# $K$-POLAR POLYNOMIALS

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**1. Introduction.** The complex polynomials

$$f(z) = \sum_{j=0}^{n} \binom{n}{j} a_j z^j , \qquad g(z) = \sum_{j=0}^{n} \binom{n}{j} b_j z^j \tag{1}$$

are called apolar if their coefficients satisfy the condition

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} a_{n-j} b_j = 0 .$$

A well known property of apolar polynomials is given [1] by

GRACE'S THEOREM. *If the polynomials $f(z)$ and $g(z)$ are apolar, then every circular domain containing all the zeros of one polynomial also contains at least one zero of the other.*

The term "circular domain" is used here to denote any region into which the circle $|z| \leqq 1$ can be transformed by a nonsingular linear fractional transformation

$$w = (ax + b)/(cx + d) ;$$

that is, a circular domain is a closed interior of a circle, a closed exterior of a circle, or a closed half plane.

It is natural to ask whether similar but more stringent conditions on the coefficients of (1) will insure that every circular domain containing all the zeros of one polynomial also contains at least $k$ zeros of the other when $k$ is integer greater than unity. We show here that this is the case. Our results can be stated more easily if we first make the

DEFINITION. The polynomials (1) are called $k$-polar ($1 \leqq k \leqq n$, $k$ an integer) if their coefficients satisfy the $k^2$ conditions

$$\sum_{j=0}^{n-k+1} (-1)^j \binom{n-k+1}{j} a_{s-j} b_{j+h} = 0 \tag{2}$$
$$(h = 0, \cdots, k-1;\ s = n, \cdots, n-k+1) .$$

We shall show that $k$-polarity of the polynomials (1) is sufficient to insure that the desired relation between their zeros does hold.

It is apparent that when $k$ is relatively large in comparison with

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$n$ there is only a restricted class of polynomials $f(z)$ for which $k$-polar polynomials $g(z)$ can exist. We shall show that when $2k + 1 \geqq n$ the $k$-polarity of the polynomials (1) is both necessary and sufficient for them to have a common, repeated zero such that the multiplicities, $p$ and $q$, with which this zero occurs in the two polynomials satisfy the inequalities $p \geqq k$, $q \geqq k$, $p + q \geqq n + k$.

**2. The polar derivative.** To prove our principal results, we shall need a lemma concerning the $(n-1)$st degree polynomial

$$f_\zeta(z) = nf(z) + (\zeta - z)f'(z) = n \sum_{j=0}^{n-1} \binom{n-1}{j} (a_{j+1}\zeta + a_j)z^j .$$

This polynomial is called the "polar derivative of $f(z)$" or the "derivative of $f(z)$ with respect to $\zeta$". It can be obtain [2] from $f(z)$ as follows:

By the linear transformation

$$z = L(w) = (aw + b)/(cw + d) \qquad (bc - ad = 1) \tag{3}$$

transform $f(z)$ into the polynomial

$$F(w) = (cw + d)^n f(L(w)) ; \tag{4}$$

then to the derivative $F'(w)$ apply the inverse transformation $w = L^{-1}(z)$, obtaining $f_\zeta(z)$. If $c \neq 0$, then $\zeta = a/c = L(\infty)$; if $c = 0$, then $\zeta = \infty = L(\infty)$.

We shall refer to the polynomial $F(w)$ defined by (4) as the transform by (3) of the polynomial $f(z)$. It is important to observe [2] that the zeros of the transform $F(w)$ are the transforms by $w = L^{-1}(z)$ of those of $f(z)$.

LEMMA 1. *Let the nth degree polynomial $f(z)$ have $n-k$ zeros in $|z| < 1$ and $k$ zeros in $|z| > r$, where $r > 1$. Then there is a point $\zeta$ (not unique) such that $f_\zeta(z)$ has exactly $k-1$ zeros in $|z| > r$.*

*Proof.* Form $F(w)$ by applying to $f(z)$ the transformation

$$z = L(w) = (\zeta w - 1)/(w - \zeta) \qquad (1 < \zeta < r) ,$$

which takes $|z| < 1$ into $|w| < 1$ and takes $|z| > r$ into the circle

$$K_2 \colon \ |w - C_2| < R_2 , \qquad C_2 = \frac{\zeta(r^2 - 1)}{r^2 - \zeta^2} , \qquad R_2 = \frac{r(\zeta^2 - 1)}{r^2 - \zeta^2} .$$

