A COMPLETE SET OF UNITARY INVARIANTS FOR OPERATORS GENERATING FINITE W^*-ALGEBRAS OF TYPE I

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1. Introduction. The principal object of this paper is to give a complete set of unitary invariants for a certain class of operators on Hilbert space. The operators considered are exactly those operators which generate a W^*-algebra which is finite of type I in the terminology of Kaplansky [6]. Such an operator is a direct sum of homogeneous n-normal operators, and a homogeneous n-normal operator can be regarded as a continuous function from a totally disconnected topological space to the full ring of $n \times n$ complex matrices. Thus it was conjectured by Kaplansky that if one could find a suitable set of invariants for complex matrices, one could also solve the unitary equivalence problem for homogeneous n-normal operators, and Brown's solution [2] of the problem in the case $n = 2$ strengthened this belief. A complete set of unitary invariants for $n \times n$ matrices was furnished by Specht. In [10] he showed that there is a collection of traces attached to every $n \times n$ complex matrix such that two matrices are unitarily equivalent if and only if the corresponding traces in this collection are equal. A generalization of the trace of a matrix to n-normal operators is given by Diximier in [3], and it was thus natural to suppose that the generalized Specht invariants would serve for homogeneous n-normal operators. (See page 20, [7].)

Unfortunately, the Specht invariants have the unpleasant feature that they are infinite in number, and for n fixed it seemed likely that some finite subset would serve. Herein it is shown (Theorems 1 and 2) that there is always a subset of less than 4^{n^2} traces which is a complete set of unitary invariants for $n \times n$ complex matrices. Furthermore, the same invariants form a set of orthogonal invariants for $n \times n$ real matrices. (One observes that Specht's proof does not generalize to the real case, due to the failure there of Burnside's theorem.)

The (local) unitary equivalence problem for homogeneous n-normal operators generating the same W^*-algebra is then considered, and it is shown that the same finite number of Dixmier traces is a (local) complete set of unitary invariants for such operators (Theorem 3). Finally the question of global unitary equivalence for operators which generate a finite W^*-algebra of type I is considered, and a global complete set of

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unitary invariants is determined (Theorem 5). In particular, to each such operator A is attached a countable collection of mutually commuting normal operators $N_i(A)$. Then A is unitarily equivalent to B if and only if there is a unitary isomorphism φ between the respective Hilbert spaces which satisfies $\varphi N_i(A)\varphi^{-1} = N_i(B)$ for all i.

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2. $n \times n$ matrices. We first obtain the result for $n \times n$ matrices, or what is the same thing, for operators on an n-dimensional (complex) Hilbert space. The reader is reminded that a ring of operators, or W^*-algebra, is a self-adjoint algebra of operators closed in the weak operator topology acting on a Hilbert space. W^*-algebras are not assumed to contain the identity operator.

Throughout this paper W will denote the free multiplicative semigroup generated by the two free variables x and y. Words in this collection are denoted by $w(x, y)$, and the collection of all words $w(x, y)$ with the property that the sum of the exponents appearing in $w(x, y)$ does not exceed n is denoted by $W(n)$. Also, if A is an operator, the notation $W_A(n)$ denotes the collection of all operators $w(A, A^*)$ with $w(x, y) \in W(n)$.

LEMMA 2.1. *If A is an operator on an n-dimensional Hilbert space, and d is a positive integer such that every operator in $W_A(d+1)$ is a linear combination of operators in $W_A(d)$, then $W_A(d)$ spans the *-algebra V generated by A.*

Proof. Clearly, V consists of all linear combinations of words $w(A, A^*)$ where $w(x, y) \in W$. If $W_A(d)$ does not span V, then there is a word $w(A, A^*)$ which is independent of $W_A(d)$ with the property that the sum of the exponents in $w(x, y)$ is a minimum. A contradiction is easily reached by factoring A (or A^*) out of $w(A, A^*)$ and writing the other factor as a linear combination of operators in $W_A(d)$.

LEMMA 2.2. *If A and V are as before, then V is spanned by the collection of operators $W_A(n^2)$.*

Proof. This follows from Lemma 2.1 and the fact that V can contain at most n^2 linearly independent operators.

We introduce the notation $\sigma(A)$ for the trace of an operator A acting on a finite dimensional space.

