THE MINIMUM BOUNDARY FOR AN ANALYTIC POLYHEDRON

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1. Introduction. If K is a compact subset of a complex analytic manifold M, then for each f, holomorphic (analytic) in a neighborhood of K, the maximum modulus of f over K is attained on the topological boundary of K. If the complex dimension of M is greater than 1, it may happen that there are proper closed subsets of the topological boundary on which each holomorphic f attains its maximum modulus. In case there are sufficiently many holomorhic functions on M to separate the points of the manifold, a general result of Šilov [6] states that there is a uniquely determined smallest closed subset of K which has this maximum modulus property. This set is known as the *Šilov boundary* for the ring of functions holomorphic in a neighborhood of K.

The Šilov theorem is valid for separating algebras of continuous complex-valued functions on a compact space, and has nothing to do with analyticity as such. Many years earlier, the pioneering work on maximum modulus sets for rings of analytic functions had been done by Bergman [1; 2; 3]. He considered principally domains in C^n which were bounded by a finite number of analytic hypersurfaces, and for these he introduced a *distinguished boundary surface*. For a wide class of such domains, he showed that his distinguished boundary was a smallest maximum modulus set. References to more recent work on these problems may be found in the second author's paper [9].

In this paper we consider the case in which M is a Stein manifold, e.g., a domain of holomorphy in C^n, and the compact set K is an *analytic polyhedron*. This means that K has the form

$$K = \{m \in D; |f_j| \leqq 1, j = 1, \cdots, k\}$$

where $f_1, \cdots, f_k$ are holomorphic functions on some open subset D of the manifold M. We consider those subsets S of K which have the property that, for every f holomorphic in a neighborhood of K, the maximum modulus of f over K is *attained* on the subset S. We prove that among all such subsets S there is a smallest one, which we call the *minimum boundary* for the polyhedron. The closure of this minimum boundary is (of course) the Šilov boundary for the ring of functions holomorphic on K. While it is difficult to give an explicit description of this Šilov boundary, such a description can be given for the minimum boundary. It is obtained by deleting from K all connected local analytic

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varieties of positive dimension which are contained in K. In terms of the functions f_j which define the polyhedron K, the description is as follows. Let $m_0 \in K$, and let $j_1, \cdots, j_r$ be those indices j for which $|f_j(m_0)| = 1$. Then m_0 belongs to the minimum boundary if and only if m_0 is an isolated point of the set (variety)

$$V = \{m \in D; f_{j_i}(m) = f_{j_i}(m_0), i = 1, \cdots, r\} .$$

In [7], the first author identified the minimum boundary for the ring of functions on K which are uniform limits of functions holomorphic in a neighborhood of K. The arguments there made essential use of a fundamental theorem of Bishop: if A is a uniformly closed separating algebra of continuous complex-valued functions on a compact metric space $K(1 \in A)$, there is a smallest subset of K on which each function in A attains its maximum modulus. This minimum boundary for A consists of the points of K at which some function in A "peaks", i.e., those points with the property that there is a function in A which attains its maximum modulus at the point, and at no other point. The description of the minimum boundary for the ring of functions holomorphic on K is exactly the one which was shown in [7] to define the minimum boundary for the uniform closure of the ring. In particular, it results that the minimum boundary for the polyhedron consists of those points of K at which some holomorphic function peaks (over K), or, it consists of the peak points for functions which are uniformly approximable by holomorphic functions. We shall use methods from [7], and we shall make essential use of Bishop's general existence theorem. This theorem is not directly applicable to the ring of functions holomorphic in a neighborhood of K, since the ring is not generally closed under uniform convergence. However, by using a technique from the second author's paper [8], based upon the solution of the second Cousin problem for Stein manifolds, we are able to show that the ring of holomorphic functions has the same peak points as does its uniform closure.