Now $F(w)$ has $k$ zeros in $K_2$ and $n-k$ zeros in $|w| < 1$. Since the maximum modulus of these latter $n-k$ zeros is less than unity, we can choose $\mu < 1$ such that these zeros also lie in $|w| < \mu$. Let $\rho =$

$(1+\mu)/2$. The circle

$$K_1\colon\ |w-(\rho-1)|<\rho$$

contains the circle $|w|<\mu$; for the line segment connecting $w=-\mu$ and $w=\mu$ is a diameter of $|w|<\mu$ and is contained in the line segment connecting $w=-1$ and $w=\mu$, which is a diameter of $K_1$. Thus $K_1$ contains $n-k$ zeros of $F(w)$. Applying the Walsh two circle theorem [5] to $K_1$ and $K_2$, we find that the zeros of $F'(w)$ lie in $K_1$, $K_2$, and the third circle

$$K\colon\ |w-C|<R\,,\quad C=\frac{(n-k)C_2+k(\rho-1)}{n}\,,\quad R=\frac{(n-k)R_2+k\rho}{n}\,.$$

Furthermore, it is an immediate consequence of the two circle theorem that if the boundaries of $K$ and $K_2$ do not intersect then there are exactly $k-1$ zeros of $F'(w)$ in $K_2$. The condition for the non-intersection of these two circles is

$$C_2-C>R_2+R\,.$$

This condition is equivalent to

$$n(C_2-C-R_2-R)=kC_2-R_2(2n-k)-k(2\rho-1)>0\,,$$

and this last inequality is equivalent to

$$\phi(\zeta)=k(r^2-1)\zeta-(2n-k)r(\zeta^2-1)-k(2\rho-1)(r^2-\zeta^2)>0\,.$$

Now

$$\phi(1)=2k(r^2-1)(1-\rho)>0\,,$$

since $r>1$ and $\rho<1$. Since $\phi(\zeta)$ is a real, continuous function of $\zeta$, it follows that $\phi(\zeta)>0$ in an interval $1\leqq\zeta\leqq 1+\varepsilon$, where $\varepsilon>0$. For any value of $\zeta$ in this interval, $K$ and $K_2$ do not intersect and $F'(w)$ has exactly $k-1$ zeros in $K_2$. Now the zeros of $f_\zeta(z)$ are the transforms by $z=L(w)$ of those of $F'(w)$. Hence exactly $k-1$ of them lie in the transform of $K_2$, that is, in $|z|>r$.

3. **Properties of the $k$-polarity conditions.** To prove our principal results, we shall need to establish first some properties of the $k$-polarity conditions.

LEMMA 2. *For $k=1,\cdots,n+1$, the polynomials* (1) *can be written in the form*

$$f(z)=\sum_{j=0}^{k-1}\binom{k-1}{j}z^j f_{k,j},$$

where

$$f_{k,j} = f_{k,j}(z) = \sum_{i=0}^{n-k+1} \binom{n-k+1}{i} a_{i+j} z^i \qquad (j = 0, \cdots, k-1) .$$

The functions $f_{k,j}$ satisfy the relation

$$zf_{k+1,j+1} + f_{k+1,j} = f_{k,j} .$$

*Proof.* We show first the property of the functions $f_{k,j}$ which is stated last in the lemma. Using the definition of $f_{k,j}$ and a well known property of the binomial coefficients, we write

$$\begin{aligned} zf_{k+1,j+1} + f_{k+1,j} &= \sum_{i=0}^{n-k} \binom{n-k}{i} a_{i+j+1} z^{i+1} + \sum_{i=0}^{n-k} \binom{n-k}{i} a_{i+j} z^i \\ &= \sum_{i=0}^{n-k+1} \binom{n-k}{i-1} a_{i+j} z^i + \sum_{i=0}^{n-k} \binom{n-k}{i} a_{i+j} z^i \\ &= a_{n-k+1+j} z^{n-k+1} + \sum_{i=1}^{n-k} \left[ \binom{n-k}{i-1} + \binom{n-k}{i} \right] a_{i+j} z^i + a_j \\ &= \sum_{i=0}^{n-k+1} \binom{n-k+1}{i} a_{i+j} z^i \\ &= f_{k,j} . \end{aligned}$$