THEOREM 1. *If A and B are operators on an n-dimensional*

Hilbert space $\mathscr{H}$, *and* $\sigma[w(A, A^*)] = \sigma[w(B, B^*)]$ *for every word* $w(x, y) \in W(2n^2)$, *then* A *and* B *are unitarily equivalent.*

Proof. Let $R(A)$ and $R(B)$ be the *-algebras generated by A and B respectively. If $A^* = \lambda A$ for some scalar λ, then it is easy to see that $B^* = \lambda B$, so that A and B are normal, and the traces assumed equal are more than enough to guarantee the unitary equivalence of A and B. Thus, we can assume that A and A^* are linearly independent, and it results from the preceding lemmas that there is a basis $\beta(A) = \{w_i(A, A^*)/w_i(x, y) \in \tau\}$ of $R(A)$ such that $\tau \subset W(n^2 - 1)$, $w_1(x, y) = x$ and $w_2(x, y) = y$. It follows easily from the hypothesis and the fact that $\sigma(CC^*) = 0$ implies $C = 0$ for arbitrary C that $\beta(B) = \{w_i(B, B^*)/w_i(x, y) \in \tau\}$ is a basis for $R(B)$. To complete the proof, it suffices to show that if $w_j(x, y)$ is any word in τ and $w_j(A, A^*)\,A = \sum_i \alpha_{ij} w_i(A, A^*)$, $w_j(A, A^*)A^* = \sum_i \gamma_{ij} w_i(A, A^*)$, then $w_j(B, B^*)\,B = \sum_i (\alpha_{ij} w_i(B, B^*)$ and $w_j(B, B^*)B^* = \sum_i \gamma_{ij} w_i(B, B^*)$. For if this is so, then it is clear that if any word $w(A, A^*)$ is formed by multiplications of appropriate powers of A by appropriate powers of A^*, and the corresponding word $w(B, B^*)$ is formed similarly, we will obtain $w(A, A^*) = \sum_i \delta_i w_i(A, A^*)$ and $w(B, B^*) = \sum_i \delta_i w_i(B, B^*)$. This implies that $\sigma[w(A, A^*)] = \sigma[w(B, B^*)]$, and the result will follow from the original theorem of Specht. Thus let $w_j(x, y) \in \tau$ and consider $L = w_j(B, B^*)B - \sum_i \alpha_{ij} w_i(B, B^*)$ and $N = w_j(B, B^*)B - \sum_i \gamma_{ij} w_i(B, B^*)$. Since LL^* and NN^* are linear combinations of words each of which is in $W_B(2n^2)$, it follows from the hypothesis that $\sigma(LL^*) = \sigma(NN^*) = 0$, so that $L = N = 0$, and the proof is complete.

It is easy to see that some of the equalities $\sigma[w(A, A^*)] = \sigma[w(B, B^*)]$, $w(x, y) \in W(2n^2)$, follow from others as a result of properties of the trace function, and thus there are smaller sets of invariants than the set indicated by Theorem 1. For example, it suffices to assume equality for words of the form x^i and $x^i y^j x^k \cdots y^t$ in view of the identity $\sigma(A^*) = [\sigma(A)]^*$ and the fact that the trace of any commutator is zero. Detailed consideration of the case $n = 3$ indicated (see § 5) that it is probably not worthwhile to pursue the question of how many words can thus be dispensed with, so we content ourselves with the observation that there are more distinct sequences of positive integers each having the property that the sum of its terms is at most $2n^2$ than there are traces needed.

THEOREM 2. *There is a complete set of unitary invariants for* $n \times n$ *complex matrices containing fewer than* 4^{n^2} *elements.*

Proof. By induction, the number of distinct sequences of positive integers each having the property that the sum of its terms is a given positive integer k is 2^{k-1}, and one sums the resulting geometric series.

The following corollary extends the above result to real matrices.

COROLLARY. *Any collection of traces which is a complete set of unitary invariants for $n \times n$ complex matrices is also a complete set of orthogonal invariants for $n \times n$ real matrices, and therefore there is a complete set of orthogonal invariants containing fewer than 4^{n^2} elements.*

Proof. We can assume A and B are real $n \times n$ matrices with $UAU^* = B$, U complex unitary. Let $U = R + iJ$ where R and J are real matrices. Then $RA = BR$, $JA = BJ$, $RA^* = B^*R$, $JA^* = B^*J$, and one can choose a real λ such that $S = R + \lambda J$ is nonsingular. It follows that $SAS^{-1} = B$ and $SA^*S^{-1} = B^*$, and the usual construction yields an orthogonal real matrix V such that $VAV^* = B$.

