ON THE ACTION OF SO(3) ON A COHOMOLOGY MANIFOLD

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1. Introduction. Let the rotation group SO(3) of the euclidean 3-space act on a locally compact Hausdorff space X such that the highest dimension of the orbits is 3. Then the following results can be found in Montgomery-Samelson [2].

THEOREM 1. *If X is an integral cohomology n-manifold and an integral cohomology n-sphere, then the principal isotropy group is either trivial or contains a dihedral subgroup of order* 4.

THEOREM 2. *If X is the n-sphere, if the action of* SO(3) *on X is differentiable and if the union of all the singular orbits is of dimension $< n-2$, then the principal isotropy group is trivial.*

The purpose of the present paper is to generalize these theorems.

Basic notations, concepts and theorems which are often used in the study of topological transformation groups will not be given in this paper. Any reader who is not familiar with them may consult [1] or the references given in [1].

Throughout the paper, Z denotes the ring of integer, R denotes the field of rational numbers, p denotes a prime and Z_p denotes the field of integers mod p. G denotes the rotation group SO(3) of the euclidean 3-space, T denotes a circle group in G and N denotes the normalizer of T. Notice that N/T is of order 2 and that every element of $N-T$ is of order 2. Also notice that the cyclic subgroup of T of order p, which we also denote by Z_p, has N as its normalizer in G. As in [1], $H_c^k(M; L)$ denotes the kth Alexander-Wallace-Spanier cohomology group of M with compact support and with coefficients in L. When M is compact, it is also written $H^k(M; L)$.

2. On the action of N.

LEMMA 1. *Let N be the normalizer of a circle group T in* SO(3) *and let N act on an orientable connected rational cohomology m-manifold Y such that*

(i) *all the orbits are* 1*-dimensional and*

(ii) *N/T acts freely on Y/T.*

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Then Y/N is a connected rational cohomology $(m-1)$-manifold which is orientable or nonorientable according as the elements of $N-T$ reverse or preserve the orientation of Y.

Proof. Since all the orbits are 1-dimensional, the sheaf

$$\mathscr{H}^1(Tx; R) = \bigcup H^1(Tx; R)$$

on Y/T is constant so that Y/T is an orientable connected rational $(m-1)$-manifold. Since N/T acts freely on Y/T, $Y/N = (Y/T)/(N/T)$ is a connected rational cohomology $(m-1)$-manifold. Moreover, Y/N is orientable or nonorientable according as $N-T$ preserves the orientation of Y/T.

Let $h \in N-T$ and let $y \in Y$. Then there is a connected slice K of the transformation group (N, Y) at y and a connected neighborhood Q of the identity in T such that the map $Q \times K \to QK$, given by $(g, x) \to gx$, is a homeomorphism onto and such that $hQh^{-1} = Q$. Clearly hK is a slice of (N, Y) at hy and the map $Q \times hK \to QhK$ $(=hQK)$, given $(g, x) \to gx$, is a homeomorphism onto. Let Y, Q, K and hK be oriented such that the maps $Q \times K \to Y$ and $Q \times hK \to Y$, given by $(g, x) \to gx$, are orientation-preserving. Since the map $Q \to Q$, given by $t \to hth^{-1}$, reverses the orientation of Q, it follows that h reverses the orientation of Y if and only if $h\colon K \to hK$ is orientation-preserving.

Let $\pi\colon Y \to Y/T$ be the canonical projection. Since the sheaf $\mathscr{H}^1(Tx; R)$ on Y/T is constant, Y/T may be oriented such that the maps $\pi \mid K$ and $\pi \mid hK$ are both orientation-preserving. Hence h reverses the orientation of Y if and only if $N - T = hT$ preserves the orientation of Y/T. This completes the proof of Lemma 1.

LEMMA 2. *Let X be a connected* mod 2 *cohomology n-manifold with $H_c^k(X; Z_2) = 0$ for $k = n-1$, $n-2$, and let $G = \mathrm{SO}(3)$ act on X. If the principal isotropy group is a finite group of even order, then it contains a dihedral subgroup of order* 4. *Moreover, the stationary point set of every cyclic subgroup of G of order 2 is a connected* mod 2 *cohomology $(n-2)$-manifold and that of every dihedral subgroup of G of order 4 is a* mod 2 *cohomology $(n-3)$-manifold.*

Proof. Let Z_2 be the cyclic subgroup of T of order 2. Since the principal isotropy group is a finite group of even order, the stationary point set $F(Z_2)$ of Z_2 intersects every principal orbit at a 1-dimensional set. Therefore $F(Z_2)$ is of mod 2 cohomology dimension $n-2$ everywhere and hence it is a mod 2 cohomology $(n-2)$-manifold [1; p. 76].

By hypothesis,

$$H_c^n(X; Z_2) = Z_2 ,$$
$$H_c^k(X; Z_2) = 0 \text{ for } k > n \text{ and for } k = n-1,\, n-2 .$$

Using Smith sequence [1; p. 41], one can easily see that

$$H_c^{n-2}(F(Z_2); Z_2) = Z_2 .$$

Hence $F(Z_2)$ is connected.