The proof of the first part of the lemma is by induction. It is true when $k = 1$, since $f_{1,0}$ reduces at once to $f(z)$. For any $k > 1$ we have

$$\begin{aligned} \sum_{j=0}^{k} \binom{k}{j} z^j f_{k+1,j} &= z^k f_{k+1,k} + \sum_{j=1}^{k-1} \left[ \binom{k-1}{j-1} + \binom{k-1}{j} \right] z^j f_{k+1,j} + f_{k+1,0} \\ &= \sum_{j=0}^{k-1} \binom{k-1}{j} z^{j+1} f_{k+1,j+1} + \sum_{j=0}^{k-1} \binom{k-1}{i} z^j f_{k+1,j} \\ &= \sum_{j=0}^{k-1} \binom{k-1}{j} z^j (zf_{k+1,j+1} + f_{k+1,j}) \\ &= \sum_{j=0}^{k-1} \binom{k-1}{j} z^j f_{k,j} . \end{aligned}$$

If the first part of the lemma is true when $k$ is replaced by $k - 1$, then the last expression above is equal to $f(z)$. It follows that the lemma is true for all values of $k$.

LEMMA 3. *The polynomials* (1) *are $k$-polar if and only if the polynomials $f_{k,j}$ and $g_{k,i}$ are apolar for all $i = 0, \cdots, k-1$ and $j = 0, \cdots, k-1$.*

*Proof.* The proof is immediate, since applying the apolarity condition to all $f_{k,j}$ and $g_{k,i}$ yields conditions (2) at once.

LEMMA 4. *The k-polarity conditions* (2) *are invariant under non-singular linear transformations of the polynomials* (1).

*Proof.* Since any non-singular linear transformation is equivalent to a succession of transformations of the forms $z = \gamma w (\gamma \neq 0)$, $z = 1/w$, $z = w + \gamma$, the lemma can be established by showing the invariance of (2) for each of these special forms.

Each sum in (2) is invariant under magnifications and rotations. For applying $z = \gamma w$ to both $f(z)$ and $g(z)$ replaces $a_{s-j}$ by $\gamma^{s-j}a_{s-j}$ and $b_{j+h}$ by $\gamma^{j+h}b_{j+h}$, whence each term of the sum is multiplied by $\gamma^{s-j}\gamma^{j+h} = \gamma^{s+h}$. The sum, therefore, remains equal to zero.

Under the transformation $z = 1/w$, the polynomials (1) are carried into

$$F(w) = \sum_{j=0}^{n} \binom{n}{j} A_j w^j \text{ and } F(w) = \sum_{j=0}^{n} \binom{n}{j} B_j w^j ,$$

where $A_j = a_{n-j}$ and $B_j = b_{n-j} (j = 0, \cdots, n)$. The entire set of conditions (2) is invariant under this transformation. For we have

$$\begin{aligned}
&\sum_{j=0}^{n-k+1} (-1)^j \binom{n-k+1}{j} A_{s-j} B_{j+h} \\
&= \sum_{j=0}^{n-k+1} (-1)_j \binom{n-k+1}{j} a_{n-s+j} b_{n-h-j} \\
&= \sum_{j=n-k+1}^{0} (-1)^{n-k+1-j} \binom{n-k+1}{n-k+1-j} a_{2n-s-k+1-j} b_{k-h-1+j} \\
&= (-1)^{n-k+1} \sum_{j=0}^{n-k+1} (-1)^j \binom{n-k+1}{j} a_{s'-j} b_{h'+j} ,
\end{aligned}$$

where $s' = 2n - s - k + 1$ and $h' = k - h - 1$, so that $s'$ takes on the values $n - k + 1, \cdots, n$ and $h'$ takes on the values $k - 1, \cdots, 0$. Hence satisfaction of (2) by $f(z)$ and $g(z)$ insures satisfaction of (2) by $F(w)$ and $G(w)$.