2. Notation and basic definitions. A *Stein manifold* is a d-dimensional complex analytic manifold M such that

(i) the global holomorphic (analytic) functions on M separate the points of M;

(ii) for each point $m \in M$ there are global holomorphic functions $h_1, \cdots, h_d$ which serve as coordinates in some neighborhood of m;

(iii) M is a countable union of compact sets;

(iv) if K is any compact set in M, the set of points $m \in M$ such that $|h(m)| \leqq \sup_K |h|$ for every holomorphic function h on M is also compact.

Let M be a Stein manifold. An *analytic polyhedron* in M is a subset P of M such that

(i) P is compact

(ii) $P = \{m \in D; |f_j(m)| \leqq 1, j = 1, \cdots, k\}$, where D is an open subset of M and $f_1, \cdots, f_k$ are holomorphic functions on D.

If P is an analytic polyhedron in M we denote by $H(P)$ the set of all functions f on P such that f is the restriction to P of a function holomorphic in some neighborhood of P. We denote by $A(P)$ the class of functions on P which can be approximated, uniformly on P, by functions in $H(P)$. Both $H(P)$ and $A(P)$ are algebras of continuous functions on P. Our task is to prove the existence of a smallest subset S of P such that, for every h in $H(P)$, the maximum modulus of h over P is attained on the set S, and then to describe the set S explicitly. For this we need to discuss briefly boundaries for algebras of continuous functions.

Let X be a compact Hausdorff space, and let A be a collection of continuous complex-valued functions on X. *A boundary* (*of* X) *for* A is a subset S of X such that

$$\max_S |f| = \max_X |f| , \qquad f \in A$$

that is, a subset S of X such that for each f in A the maximum modulus of f over X is *attained* at some point of S. If

(a) A is a complex-linear algebra, using pointwise operations

(b) the constant functions are in A

(c) the functions in A separate the points of X,

then among all *closed* boundaries for A there is a smallest one, i.e., the intersection of all closed boundaries for A is a boundary for A. This smallest closed boundary for A we call the *Šilov boundary* for A, in honor of G. E. Šilov who first proved its existence [6]. If, in addition to (a), (b), and (c) we have

(d) A is closed under uniform convergence

(e) X is metrizable

then the intersection of *all* boundaries for A is a boundary for A. This smallest of all boundaries we shall call the *minimum boundary* for A. Its existence was proved by Bishop [4], who also showed that it consists of those points $x \in X$ which are *peak points* for A. We call x a peak point for A if there exists an f in A such that $|f(x)| > |f(y)|$ for all points y in X which are different from x. Evidently, the Šilov boundary for A is the closure of the minimum boundary for A.

Both $H(P)$ and $A(P)$ are algebras of continuous functions on the compact space P which satisfy conditions (a), (b), (c) above. Since $A(P)$ is the uniform closure of $H(P)$, these algebras have the same Šilov boundary. Now P is metrizable, as is easy to see from the countability condition imposed on a Stein manifold. Therefore, there exists a minimum boundary for the uniformly closed algebra $A(P)$. The general function algebra results cited above do not guarantee the existence of a minimum boundary for $H(P)$; thus, as we proceed now to prove the existence of such a boundary, we shall make heavy use of explicit properties of analytic polyhedra.

3. The minimum boundary. Now suppose we are given the analytic polyhedron

$$P = \{m \in D; |f_j(m)| \leqq 1, j = 1, \cdots, k\}$$

in the Stein manifold M. With each point m_0 in the polyhedron P we associate an analytic variety V_{m_0} in the ambient neighborhood D by

$$V_{m_0} = \{m \in D; f_{j_i}(m) = f_{j_i}(m_0), i = 1, \cdots, r\} \tag{3.1}$$

where $j_1, \cdots, j_r$ are those indices j such that $|f_j(m_0)| = 1$.