3. Homogeneous n-normal operators. Terms such as abelian projections, equivalence of projections, and homogeneity of projections are taken as defined in [6]. A W^*-algebra R is *n-normal* if it satisfies the identity

$$(*) \qquad \sum \pm X_1 X_2 \cdots X_{2n} = 0$$

where the sum is taken over all permutations on $2n$ objects, and the sign is determined by the parity of the permutation. An n-normal algebra is *homogeneous n-normal* (also called type I_n) if the unit is homogeneous of order n, and an operator is (homogeneous) n-normal if the W^*-algebra it generates is (homogeneous) n-normal.

The imposition of (*) on an algebra R restricts the number of nonzero, orthogonal, equivalent projections in R to a maximum of n, and since every direct summand of an n-normal algebra contains an abelian projection [2], it follows easily that any n-normal algebra is a direct sum of algebras of type I_k where $k \leqq n$. Kaplansky [5] and Brown [2] gave a structure theory for these algebras, and according to [2], if R is a homogeneous n-normal algebra, then R is unitarily isomorphic to the algebra of all $n \times n$ matrices with entries from an abelian W^*-algebra Z' containing 1. By applying the representation theorem for abelian C^*-algebras to Z', one obtains that Z' is C^*-isomorphic to the C^*-algebra $C(\mathfrak{X})$ of all continuous complex-valued functions on a compact Hausdorff space $\mathfrak{X}$. Now Z' is weakly closed, and it has been shown that this gives $\mathfrak{X}$ the additional properties that the closure of every open set is open, and the compact open sets form a base for the topology [11]. It results that R is C^*-isomorphic to the C^*-algebra $M_n(\mathfrak{X})$ of all continuous functions from $\mathfrak{X}$ to the full ring M_n of $n \times n$ complex matrices, where $\| A(\cdot) \| = \sup_{t \in \mathfrak{X}} \| A(t) \|$. If $A = (A_{ij}) \in R$, then A corresponds to the function $A(\cdot) \in M_n(\mathfrak{X})$ whose value at $t \in \mathfrak{X}$ is $A(t) = (a_{ij}(t))$, where $a_{ij}(\cdot)$ is the function in $C(\mathfrak{X})$ corresponding to A_{ij} in Z'. See [2] for details.

It will be useful from here on to have a notation for a diagonal matrix which has the same entry E in every position on the main diagonal and zeros elsewhere. We hereby adopt the notation Diag (E) for this matrix whose size will always be clear from context.

Dixmier [3] has demonstrated the existence of a unique center valued trace-like function (called by him *"l'application ≒ canonique"*) defined on finite W^*-algebras. This function, which we denote by $D(\cdot)$, is linear, a unitary invariant, constant on the center of the algebra, preserves the *-operation, and has the property that if A_λ is a net of uniformly bounded operators converging weakly to A, then $D(A_\lambda)$ converges weakly to $D(A)$. For more information concerning this function, see [3].

Our intention is to use operators of the form $D[w(A, A^*)]$ as unitary invariants for operators A generating finite W^*-algebras of type I. To this end, let R be a homogeneous n-normal W^*-algebra. Then as mentioned, we can take R to be the W^*-algebra of all $n \times n$ matrices over an abelian algebra Z', and any $A \in R$ has the form $A = (A_{ij})$. Thus one can define a mapping $A \to \text{Diag}\,(1/n \sum_i A_{ii})$ from R to the center of R, and it is not hard to see that this mapping has all of the afore mentioned properties of $D(\cdot)$, and in addition is globally weakly and uniformly continuous. From these considerations and from Theorem 3, page 267, [3], it follows that $D(A) = \text{Diag}\,(1/n \sum_i A_{ii})$. The usefulness of this fact is that under the C^*-isomorphism between R and $M_n(\mathfrak{X})$, any operator $D[w(A, A^*)] \in R$ corresponds to the function Diag $(1/n\, \sigma[w(A(\cdot), A^*(\cdot)])$ in $M_n(\mathfrak{X})$.

We solve the local unitary equivalence problem first in the simplest case where the n-normal operators A and B under consideration are both in the homogeneous W^*-algebra R and where A is assumed to generate R. We begin by supposing that $D[w(A, A^*)] = D[w(B, B^*)]$ for $w(x, y) \in C$, where C is any collection of words $w(x, y)$ furnishing a complete set of unitary invariants for $n \times n$ matrices. Then it follows that $A(t)$ is unitarily equivalent to $B(t)$ for each $t \in \mathfrak{X}$, and as a result $D[w(A, A^*)] = D[w(B, B^*)]$ for all words $w(x, y) \in W$. At this point we make two observations. The first is that the problem of finding a unitary operator in R satisfying $UAU^* = B$ is equivalent to being able to choose the unitary matrix $U(t)$ implementing the equivalence of $A(t)$ and $B(t)$ in a continuous fashion. The second follows: consider the mapping $\phi: p(A, A^*) \leftrightarrow p(B, B^*)$ between the algebraic *-algebras generated by A and B. It is clear, since ϕ was shown above to be trace preserving, that ϕ is in fact a norm-preserving *-algebra isomorphism, and as such can be extended to a C^*-isomorphism between the C^*-algebras generated by A and B. Thus the question of whether A is unitarily equivalent to B is exactly the question of whether the isomorphism ϕ