To prove the invariance of (2) under translations, we first make use of Lemma 2 and show that if $f(z)$ is transformed into $F(w)$ by $z = w + \gamma$, then each polynomial $F_{k,j}(w)$ is a linear combination of the polynomials $f_{k,j}(w + c) (j = 0, \cdots, k - 1)$. Precisely, we show that the equations

$$F_{k,j}(w) = \sum_{h=0}^{k-j-1} \binom{k-j-1}{h} \gamma^h f_{k,j+h}(w+\gamma) \qquad (j = 0, \cdots, k-1) \tag{5}$$

hold for every $k = 1, \cdots, n + 1$. The proof is by induction on $k$. We show first that the desired relations hold for the highest value of $k$, that is, $k = n + 1$. When $k = n + 1$, the equations defining $f_{k,j}$ and $F_{k,j}$ reduce to $f_{n+1,j} = a_j$ and $F_{n+1,j} = A_j$, so that (5) becomes

$$A_j = \sum_{h=0}^{n-j} \binom{n-j}{h} \gamma^h a_{j+h} \qquad (j = 0, \cdots, k-1) .$$

To see that this holds, we find $A_j$ by collecting the coefficients of the powers of $w$ in the polynomial $f(w+\gamma)$. We have

$$\begin{aligned}
F(w) = f(w+\gamma) &= \sum_{i=0}^{n} \binom{n}{i} a_i (w+\gamma)^i \\
&= \sum_{i=0}^{n} \binom{n}{i} a_i \sum_{j=0}^{i} \binom{i}{j} \gamma^{i-j} w^j \\
&= \sum_{j=0}^{n} \binom{n}{j} w^j \sum_{i=j}^{n} \frac{\binom{n}{i}\binom{i}{j}}{\binom{n}{j}} \gamma^{i-j} \\
&= \sum_{j=0}^{n} \binom{n}{j} w^j \sum_{i=j}^{n} \binom{n-j}{i-j} a_i \gamma^{i-j} ,
\end{aligned}$$

so that

$$A_j = \sum_{i=j}^{n} \binom{n-j}{i-j} a_i \gamma^{i-j} = \sum_{h=0}^{n-j} \binom{n-j}{h} \gamma^h a_{j+h} .$$

Thus equations (5) hold when $k = n+1$. Next, we assume that they hold for general index $k+1$ and show that they also hold for index $k$. For convenience, we shall temporarily let $\phi_{k,j}$ denote $f_{k,j}(w+\gamma)$. ($F_{k,j}$ will denote $F_{k,j}(w)$ as usual.) Using the property of $F_{k,j}$ and $f_{k,j}$ established in Lemma 2 and assuming that equations (5) hold for $k+1$, we can write

$$\begin{aligned}
F_{k,j} &= wF_{k+1,j+1} + F_{k+1,j} \\
&= w \sum_{h=0}^{k-j-1} \binom{k-j-1}{h} \gamma^h \phi_{k+1,j+h+1} + \sum_{h=0}^{k-j} \binom{k-j}{h} \gamma^h \phi_{k+1,j+h} \\
&= w \sum_{h=0}^{k-j-1} \binom{k-j-1}{h} \gamma^h \phi_{k+1,j+h+1} \\
&\quad + \sum_{h=0}^{k-j-1} \left\{ \binom{k-j-1}{h-1} + \binom{k-j-1}{h} \right\} \gamma^h \phi_{k+1,j+h} \\
&\quad + \phi_{k+1,j} + \gamma^{k-j} \phi_{k+1,k} \\
&= w \sum_{h=1}^{k-j-1} \binom{k-j-1}{h} \gamma^h \phi_{k+1,j+h+1} \\
&\quad + \sum_{h=0}^{k-j-1} \binom{k-j-1}{h} \gamma^{h+1} \phi_{k+1,j+h+1} \\
&\quad + \sum_{h=0}^{k-j-1} \binom{k-j-1}{h} \gamma^h \phi_{k+1,j+h} \\
&= \sum_{h=0}^{k-j-1} \binom{k-j-1}{h} \gamma^h \{ (w+\gamma) \phi_{k+1,j+h+1} + \phi_{k+1,j+h} \} \\
&= \sum_{h=0}^{k-j-1} \binom{k-j-1}{h} \gamma^h \phi_{k,j+h} .
\end{aligned}$$

Thus equations (5) hold for $k = n+1, \cdots, 1$.

