}^{\infty} T^i A_n\Big) = 1$$

and consequently for each n there is a unique first integer k_n such that

$$1/4 \leqq P\Big(\bigcup_{-k_n}^{k_n} T^i A_n\Big) \leqq 3/4 .$$

Let

$$B_n = \bigcup_{-k_n}^{k_n} T^i A_n$$

and let $b_n = P(B_n)$. Define the sequence $\{Q_n\}$ by $Q_n(A) = P(AB_n)/b_n$. Since $b_n \geqq 1/4$ it follows that the probability measures Q_n are uniformly absolutely continuous with respect to P. Furthermore

$$\begin{aligned} | Q_n(A) - Q_n(TA) | &= (1/b_n) | P(AB_n) - P(TAB_n) | \\ &\leqq 4 | P(TATB_n) - P(TAB_n) | \\ &\leqq 4[P(TA(TB_n - B_n)) + P(TA(B_n - TB_n))] \\ &\leqq 4[P(TB_n - B_n) + P(B_n - TB_n)] . \end{aligned}$$

Now

$$TB_n - B_n \subset T^{k_n+1} A_n$$

and

$$B_n - TB_n \subset T^{-k_n} A_n .$$

Hence

$$| Q_n(A) - Q_n(TA) | \leqq 8P(A_n) = 8/n .$$

Thus

$$\lim_n \sup_{A \in \mathscr{A}} | Q_n(A) - Q_n(TA) | = 0 .$$

On the other hand $Q_n(B_n) - P(B_n) \geqq 1 - 3/4 = 1/4$ and we do not have uniform convergence. The remainder of this section remain is devoted to exhibiting conditions under which one does obtain uniform convergence of the sequence $\{Q_n\}$ to P. For this purpose we shall need several lemmas.

LEMMA 1. *Let P be an invariant measure and Q be an arbitrary measure. Then*

$$\sup_{\substack{i \\ A \in \mathscr{A}}} | Q(A) - Q(T^i A) | \leqq 2 \sup_{A \in \mathscr{A}} |Q(A) - P(A)| .$$

Proof.

$$\begin{aligned} | Q(A) - Q(T^i A) | &\leqq | Q(A) - P(A) + | P(A) - P(T^i A) | \\ &\quad + | P(T^i A) - Q(T^i A) | . \end{aligned}$$

Since P is invariant the middle term on the right vanishes and the lemma follows.

LEMMA 2. *Let P be an ergodic measure and f be a non-negative measurable function which is integrable with respect to P. Then for every $A \in \mathscr{A}$ and $\alpha > 0$ there exist infinitely many values of n such that*

$$\int_{T^n A} f(x)dP(x) < \left(\int_{\Omega} f(x)dP(x)\right) P(A) + \alpha .$$

Proof. Let $\beta = \int_{\Omega} fdP$. If $\beta = 0$ there is nothing to prove. Consequently assume $\beta > 0$. Define the measure Q by $Q(A) = \int_A fdP/\beta$ for $A \in \mathscr{A}$, and the sequence $\{Q_n\}$ by

$$Q_n(A) = \sum_{i=0}^{n-1} Q(T^i A)/n .$$

Since Q is absolutely continuous with respect to P it follows Theorem 2 that $\lim_n Q_n(A) = P(A)$ for $A \in \mathscr{A}$. If the conclusion of the lemma is false then for some $A \in \mathscr{A}$ and $\alpha > 0$ we have for sufficiently large n, $\int_{T^n A} fdP/\beta = Q(T^n A) \geqq P(A) + \alpha/2$. But then $\lim_n Q_n(A) > P(A)$ which is a contradiction.