is implemented by a unitary operator. To answer this question, it is useful to consider the problem locally in $M_n(\mathfrak{X})$, where the appropriate AW^*-algebras are more accessible. The following lemmas lead to the result.

LEMMA 3.1. *Suppose A generates the homogeneous n-normal W^*-algebra R, and as such corresponds to the function $A(\cdot) \in M_n(\mathfrak{X})$. If $\mathscr{U}$ is any compact open subset of $\mathfrak{X}$, then there is a $t \in \mathscr{U}$ such that $A(t)$ generates the full*-algebra M_n of $n \times n$ complex matrices.*

Proof. Suppose there is some compact open $\mathscr{U}$ such that for each $t \in \mathscr{U}$, the *-algebra of matrices generated by $A(t)$ [which is, of course, a direct sum of factors] is not the full algebra M_n. Let $(^{**})$ be the polynomial identity obtained from $(^*)$ by replacing n by $n-1$. It follows from the facts about polynomial identities in [1] that for any $t \in \mathscr{U}$, the *-algebra of matrices generated by $A(t)$ satisfies $(^{**})$. Now the characteristic function of $\mathscr{U}$ corresponds to a projection E' in Z', and thus the operator $E = \text{Diag}(E')$ is a central projection in R. What we have just proved is that the algebraic *-algebra generated by EA satisfies $(^{**})$. It follows by continuity that ER satisfies $(^{**})$ also, which is impossible because ER is homogeneous n-normal with R and thus contains n^2 matrix units which cannot satisfy $(^{**})$.

The next lemma uses the fact that if $\mathscr{U}$ is any compact open subset of $\mathfrak{X}$, then the algebra $M_n(\mathscr{U})$ of continuous functions from $\mathscr{U}$ to M_n, considered as a normed algebra with sup norm, is a C^*-algebra and, in fact, an AW^*-algebra.

LEMMA 3.2. *Suppose that $A(\cdot)$ and $B(\cdot)$ are elements of $M_n(\mathfrak{X})$ such that for every word $w(x, y) \in W$ and for every $t \in \mathfrak{X}$, $\sigma[w(A(t), A^*(t))] = \sigma[w(B(t), B^*(t))]$. Suppose further that $s \in \mathfrak{X}$ is such that $A(s)$ generates M_n. Then there is a compact open set $\mathscr{U}$ containing s and a unitary element $V(\cdot) \in M_n(\mathscr{U})$ such that for each $t \in \mathscr{U}$, $B(t) = V(t)A(t)V^*(t)$.*

Proof. Since $A(s)$ generates M_n, there are n^2 words $w_i(x, y)$ such that the matrices $w_i(A(s), A^*(s))$ are linearly independent, and we can take $w_1(A(s), A^*(s)) = A(s)$ and $w_2(A(s), A^*(s)) = A^*(s)$. Since $A(\cdot)$ can be regarded as a matrix with continuous functions as entries, there is a compact open set $\mathscr{U}$ containing s such that for $t \in \mathscr{U}$, the n^2 matrices $w_i(A(t), A^*(t))$ remain linearly independent. Thus for each $t \in \mathscr{U}$, one obtains, just as in the proof of Theorem 1, that the n^2 matrices $w_i(B(t), B^*(t))$ are linearly independent. Furthermore, if

$$w_i(A(t), A^*(t))w_j(A(t), A^*(t)) = \sum_k d_{ij}^k(t)w_k(A(t), A^*(t)) ,$$

then the same equation holds with A everywhere replaced by B. Now any element $T(\cdot) \in M_n(\mathscr{U})$ is such that $T(t) = \sum_{i=1}^{n^2} c_i(t) w_i(A(t), A^*(t))$ for $t \in \mathscr{U}$, where the $c_i(\cdot)$ are uniquely determined continuous complex-valued functions on $\mathscr{U}$. This is the crucial fact, for it allows us to define the mapping