We have now established that each polynomial $F_{k,j}(w)$ is a linear combination of the polynomials $f_{k,j}(w+\gamma)$. To finish the proof of the invariance of (2) under translations, we recall the known facts (i) that apolarity is invariant under translations of the polynomials [1] and (ii) that if $E_1$ and $E_2$ are two sets of polynomials such that every polynomial of $E_1$ is apolar to every polynomial of $E_2$, then any linear combination of polynomials from $E_1$ is apolar to any linear combination of polynomials [1] from $E_2$. By Lemma 3, the $k$-polarity of $f(z)$ and $g(z)$ implies the apolarity of each polynomial in the set $E_1$: $\{f_{k,k-1}(w), \cdots, f_{k,0}(w)\}$ to each polynomial in the set $E_2$: $\{g_{k,k-1}(w), \cdots, g_{k,0}(w)\}$. Property (i) therefore implies that all polynomials of $E_1'$: $\{f_{k,k-1}(w+\gamma), \cdots, f_{k,0}(w+\gamma)$ are apolar to all polynomials of $E_2'$: $\{g_{k,k-1}(w+\gamma), \cdots, g_{k,0}(w+\gamma)\}$. We have just shown that each polynomial $F_{k,j}(w)$ is a linear combination of polynomials from $E_1'$ and each $G_{k,j}(w)$ is a linear combination of polynomials from $E_2'$. Thus property (ii) implies the apolarity of all the $F_{k,j}(w)$ to all the $G_{k,j}(w)$. Lemma 3 now gives the $k$-polarity of $F(w)$ and $G(w)$.

For convenience, we shall denote the repeated polar derivative $f_{\zeta_1\cdots\zeta_s}(z)$ as $f(z; \zeta, s)$.

LEMMA 5. *Let $k \geqq 2$ and $1 \leqq s \leqq k-1$. The $k$-polarity of $f(z)$ and $g(z)$ is necessary and sufficient for the $(k-s)$-polarity of the repeated polar derivatives $f(z; \zeta, s)$ and $g(z; \eta, s)$ for arbitrary points $\zeta_1, \cdots, \zeta_s$ and $\eta_1, \cdots, \eta_s$.*

*Proof.* It suffices to make the proof for $s = 1$, since re-application of this proof will then establish the lemma for all values of $s$ concerned. Letting $\phi(z) = f(z; \zeta, 1)$ and $\psi(z) = g(z; \eta, 1)$, we have

$$\phi(z) = \sum_{j=0}^{n-1} \binom{n-1}{j} (a_{j+1}\zeta_1 + a_j) z^j ,$$

whence

$$\begin{aligned} \phi_{k-1,j}(z) &= \sum_{i=0}^{n-k+1} \binom{n-k+1}{i} (a_{i+j}\zeta_1 + a_{i+j}) z^i \\ &= \zeta_1 f_{k,j+1}(z) + f_{k,j}(z) \qquad (j = 0, \cdots, k-2) . \end{aligned}$$

Similarly,

$$\psi_{k-1,j}(z) = \eta_1 g_{k,j+1}(z) + g_{k,j}(z) \qquad (j = 0, \cdots, k-2) .$$

The $k$-polarity of $f(z)$ and $g(z)$ implies the apolarity of both $f_{k,j+1}(z)$ and $f_{k,j}(z)$ to both $g_{k+1,j}(z)$ and $g_{k,j}(z)$. Thus $\phi_{k-1,j}(z)$ and $\psi_{k-1,j}(z)$, which are linear combinations of these polynomials, are apolar. The $(k-1)$-polarity of $\phi(z)$ and $\psi(z)$ now follows at once from Lemma 3.

If, on the other hand, $f(z; \zeta, 1) = f_{\zeta_1}(z)$ and $g(z; \eta, 1) = g_{\eta_1}(z)$ are $(k-1)$-polar for arbitrary values of $\zeta_1$ and $\eta_1$, then, in particular, both $f_0(z)$ and $f_\infty(z)$ are $(k-1)$-polar to both $g_0(z)$ and $g_\infty(z)$. For convenience, denote $f(z; \zeta, 1)$ by $\phi(z; \zeta_1)$ and $g(z; \eta, 1)$ by $\psi(z; \eta_1)$. We have

$$\phi(z; 0) = f_0(z) = n \sum_{j=0}^{n-1} \binom{n-1}{j} a_j z^j ,$$
$$\phi(z; \infty) = f_\infty(z) = n \sum_{j=0}^{n-1} \binom{n-1}{j} a_{j+1} z^j ,$$

whence

$$\phi_{k-1,j}(z; 0) = n \sum_{i=0}^{(n-1)-(k+1)-1} \binom{n-k+1}{i} a_{i+j} z^i = f_{k,j}(z) ,$$
$$\phi_{k-1,j}(z, \infty) = n \sum_{i=0}^{n-k+1} \binom{n-1}{i} a_{i+j+1} z^i = f_{k,j+1}(z) \qquad (j = 0, \cdots, k-2) .$$