LEMMA 3. *Let P be an ergodic measure and Q be a measure which is absolutely continuous with respect to P. Then*

$$\sup_{A \in \mathscr{A}} | Q(A) - P(A) | \leqq 2 \sup_{\substack{i \\ A \in \mathscr{A}}} | Q(A) - Q(T^i A) | .$$

Proof. Let f be the Radon-Nikodym derivative of Q with respect to P, and let $B = \{x \mid f(x) \geqq 1\}$. Then

$$\sup_{A \in \mathscr{A}} | Q(A) - P(A) | = \int_B [f(x) - 1]dP(x) .$$

Assume that $P(B) \leqq 1/2$; in the contrary case we can use B^c. Now if i is any integer we have

$$\begin{aligned} \sup_{\substack{i \\ A \in \mathscr{A}}} | Q(A) - Q(T^i A) | &\geqq Q(B) - Q(T^i B) \\ &= [Q(B) - P(B)] - [Q(T^i B) - P(T^i B)] \\ &= \int_B [f - 1]dP - \int_{T^i B} [f - 1]dP . \end{aligned}$$

Hence

$$\begin{aligned} \sup_{A \in \mathscr{A}} |Q(A) - P(A) | &\leqq \sup | Q(A) - Q(T^i A) | + \int_{T^i B} [f - 1]dP \\ &\leqq \sup_{\substack{i \\ A \in \mathscr{A}}} | Q(A) - Q(T^i A) | + \int_{B \cap T^i B} [f - 1]dP . \end{aligned}$$

Now let

$$g(x) = \begin{cases} f(x) - 1, x \in B \\ 0, x \in B^c \end{cases} .$$

Then

$$\int_{B \cap T^i B} [f - 1] dP = \int_{T^i B} g dP .$$

Let $\alpha > 0$. Then from Lemma 2 there exist integers i such that

$$\int_{T^i B} g dP < \left(\int_{\Omega} g dP\right) P(B) + \alpha .$$

But $P(B) \leqq 1/2$ and

$$\int_{\Omega} g dP = \int_{B} [f - 1]\, dP \leqq \sup_{A \in \mathscr{A}} |\, Q(A) - P(A) \,| .$$

Hence

$$\int_{T^i B} g dP < 1/2 \sup_{A \in \mathscr{A}} |\, Q(A) - P(A) \,| + \alpha$$

and we obtain

$$\sup_{A \in \mathscr{A}} |\, Q(A) - P(A) \,| \leqq 2 \sup_{\substack{i \\ A \in \mathscr{A}}} |\, Q(A) - Q(T^i A) \,| + \alpha$$

for abritrary $\alpha > 0$.

THEOREM 3. *Let P be an ergodic measure and let $\{Q_n\}$ be a sequence of measures each of which is absolutely continuous with respect to P. Then*

$$\lim_{n} \sup_{A \in \mathscr{A}} |\, Q_n(A) - P(A) \,| = 0$$

if and only if

$$\lim_{n} \sup_{\substack{i \\ A \in \mathscr{A}}} |\, Q_n(A) - Q_n(T^i A) \,| = 0 .$$

Proof. The theorem follows from Lemmas 1 and 3. Theorem 3 may also be formulated in terms of L_1 convergence. For if f_n is the Radon-Nikodym derivative of Q_n with respect to P, then

$$\sup_{A \in \mathscr{A}} |\, Q_n(A) - P(A) \,| = \int_{\{f_n > 1\}} [f_n - 1] dP = \int_{\{f_n < 1\}} [1 - f_n] dP .$$

Thus

$$\lim_{n} \sup_{A \in \mathscr{A}} |\, Q_n(A) - P(A) \,| = 0$$

if and only if

$$\lim_n \int_\Omega |f_n - 1| \, dP = 0 .$$

Similarly we have

$$\lim_n \sup_{\substack{i \\ A \in \mathscr{A}}} | Q_n(A) - Q_n(T^i A) | = 0$$

if and only if

$$\lim_n \sup_i \int_\Omega |f_n(x) - f_n(T^i x)| \, dP(x) = 0 .$$

Consequently we have the

COROLLARY. *Let P be an ergodic measure, let $\{Q_n\}$ be a sequence of measures each of which is absolutely continuous with respect to P, and let $\{f_n(x)\}$ be the corresponding sequence of Radon-Nikodym derivaties. Then*

$$\lim_n \int_\Omega |f_n(x) - 1| \, dP(x) = 0$$

if and only if

$$\lim_n \sup_i \int_\Omega |f_n(x) - f_n(T^i x)| \, dP(x) = 0 .$$

5. Uniform convergence of densities. In this section we shall be concerned with probability density functions with respect to an ergodic measure P, i.e., a function f is a probability if f is measurable, non-negative, and $\int_\Omega f dP = 1$. We begin with

LEMMA 4. *Let P be an ergodic measure and let f be a probability density with respect to P. Let $\alpha > 0$ and define the sets A and B by*

$$A = \left\{ x \,\middle|\, \sup_{i,j} | f(T^i x) - f(T^j x) | < \alpha \right\}$$

and

$$B = \{ x \mid | f(x) - 1 | > \alpha \} .$$

Then $P(AB) = 0$.