$$\phi: \sum c_i(\cdot) w_i(A(\cdot), A^*(\cdot)) \to \sum c_i(\cdot) w_i(B(\cdot), B^*(\cdot))$$

of $M_n(\mathscr{U})$ onto itself. Using the facts mentioned above, it is not hard to see that ϕ is in fact a *-algebra automorphism of $M_n(\mathscr{U})$ which leaves the center elementwise fixed. It follows from Theorem 3, [5] that there is a unitary element $V(\cdot) \in M_n(\mathscr{U})$ implementing ϕ, and since ϕ maps $A(\cdot)$ to $B(\cdot)$ we have the desired result. (It is perhaps worth remarking that instead of using Kaplansky's theorem above, the desired unitary element $V(\cdot)$ could have been constructed via a construction from standard algebra.)

THEOREM 3. *Suppose A is a homogeneous n-normal operator generating the W^*-algebra R, and suppose B is any operator in R. Suppose also that C is any collection of words $w(x, y)$ with the property that the associated traces form a complete set of unitary invariants for $n \times n$ complex matrices. Finally, suppose that $D[w(A, A^*)] = D[w(B, B^*)]$ for each $w(x, y) \in C$. Then there is a unitary element $U \in R$ such that $UAU^* = B$.*

Proof. Consider collections of nonzero, orthogonal, central projections E_λ in R for which there exists some unitary operator V_λ in R satisfying $BE_\lambda = V_\lambda A V_\lambda^* E_\lambda$. By Zorn one obtains a maximal collection $\{E_\lambda\}$. Let $F = \sup_\lambda \{E_\lambda\} = \sum_\lambda E_\lambda$. To show that F is the unit of R, suppose not. Then the central projection $1 - F$ is nonzero, and thus is of the form Diag (E') where E' is a projection in Z'. Now E' corresponds to the characteristic function of a compact open subset $\mathscr{U}_1$ of $\mathfrak{X}$, and by Lemmas 3.1 and 3.2 we can drop down to a compact open subset $\mathscr{U}$ of $\mathscr{U}_1$ such that there is a unitary $V(\cdot) \in M_n(\mathscr{U})$ with $B(t) = V(t)A(t)V^*(t)$ for every $t \in \mathscr{U}$. Then of course $V(\cdot)$ can be extended to a unitary element $V(\cdot) \in M_n(\mathfrak{X})$, and if E is the central projection in R corresponding to the set $\mathscr{U}$, we have $BE = VAV^*E$. This contradicts the maximality of the collection $\{E_\lambda\}$, and thus $\sum_\lambda E_\lambda = 1$. If U is defined as $\sum_\lambda E_\lambda \cdot V_\lambda$, it is an easy matter to verify that U is a unitary operator in R and that $UAU^* = B$.

We can remove the restriction in Theorem 3 that A generates a homogeneous algebra, provided we maintain the requirement that A generates a W^*-algebra of type I, finite. For it is known that any such algebra R is a direct sum $\sum_{i \in I} \oplus R_i$ where I is some (perhaps

infinite) subset of the positive integers and each R_i is homogeneous i-normal, and it is easy to see that the Dixmier trace $D(\cdot)$ on R is the direct sum of the functions $D_i(\cdot)$ defined as previously on the homogeneous summands R_i. Thus we get immediately.

THEOREM 4. *If A generates the finite W^*-algebra R of type I, B is any operator in R, and $D[w(A, A^*)] = D[w(B, B^*)]$ for each $w(x, y) \in W$, then there is a unitary operator $U \in R$ such that $UAU^* = B$.*

4. Global unitary equivalence. We now shift our attention from the question of local unitary equivalence to the question of global unitary equivalence. In other words, if A and B are operators on the Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ respectively, and each generates a finite W^*-algebra of type I, we wish to set forth necessary and sufficient conditions for the existence of a unitary isomorphism φ mapping $\mathscr{H}$ onto $\mathscr{K}$ and satisfying $\varphi A\varphi^{-1} = B$. Suppose A and B generate the W^*-algebras $R(A)$ and $R(B)$ respectively. Let $D_a(\cdot)$ be the Dixmier trace defined on the algebra $R(A)$, and similarly let $D_b(\cdot)$ be the trace on the algebra $R(B)$. In order to eventually arrive at a complete set of unitary invariants for A and B, we must set forth conditions which will ensure that the algebras $R(A)$ and $R(B)$ are unitarily equivalent, and the following lemma begins this program.