Similarly,

$$\psi_{k-1,j}(z; 0) = g_{k,j}(z) ,$$
$$\psi_{k-1,j}(z; \infty) = g_{k,j+1}(z) \qquad (j = 0, \cdots, k-2) .$$

The $(k-1)$-polarity of $\phi(z; 0)$ and $\phi(z; \infty)$ to $\psi(z; 0)$ and $\psi(z; \infty)$ implies the apolarity of all the $\phi_{k-1,j}(z; 0)$ and $\phi_{k-1,j}(z; \infty)$ to all the $\psi_{k-1,j}(z; 0)$ and $\psi_{k-1,j}(z; \infty)$ for $j = 0, \cdots, k-2$. The apolarity of all the $f_{k,j}(z)$ to all the $g_{k,j}(z)$ for $j = 0, \cdots, k-1$ now follows at once. This, in turn, implies the $k$-polarity of $f(z)$ and $g(z)$.

LEMMA 6. *Let the nth degree polynomials $f(z)$ and $g(z)$ be k-polar. Let $\zeta_1, \cdots, \zeta_{n-k+1}$ be the zeros of any one of the polynomials $g_{k,k-1}(z), \cdots, g_{k,0}(z)$, and let all these zeros be finite. Then $f(z; \zeta, n-k+1)$ vanishes identically.*

*Proof.* If $\zeta_1, \cdots, \zeta_{n-k+1}$ are the zeros of

$$g_{k,h}(z) = \sum_{i=0}^{n-k+1} \binom{n-k+1}{i} b_{i+h} z^i ,$$

then their elementary symmetric functions can be expressed in terms of the coefficients. Let $S_0^{(m)} = 1$ and for $i = 1, \cdots, m$ let $S_i^{(m)}$ denote the sum of all possible products of $\zeta_1, \cdots, \zeta_m$ taken $i$ at a time. (Note that $b_{n-k+1+h} \neq 0$ since it is the leading coefficient of $g_{k,h}(z)$ and all the zeros of this polynomial are finite.) We have

$$S_i^{(n-k+1)} = (-1)^i \binom{n-k+1}{i} \frac{b_{n-k+1+h-i}}{b_{n-k+1+h}} \qquad (i = 0, \cdots, n-k+1) .$$

Thus we can write

$$b_{n-k+1+h} \sum_{i=0}^{n-k+1} a_{j+i} S_i^{(n-k+1)}$$
$$\sum_{i=0}^{n-k+1} (-1)^i \binom{n-k+1}{i} a_{j+i} b_{n-k+1+h-i} \qquad (j = 0, \cdots, k-1).$$

Now for each value of $j$, the last expression above is the left side of one of the conditions (2). Consequently the $k$-polarity of $f(z)$ and $g(z)$ gives

$$\sum_{i=0}^{n-k+1} a_{j+i} S_i^{(n-k+1)} = 0 \qquad (j = 0, \cdots, k-1).$$

Now it is known [3] that $f(z; \zeta, t)$ can be written in the form

$$f(z; \zeta, t) = \frac{n!}{(n-t)!} \sum_{j=0}^{n-t} \binom{n-t}{j} \sum_{i=0}^{t} a_{j+i} S_i^{(t)} z^j .$$

For $t = n - k + 1$, we have just shown that the sum which appears in the coefficient of each $z^j$ vanishes. Consequently, we have $f(z; \zeta, n - k + 1) \equiv 0$, as we wanted to show.

## 4. K-polar polynomials.

We are now ready to prove our principal results.

THEOREM 1. *If the polynomials $f(z)$ and $g(z)$ are $k$-polar, then every circular domain containing all the zeros of one polynomial also contains at least $k$ zeros of the other.*

*Proof.* The proof will be by induction on $k$. For $k = 1$, this theorem is simply Grace's theorem.