Proof. Let $B' = \{x \mid f(x) > 1 + \alpha\}$. Suppose $P(AB') > 0$. Let $C = \bigcup_{-\infty}^{\infty} T^i(AB')$. Since P is ergodic $P(C) = 1$. If $x \in C$ there exists an integer m such that $T^m x \in AB'$. Hence

$$\sup_{i,j} |f(T^i x) - f(T^j x)| \leqq \sup_{\substack{i,j \\ x \in A}} |f(T^i x) - f(T^j x)| \leqq \alpha .$$

In particular $|f(x) - f(T^m x)| \leqq \alpha$ or $f(x) \geqq f(T^m x) - \alpha$. But $T^m x \in B'$ which means $f(x) > 1$. Since an integer m can be found for each $x \in C$ we have $f(x) > 1$ for all $x \in C$. Then $\int_\Omega f dP = \int_C f dP > 1$, a contradiction to the fact that f is a probability density. A similar argument applies to the set $B'' = \{x \mid f(x) < 1 - \alpha\}$.

THEOREM 4. *Let P be an ergodic measure and let $\{f_n\}$ be a sequence of probability densities with respect to P. Then the following statements are equivalent:*

(i) $P\left(\lim_n \sup_{i,j} |f_n(T^i x) - f_n(T^j x)| = 0\right) > 0$.

(ii) $P\left(\lim_n \sup_{i,j} |f_n(T^i x) - f_n(T^j x)| = 0\right) = 1$.

(iii) $P\left(\lim_n \sup_x |f_n(x) - 1| = 0\right) = 1$.

Proof.

(a) (i) implies (ii). Suppose (i) is true. Let B be a set such that $P(B) > 0$ and such that

$$\lim_n \sup_{i,j} |f_n(T^i x) - f_n(T^j x)| = 0$$

for $x \in B$. But clearly this is also true for

$$x \in C = \bigcup_{i=-\infty}^{\infty} T^i B ,$$

and $P(C) = 1$. Thus (ii) holds.

(b) (ii) implies (iii). Let C be the set of measure one such that for $x \in C$ we have

$$\lim_n \sup_{i,j} |f_n(T^i x) - f_n(T^j)| = 0 .$$

Then for $x \in C$ and every positive integer k there exists a positive integer N_k such that

$$\sup_{i,j} |f_n(T^i x) - f_n(T^j x)| < 1/k$$

for $n \geqq N_k$. Let

$$A_k = \bigcup_{n \geqq N_k} \left\{x \,\middle|\, |f_n(x) - 1| > 1/k\right\} .$$

It follows from Lemma 4 that $P(A_k) = 0$ for $k = 1, 2, \cdots$. Let $A =$

$C - \bigcup_k A_k$. Then $P(A) = 1$, and for $x \in A$ we have $|f_n(x) - 1| \leqq 1/k$ for $n \geqq N_k$. Consequently

$$\lim_n \sup_{x \in A} |f_n(x) - 1| = 0$$

and (iii) follows.

(c) (iii) implies (i). Let A be the set of measure one such that

$$\lim_n \sup_{x \in A} |f_n(x) - 1| = 0 .$$

Let

$$A_0 = \bigcap_{i=-\infty}^{\infty} T^i A .$$

Then $P(A_0) = 1$ and for $x \in A_0$ we have

$$\sup_{i,j} |f_n(T^i x) - f_n(T^j x)| \leqq 2 \sup_i |f_n(T^i x) - 1|$$

and the last quantity approaches zero. Thus (i) holds and the theorem is proved.

Reference

1. J. R. Blum and D. L. Hanson, *On Invariant Probability Measures I*, Pacific J. Math., **10** (1960), 1125–1129.

Sandia Corporation and Indiana University