LEMMA 4.1. *If A and B are as above, and if there is a unitary isomorphism φ mapping $\mathscr{H}$ onto $\mathscr{K}$ such that $\varphi D_a[w(A, A^*)]\varphi^{-1} = D_b[w(B, B^*)]$ for each $w(x, y) \in W$, then $\varphi Z(A)\varphi^{-1} = Z(B)$, where $Z(A)$ and $Z(B)$ are the centers of $R(A)$ and $R(B)$ respectively.*

Proof. It clearly suffices to demonstrate that the W^*-algebra $Z(\Gamma)$ which the collection $\Gamma = \{D_a[w(A, A^*)]/w(x, y) \in W\}$ generates is $Z(A)$. By the fundmental density theorem, $Z(\Gamma)$ is the ultraweak ("*ultrafaible*") closure of the algebraic *-algebra generated by Γ, and $R(A)$ is the ultraweak closure of the algebraic *-algebra generated by A. If $K \in Z(A)$, then there is a net of polynomials $p_\lambda(A, A^*)$ converging ultraweakly to K, and $D_a[p_\lambda(A, A^*)]$ converges ultraweakly to $D_a(K) = K$.

One conjectures that the number of traces required in the previous lemma can be reduced somewhat if it is assumed that $R(A)$ and $R(B)$ are both n-normal for some n. (This is equivalent to supposing that there exists a positive integer n such that neither $R(A)$ nor $R(B)$ has a nontrivial i-homogeneous summand with $i > n$.) The following lemma affirms this conjecture.

LEMMA 4.2. *If $R(A)$ and $R(B)$ are both n-normal W^*-algebras, C is any collection of words $w(x, y)$ furnishing a complete set of unitary*

invariants for $n \times n$ complex matrices, and $\varphi D_a[w(A, A^*)]\varphi^{-1} = D_b[w(B, B^*)]$ for $w(x, y) \in C$, then $\varphi Z(A)\varphi^{-1} = Z(B)$.

Proof. As before, it suffices to demonstrate that the W^*-algebra which the collection $\Omega = \{D_a[w(A, A^*)]/w(x, y) \in C\}$ generates is $Z(A)$. On the other hand, we know from Lemma 4.1 that the collection $\Gamma = \{D_a[w(A, A^*)]/w(x, y) \in W\}$ generates $Z(A)$. Thus it suffices to show that Ω and Γ generate the same W^*-algebra, or even less, the same C^*-algebra. Now $R(A)$ is a direct sum of homogeneous algebras R_i, and thus is C^*-isomorphic to an algebra of the form $\sum_{i \in I} \oplus M_i(\mathfrak{X}_i)$ where I is a subset of the first n positive integers. Consider the compact Hausdorff space $\mathfrak{X} = \bigcup_{i \in I} \mathfrak{X}_i$, defined by agreeing that a set $\mathscr{U}$ is open in $\mathfrak{X}$ if and only if $\mathscr{U} \cap \mathfrak{X}_i$ is open in $\mathfrak{X}_i$ for each $i \in I$. Clearly, $Z(A)$ is C^*-isomorphic to the C^*-algebra F of all continuous complex-valued functions on $\mathfrak{X}$. For any $w(x, y) \in W$, let $f_w \in F$ be the element corresponding to $D_a[w(A, A^*)]$ in $Z(A)$. If A corresponds to $\sum_{i \in I} \oplus A_i(\cdot)$, then it is easy to see that $f_w/\mathfrak{X}_i$ (the restriction of f_w to $\mathfrak{X}_i$) is equal to $1/i\, \sigma[w(A_i(\cdot), A_i^*(\cdot))]$. We want to prove that $\Omega_1 = \{f_w \in F/w(x, y) \in C\}$ and $\Gamma_1 = \{f_w \in F/w(x, y) \in W\}$ generate the same closed subalgebra of F. Define $g_w/\mathfrak{X}_i = i \cdot f_w/\mathfrak{X}_i$ for $i \in I$. Then Ω_1 and Γ_1 generate the same closed subalgebra of F if and only if $\Omega_2 = \{g_w/w(x, y) \in C\}$ and $\Gamma_2 = \{g_w/w(x, y) \in W\}$ do also. We apply the Stone-Weierstrass Theorem to prove that Ω_2 and Γ_2 do indeed generate the same C^*-subalgebra of F, and thus complete the argument. Suppose $t_1, t_2 \in \mathfrak{X}$, and suppose $g_w(t_1) = g_w(t_2)$ for each $w(x, y) \in C$. Say $t_1 \in \mathfrak{X}_i$ and $t_2 \in \mathfrak{X}_j$. Then the matrices $A_i(t_1)$ and $A_j(t_2)$ can both be made into $n \times n$ matrices by forming direct sums with appropriate sized zero matrices, and one sees by virtue of the hypothesis on C that the resulting $n \times n$ matrices are unitarily equivalent. Thus $g_w(t_1) = g_w(t_2)$ for all $w(x, y) \in W$, and it remains only to show that if $t \in \mathfrak{X}_i$ is such that $g_w(t) = 0$ for all $w(x, y) \in C$, then $g_w(t) = 0$ for all $w(x, y) \in W$. This is immediate, however, since then $A_i(t)$ is unitarily equivalent to the zero matrix and thus is equal to zero.