THEOREM 1. *Let $m_0 \in P$ and suppose m_0 is a local peak point for the algebra $A(P)$. Then m_0 is an isolated point of the variety V_{m_0} (3.1).*

Proof. This is proved in [7; Theorem 4.1]; here, we merely outline the proof. By the statement that m_0 is a local peak point for the algebra $A(P)$ we mean that there is a function $h \in A(P)$ and a neighborhood N of m_0 such that $f(m_0) = 1$ and $|f(m)| < 1$ for all other points in $N \cap P$. Given such an f and N, we may assume that those functions f_j (occurring in the definition of P) which are of modulus less than 1 at m_0 are of modulus less than 1 on the open set N. Then $N \cap V_{m_0} = V$ is an analytic variety in N and this variety is contained in the polyhedron P. Since f is a uniform limit on P of functions holomorphic in a neighborhood of P, f is 'analytic' on the variety V. Also, f has a local maximum over V at the point m_0. The maximum modulus principle for analytic varieties then states that m_0 is an isolated point of V.

THEOREM 2. *Let $m_0 \in P$ and suppose that m_0 is an isolated point of the variety V_{m_0} (3.1). Then m_0 is a peak point for the algebra $H(P)$.*

Proof. Let $g_i = \frac{1}{2}(1 + \overline{f_{j_i}(m_0)} f_{j_i})$, $i = 1, \cdots, r$. Then the function g_i is bounded by 1 on P, and has the value 1 at any point of P where it is of modulus 1. Now let $h = g_1 \cdots g_r$. Then h is bounded by 1 on P, has the value 1 at any point of P where it is of modulus 1, and, furthermore, the set of points in P where h has the value 1 is precisely the intersection of P with the analytic variety V_{m_0}. Thus, m_0 is an isolated point of the set on which $h = 1$. We should also remark that h is holomorphic in the open set D which occurs in the definition of P.

Since M is a Stein manifold, we can find functions $h_1, \cdots h_n$, holomorphic on all of M, such that the map $m \to (h_1(m), \cdots, h_n(m))$ is biholomorphic on P, and the image of P under this map is a polynomial convex subset of C^n. See [5]. Now we consider the map

$$\phi(m) = (h_1(m), \cdots, h_n(m), h(m))$$

from D into C^{n+1}. This map is biholomorphic, and the set $K = \phi(P)$ is polynomial convex. (polynomial convexity of K means that if z is a point of C^{n+1} which is not in K, there exists a polynomial p in $(n+1)$ variables such that $|p(z)| > \sup_K |p|$.) Let $z^0 = \phi(m_0)$. The coordinate function z_{n+1} is bounded by 1 on K, is equal to one at any point of K where it is of modulus 1, and z^0 is an isolated point of $K \cap \{z_{n+1} = 1\}$.

Choose a neighborhood U of the point z^0 such that for any point $z = (z_1, \cdots, z_{n+1})$ in $U \cap K$ which is different from z^0 we have $|z_{n+1}| < 1$. By [8; Theorem 2.4] there exists a function g, holomorphic in a neighborhood W of K, such that g never vanishes on $W - U$ and $g/(1 - z_{n+1})$ is holomorphic and without zeros on $W \cap U$.

On the neighborhood $W \cap U$, the function g has the form $g = (z_{n+1} - 1)k$, where k is holomorhpic and never vanishes on $W \cap U$. In particular $k(z^0) \neq 0$. Thus, by shrinking U, we may assume that k has a single-valued logarithm; say $k = e^l$ where l is holomorphic on $W \cap U$. Since K is polynomial convex, it is an intersection of polynomial convex open sets. Therefore, it may be assumed that W is polynomial convex, and hence is itself a Stein manifold. Similarly, a small contraction of W will assure that the part of the intersection of W with the hyperplane $\{z_{n+1} = 1\}$ which lies in U is a closed analytic variety in W. Since W is a Stein manifold, there is a holomorphic function p on W such that $p = l$ on that variety [5]. Now let $\tilde{g} = ge^{-p}$. Then $\tilde{g}$ is holomorphic on W and has no zeros on $W - U$. Also $\tilde{g} = (z_{n+1} - 1)ke^{-p}$ on $W \cap U$. Now $ke^{-p} = ke^{-l} = 1$ on that part of the hyperplane $\{z_{n+1} = 1\}$ which lies in $W \cap U$. Thus, on $W \cap U$, $ke^{-p} = 1 + (z_{n+1} - 1)\tilde{k}$. Finally we have a function $\tilde{g}$, holomorphic on W, which has no zeros on $W - U$ and has the form

$$\tilde{g} = (z_{n+1} - 1) + (z_{n+1} - 1)^2\tilde{k}$$

on $W \cap U$ ($\tilde{k}$ holomorphic on $W \cap U$).