Assume that the theorem holds for $k = m$, and let $f(z)$ and $g(z)$ be $(m + 1)$-polar. Let $C$ be a closed circular domain containing all the zeros of $g(z)$ and exactly $s$ zeros of $f(z)$. Then $C$ is contained in an open circular domain $C'$ whose closure also contains exactly $s$ zeros of $f(z)$. Since $k$-polarity is invariant under linear transformations, we can take $|z| > 1$ as $C'$. Then for a suitable $r > 1$, all the zeros of $g(z)$ and exactly $s$ zeros of $f(z)$ lie in $|z| > r$, while $n - s$ zeros of $f(z)$ lie in $|z| < 1$. By Lemma 1, therefore, there is a point $\zeta$ such that exactly $s - 1$ zeros of $f_\zeta(z)$ lie in $|z| > r$. Also, by Laguerre's theorem [2], all the zeros of $g_\eta(z)$ lie in $|z| > r$ whenever $\eta$ lies in $|z| \leqq r$. By Lemma 5, the $(m + 1)$-polarity of $f(z)$ and $g(z)$ implies the $m$-polarity of $f_\zeta(z)$ and $g_\eta(z)$ for all values of $\zeta$ and $\eta$. Consequently, the assumption that the theorem holds for $k = m$ implies that the circular domain $|z| > r$, which contains all the zeros of $g_\eta(z)$, must contain at least $m$ zeros of $f_\zeta(z)$. Since we know that this domain contains exactly $s-1$ zeros of $f_\zeta(z)$, we have $s-1 \geqq m$. That is, $s \geqq m+1$, so that the theorem holds for $k = m+1$.

THEOREM 2. *For $(n+1)/2 \leqq k \leqq n$, the $k$-polarity of the nth degree polynomials $f(z)$ and $g(z)$ is necessary and sufficient for them to have a common, repeated zero whose multiplicities, $p$ and $q$, satisfy the inequalities $p \geqq k$, $q \geqq k$, $p+q \geqq n+k$.*

*Proof.* Suppose that two polynomials have a common repeated root whose multiplicities satisfy the given inequalities. A linear transformation will take the polynomials into

$$z^p\phi(z) = \sum_{i=0}^{n}\binom{n}{i}a_i z^i$$

and

$$z^q\psi(z) = \sum_{i=0}^{n}\binom{n}{i}b_i z^i$$

where $a_0 = \cdots = a_{p-1} = 0$ and $b_0 = \cdots = b_{q-1} = 0$. Now every product $a_ib_j$ which occurs in the $k$-polarity conditions (2) vanishes. For if $a_ib_j$ is to be nonzero, we must have $i \geqq p$ and $j \geqq q$, so that $i+j \geqq p+q$ whence $i+j \geqq n+k$. The maximum value which $i+j$ can assume for any $a_ib_j$ in (2), however, is $n+k-1$. Thus conditions (2) are satisfied and the polynomials are $k$-polar.

Suppose now that $f(z)$ and $g(z)$ are $k$-polar, with $k \geqq (n+1)/2$. We can, if necessary, perform a linear transformation on the polynomials to make $b_n \neq 0$ and $b_0 = 0$; that is, we can make all the zeros $\zeta_1, \cdots, \zeta_{n-k+1}$ of $g_{k,k-1}(z)$ finite and put one of these zeros at the origin. By Lemma 6, $f(z; \zeta, n-k+1) \equiv 0$. Thus [4] either $f(z; \zeta, n-k) \equiv 0$ or $f(z; \zeta, n-k) = c(z-\eta_{n-k+1})^k$. In either event, there is an $h$ in the range $k \leqq h \leqq n$ such that $f(z; \zeta, n-h+1) \equiv 0$ and $f(z; \zeta, n-h) = c(z-\zeta_{n-h+1})^h$. (Note that $f(z; \zeta, 0) = f(z)$.) We can assume that $\zeta_{n-h+1}$ is at the origin, so that $f(z; \zeta, n-h) = cz^h$. By Lemma 5, the $k$-polarity of $f(z)$ and $g(z)$ guarantees the $(k+h-n)$-polarity of $f(z; \zeta, n-h)$ and $g(z; \eta, n-h)$ for arbitrary $\eta_1, \cdots, \eta_{n-h}$. Let

$$f(z; \zeta, n-h) = \sum_{j=0}^{h}\binom{h}{j}A_j z^j\,,$$

$$g(z; \eta, n-h) = \sum_{j=0}^{h}\binom{h}{j}B_j z^j\,.$$

Then we have $A_0 = \cdots = A_{h-1} = 0$, $A_h \neq 0$; and the $(k+h-n)$-polariy conditions which involve $A_h$ reduce to