One knows (Theorem 3, [2]) that two n-homogeneous W^*-algebras whose centers are unitarily isomorphic are then themselves unitarily isomorphic, and the next lemma gives conditions under which the homogeneous summands of two finite W^*-algebras of type I can be aligned.

LEMMA 4.3. *Suppose A generates the n-homogeneous W^*-algebra $R(A)$ with center Z, and suppose B generates the m-homogeneous W^*-algebra $R(B)$ whose center is also Z. Suppose also that $D_a[w(A, A^*)] = D_b[w(B, B^*)]$ for each $w(x, y) \in W(\max[2n^2, 2m^2])$. Then $m = n$.*

Proof. We can regard $R(A)$ and $R(B)$ as matrix algebras over the common center Z, and if $\mathfrak{X}$ is taken to be the maximal ideal space of Z, we obtain C^*-isomorphisms of Z onto $C(\mathfrak{X})$, $R(A)$ onto $M_n(\mathfrak{X})$, and $R(B)$ onto $M_m(\mathfrak{X})$. If $A \leftrightarrow A(\cdot)$ and $B \leftrightarrow B(\cdot)$ under these isomorphisms, then as usual $D_a[w(A, A^*)]$ and $D_b[w(B, B^*)]$ correspond respectively to the $n \times n$ matrix Diag $(1/n\,\sigma[w(A(\cdot), A^*(\cdot))])$ and the $m \times m$ matrix Diag $(1/m\sigma[w(B(\cdot), B^*(\cdot))])$. It follows from the hypothesis and the isomorphism between $C(\mathfrak{X})$ and Z that $m/n\sigma[w(A(t),A^*(t))]=\sigma[w(B(t),B^*(t))]$ for each $t \in \mathfrak{X}$ and for each $w(x, y) \in W(\max [2n^2, 2m^2])$. By Lemma 3.1 we can choose a point $s \in \mathfrak{X}$ such that $A(s)$ generates M_n, and thus find n^2 words $w_i(x, y) \in W(n^2)$ such that the matrices $w_i(A(s), A^*(s))$ are linearly independent. Proceeding just as in the proof of Theorem 1, one concludes that the n^2 matrices $w_i(B(s), B^*(s))$ are linearly independent, and thus $m \geqq n$. The result follows by symmetry.

We are now in a position to prove the central result of the paper.

THEOREM 5. *Suppose A is an operator acting on the Hilbert space $\mathscr{H}$ and generating the finite W^*-algebra $R(A)$ of type I. Let $D_a(\cdot)$ be the Dixmier trace defined on the algebra $R(A)$. Then A is unitarily equivalent to an operator B acting on the Hilbert space $\mathscr{K}$ if and only if*

(1) *B generates a W^*-algebra $R(B)$ which is finite of type I, and*

(2) *there is a unitary isomorphism φ of the Hilbert space $\mathscr{H}$ onto the Hilbert space $\mathscr{K}$ satisfying $\varphi D_a[w(A, A^*)]\varphi^{-1} = D_b[w(B, B^*)]$ for each $w(x, y) \in W$, where $D_b(\cdot)$ is the Dixmier trace on the algebra $R(B)$.*