From $\tilde{g}$ we shall now construct a function holomorphic in a neighborhood of K, which has the property that its maximum modulus over K is attained at z^0 and at no other point of K. This function will be an analytic function of $\tilde{g}$. Thus we shall examine the range of $\tilde{g}$ on K. The crucial fact about the set $\tilde{g}(K)$ is that it lies outside a simply connected domain in the plane which has an analytic boundary containing the origin. To see this, we argue as follows. Choose a neighborhood N of z^0 such that the function $\tilde{k}$ is bounded on $W \cap N$. We shall then have

$$|\tilde{g} - (z_{n+1} - 1)| \leqq c\,|z_{n+1} - 1|^2 \quad \text{on} \quad W \cap N$$

where c is some positive constant. The range of $(z_{n+1} - 1)$ on K lies in the left half-plane. For each point w in the left halfplane, we consider the disc

$$|\zeta - w| \leqq c\,|w|^2 .$$

It is easy to see that there is an analytic curve γ through the origin such that for all points w near the origin this disc lies to the left of γ. Because, a short computation shows that the envelope of the family of discs is an analytic curve through the origin. We may assume that N is sufficiently small that for any point z in $K \cap N$ the point $w = z_{n+1} - 1$ has this property. Therefore, the range of $\tilde{g}$ on $N \cap K$ lies to the left of γ. On $K - N$ the function $\tilde{g}$ has no zeros. Choose $\varepsilon > 0$ such that $|\tilde{g}| > \varepsilon$ on $K - N$. If ε is sufficiently small, the circle $|w| = \varepsilon$ will intersect γ in precisely two points. Let D be the domain bounded by γ and $|w| = \varepsilon$. Let τ be the Rieman map of the complement of D onto the unit disc, which carries the origin onto 1. Since γ is an analytic curve, τ will extend analytically across that part of γ which is on the boundary of D. The composition $\tau \circ \tilde{g} = F$ is then holomorphic on a neighborhood of K, is of modulus less than 1 on $K - \{z^0\}$, and $F(z^0) = 1$.

Now we return to the polyhedron P which we mapped holomorphically onto K via the map ϕ. If we let $f = F \circ \phi$, then f is holomorphic in a neighborhood of P, and the maximum modulus of f over P is attained at m_0 and at no other point of P. Thus m_0 is a peak point for the algebra $H(P)$.

COROLLARY. *Let M be a Stein manifold, and Let P be an analytic polyhedron in M. Then there is a (unique) smallest subset S of P such that for every function f, holomorphic in a neighborhood of P, the maximum modulus of f over P is attained on the set S. A necessary and sufficient condition that a point m_0 in P should belong to this minimum boundary S is any one of the following.*

(i) *m_0 is a peak point for the algebra $H(P)$.*

(ii) *m_0 is a peak point for the algebra $A(P)$.*

(iii) *m_0 is a local peak point for the algebra $H(P)$.*

(iv) *m_0 is a local peak point for the algebra $A(P)$.*

(v) *There is no connected local analytic variety of positive dimension which passes through m_0 and is contained in P.*

(vi) *m_0 is an isolated point of the variety V_{m_0}, defined by* (3.1).

Proof. By Bishop's theorem [4], there is a minimum boundary for the algebra $A(P)$, and it consists of those points of P which are peak point for $A(P)$. In Theorem 1 we showed that any local peak point for the algebra $A(P)$ satisfies (vi). Indeed, the proof showed that the point satisfies (v), which clearly implies (vi). Theorem 2 states that (vi) implies (i). From this it is clear that the six statements about m_0 are equivalent. Furthermore, it is evident that the minimum boundary for $A(P)$ is a boundary for $H(P)$; and, since each point of this boundary is a peak point for $H(P)$, this boundary is the smallest boundary for $H(P)$.

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