$$A_h B_0 = \cdots = A_h B_{k+h-n-1} = 0 \ ,$$

whence

$$B_0 = \cdots = B_{k+h-n-1} = 0 \ . \tag{6}$$

We know [3] that

$$B_j = \mu \sum_{i=0}^{n-h} b_{j+i} S_i^{(n-h)} \qquad (j = 0, \cdots, h) \ ,$$

where $\mu = (n!)/(h!)$. Now equations (6) hold for arbitrary values of $\eta_1, \cdots, \eta_{n-h}$. Hence they hold in particular for $\eta_1 = \cdots = \eta_{n-h} = 0$. For these values, we have $S_1^{(n-h)} = \cdots = S_{n-h}^{(n-h)} = 0$, so that $B_0 = \mu b_0$, whence $B_0 = 0$ implies $b_0 = 0$. We can now use $\eta_1 = 1$, $\eta_2 = \cdots = \eta_{n-h} = 0$, so that $S_1^{(n-h)} = 1$, $S_2^{(n-h)} = \cdots = S_{n-h}^{(n-h)} = 0$, $B_0 = \mu b_1$, whence $b_1 = 0$. Using $\eta_1 = \eta_2 = 1$, $\eta_3 = \cdots = \eta_{n-h} = 0$ gives $S_1^{(n-h)} = 2$, $S_2^{(n-h)} = 1$, $S_3^{(n-h)} = \cdots = S_{n-h}^{(n-h)} = 0$, $B_0 = \mu b_2$, whence $b_2 = 0$. It is clear that we can proceed in this way to establish $b_3 = \cdots = b_{n-h} = 0$. We now have $B_1 = \mu b_{n-h+1} S_{n-h}^{(n-h)}$, whence we can conclude that $b_{n-h+1} = 0$. It then follows that $B_2 = \mu b_{n-h+2} S_{n-h}^{(n-h)}$, whence $b_{n-h+2} = 0$. We can proceed in this way to show that successive values of $b_j$ vanish until we arrive at $B_{k+h-n-1} = \mu b_{k-1} S_{n-h}^{(n-h)} = 0$, whence $b_{k-1} = 0$. Thus $g(z)$ has at least a $k$-fold zero at the origin. Let $q$ be the multiplicity of this zero, so that $b_0 = \cdots = b_{q-1} = 0$, $b_q \neq 0$. Since $q \geqq k = 2k - k \geqq n + 1 - k$, it follows that $b_q$ appears as the $b_j$ of highest index in $k$ of the $k$-polarity conditions. Since it is the only nonvanishing $b_j$ in any of these $k$ conditions, they reduce to

$$b_q a_0 = \cdots = b_q a_{k-1} = 0 \ ,$$

whence

$$a_0 = \cdots = a_{k-1} = 0 \ .$$

Thus $f(z)$ has a $p$-fold zero at the origin with $p \geqq k$. To finish the proof, we have left only to show that $p + q \geqq n + k$. Now the product $a_p b_q$ is nonvanishing. If it were to appear in any of the $k$-polarity equations (2), then the indices of every product $a_i b_j$ appearing in the same equation would have to satisfy $i + j = p + q$. But this means that if $i > p$ so that $a_i \neq 0$, then $j < q$ so that $b_j = 0$. Thus, if $a_p b_q$ did appear in any equation of (2), it would be the only non-vanishing product in this equation, whence the equation would not hold. Hence the product $a_p b_q$ cannot appear in any of the equations (2). But every product $a_i b_j$ does appear for which

$$n - k + 1 \leqq i + j \leqq n + k - 1 \ .$$

Therefore, either $p+q<n-k+1$ or $p+q>n+k-1$. But $p+q \geqq k+k \geqq n+1>n+1-k$. Consequently, we must have $p+q>n+k-1$, that is, $p+q \geqq n+k$.

## REFERENCES

1. J. H. Grace, *The zeros of a polynomial*, Proc. Cambridge Philos. Soc., **11** (1902), 352-357.
2. E. Laguerre, Oeuvres, Paris: Gauthier-Villars, 1898, Vol. 1, pp. 48-66.
3. M. Marden, *The geometry of the zeros of a polynomial in a complex variable*, A. M. S. Math. Surveys No. III, New York: (1949), 38-43.
4. G. Pólya and G. Szegö, *Aufgaben und Lehrsätz aus der Analysis*, Berlin: Julius Springer, Vol. II, (1925), 61-64.
5. J. L. Walsh, *On the location of the roots of the derivative of a polynomial*, C. R. du Congress international des Mathematiciens, Strasbourg (1920), 339-342.

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