Proof. If there is a unitary isomorphism φ of $\mathscr{H}$ onto $\mathscr{K}$ satisfying $\varphi A\varphi^{-1} = B$, then $\varphi R(A)\varphi^{-1} = R(B)$, and $\varphi Z(A)\varphi^{-1} = Z(B)$, where $Z(A)$ and $Z(B)$ are the centers of the respective algebras $R(A)$ and $R(B)$. That $\varphi D_a[w(A, A^*)]\varphi^{-1} = D_b[w(B, B^*)]$ for each $w(x, y) \in W$ follows easily from the uniqueness of the Dixmier trace (Theorem 3, page 267, [3]). Going the other way, suppose B generates the finite W^*-algebra $R(B)$ of type I, and suppose there is a unitary isomorphism φ of $\mathscr{H}$ onto $\mathscr{K}$ such that $\varphi D_a[w(A, A^*)]\varphi^{-1} = D_b[w(B, B^*)]$ for $w(x, y) \in W$. Let A_1 be the operator $\varphi A\varphi^{-1}$ acting on $\mathscr{K}$, and suppose A_1 generates the W^*-algebra $R(A_1)$ with center $Z(A_1)$ and Dixmier trace $D_{a_1}(\cdot)$. Then another uniqueness argument shows that $D_b[w(B, B^*)] = D_{a_1}[w(A_1, A_1^*)]$ for $w(x, y) \in W$, and from Lemma 4.1 we obtain $Z(B) = Z(A_1)$. Write $R(A_1) = \sum_{i \in I} \oplus R_i$ and $R(B) = \sum_{j \in J} \oplus T_j$ where R_i and T_i are homogeneous i-normal algebras, and I and J are subsets of the positive integers. (It is convenient to regard the above direct sums as internal in this situation, and we do so.) If E_i is the unit of the algebra R_i, then at least E_i is a central projection in $Z(B)$, and we show that E_i

is in fact the unit of T_i (thus proving $I \subset J$). Write $E_i = \sum_{j \in J} \oplus F_j$, where each F_j is a central projection in T_j. If F_j is nonzero, then the algebra $F_jR(B) = F_jT_j$ is j-homogeneous, and since $F_j \leqq E_i$, $F_jR(A_1) = F_jR_i$ is i-homogeneous. It is easy to see that Lemma 4.3 is applicable to the operators F_jA_1 and F_jB, and it results that $j = i$ and hence $E_i = F_i$. Thus E_i is dominated by the unit of the algebra T_i, and from symmetry considerations one can conclude that E_i is the unit of the algebra T_i and that $I = J$. In other words, for $i \in I$, the homogeneous algebras R_i and T_i have the common center $E_iZ(B)$. If Theorem 3, [2], is now applied for each $i \in I$, there results a unitary operator V such that $VR(A_1)V^* = R(B)$ and V commutes with $Z(A_1)$. Consider $A_2 = VA_1V^*$ which clearly generates the W^*-algebra $R(B)$. This fact and another uniqueness argument yield $D_b[w(A_2, A_2^*)] = D_b[w(B, B^*)]$ for $w(x, y) \in W$, and it follows from Theorem 4 that there is a unitary operator $Y \in R(B)$ satisfying $YA_2Y^* = B$. Thus $(YV\varphi)A(YV\varphi)^{-1} = B$, and the argument is complete.

As was the case in Lemma 4.2, if it is known that the operators A and B of Theorem 5 each generate an n-normal W^*-algebra, it is possible to get by with assumptions on fewer traces.

THEOREM 6. *If the W^*-algebras $R(A)$ and $R(B)$ of Theorem 5 are n-normal, and $\varphi D_a[w(A, A^*)]\varphi^{-1} = D_b[w(B, B^*)]$ for $w(x, y) \in W(2n^2)$, then A and B are unitarily equivalent.*

The proof is similar to that of Theorem 5 and is omitted.

5. Remarks.

(1) It is of interest to ask how near the upper bound 4^{n^2} obtained in § 2 is to the least upper bound on the number of traces required to form a complete set of unitary invariants for $n \times n$ matrices. In this connection, it is well known that for $n = 2$ the collection $\{\sigma(A), \sigma(A^2), \sigma(AA^*)\}$ is a complete set of invariants, and also the author has shown [9] that for $n = 3$, a collection of nine traces suffices. Thus it would appear that the estimate 4^{n^2} is not very good, but it is thought that to obtain any substantial improvement, a completely new approach will be necessary.

(2) (*Added in proof*) I wish to acknowledge my indebtedness to Don Deckard for pointing out a slight simplification in my original proof of Theorem 1 which enabled me to reduce the number of traces needed from 16^{n^2} to 4^{n^2}.

(3) Whether the sets of invariants provided by Theorem 5 and 6 are satisfactory is, of course, open to question. We present the following facts in support of their reasonableness:

(a) Two normal operators are unitarily equivalent if and only

if their associated spectral measures are, and thus a solution of this simpler problem requires the simultaneous unitary equivalence of the corresponding elements in two infinite families of commuting projections.

(b) Brown [2] has given a complete set of unitary invariants for homogeneous binormal operators which requires the simultaneous unitary equivalence of four commuting normal functions of the operators. Furthermore, he shows by example that one cannot do away with the simultaneity of this unitary equivalence.

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