Let D_2 be a dihedral subgroup of N of order 4. We assert that the stationary point set $F(D_2)$ of D_2 is not empty. Assume that the assertion is false; then we have a fibre map

$$\lambda\colon X \to G/N$$

given by $\lambda(gF(Z_2)) = gN$, $g \in G$. Therefore we have a spectral sequence $\{E_r\}$ whose E_2-term is given by

$$E_2^{s,t} = H^s(G/N; \mathscr{H}_c^t(gF(Z_2); Z_2))$$

and whose E_∞-term is associated with $H_c^*(X; Z_2)$. Clearly

$$E_2^{1,n-2} = Z_2 ,$$
$$E_2^{s,t} = 0 \text{ for } s < 0 \text{ and for } s > 2 .$$

It follows that

$$E_\infty^{1,n-2} = E_2^{1,n-2} = Z_2 .$$

This is impossible as we know that $H_c^{n-1}(X; Z_2) = 0$.

By Borel theorem [1; p. 182], $F(D_2)$ is a mod 2 cohomology $(n-3)$-manifold. Since $F(D_2)$ intersects every orbit at a finite set, it must intersect principal orbits. Hence the principal isotropy group contains a dihedral subgroup of order 4.

LEMMA 3. *Let N be the normalizer of a circle group T in* SO(3) *and let Y be a connected* mod 2 *cohomology m-manifold with $H_c^{m-1}(Y; Z_2) = 0$. If N acts on Y such that the principal orbits are* 1-*dimensional and such that for some $h \in N - T$, the fixed point set $F(h)$ of h is a* mod 2 *cohomology $(m-1)$-manifold, then the stationary point set $F(N)$ of N is a* mod 2 *cohomology $(m-2)$-manifold.*

Proof. Since the principal orbits are 1-dimensional, $m \geqq 1$ and the stationary point set of T is of mod 2 cohomology dimension $\leqq m-2$. Hence for $m = 1$, $F(N) = \phi$ so that our conclusion is trivial.

Let $m \geqq 2$. Let D_2 be a dihedral subgroup of N of order 4 and let $D_2 - T = \{h_1, h_2\}$. Since all the elements of $N - T$ are conjugate to one another in N, it follows from our hypothesis that the fixed point set $F(h_i)$ of h_i is a mod 2 cohomology $(m-1)$-manifold, $i=1,2$. By hypothesis, $H_c^m(Y; Z_2) = Z_2$ and $H_c^k(Y; Z_2) = 0$ for $k > m$ and for $k = m-1$. We infer that $H_c^{m-1}(F(h_1); Z_2) = Z_2$ so that $Y - F(h_1)$ has exactly two components each of which is mapped into the other under h_1. Similarly $H_c^{m-1}(F(h_2); Z_2) = Z_2$ so that $F(h_2)$ is connected. Since $h_1h_2 = h_2h_1$, $h_1F(h_2) = F(h_2)$. Hence $F(h_1) \cap F(h_2) \neq \phi$.

By Borel theorem, $F(D_2) = F(h_1) \cap F(h_2)$ is a mod 2 cohomology $(m-2)$-manifold. Moreover, in the vicinity of $F(D_2)$, $F(D_2)$ coincides with the fixed point set of h_1h_2 which is independent of the choice of D_2 in N. Hence $F(N) = F(D_2)$ is a mod 2 cohomology $(m-2)$-manifold.

3. Main theorems.

THEOREM 1. *Let X be an orientable connected integral cohomology n-manifold with $H_c^k(X; Z) = 0$ for $k = n-1, n-2$* and *let $G = \mathrm{SO}(3)$ act on X. If the principal isotropy group is finite, then it is either trivial or contains a dehedral subgroup of order* 4.

Proof. Suppose that the principal isotropy group does not contain a dihedral subgroup of order 4. Then, by Lemma 2, it is of odd order so that it contains a subgroup Z_p for some odd prime p. Let $F(Z_p)$ be the stationary point set of Z_p and let N be the normalizer of Z_p. Then N acts on $F(Z_p)$ and $F(Z_p)/N$ can be canonically imbedded into X/G.

Since $F(Z_p)$ intersects every principal orbit at a 1-dimensional set, $F(Z_p)$ is a mod p cohomology $(n-2)$-manifold and $F(Z_p)/N = X/G$. By hypothesis, $H_c^n(X; Z_p) = Z_p$ and $H_c^k(X; Z_p) = 0$ for $k > n$ and for $k = n-1$, $n-2$. It follows from Smith sequence that

$$H_c^{n-2}(F(Z_p); Z_p) = Z_p$$

so that $F(Z_p)$ is connected and orientable.

Let B be the union of all the singular orbits. Then $F(Z_p) \cap B$ is the stationary point set of T so that it is of integral cohomology dimension $\leqq n-4$. Hence

$$Y = F(Z_p) - B$$

is an orientable connected mod p cohomology $(n-2)$-manifold.

Let $y \in Y$. It is clear that a slice K of the transformation group (G, X) at y is an integral cohomology $(n-3)$-manifold and it is also a slice of (N, Y) at y. Since the isotropy group N_y is finite, there is a neighborhood of y in Y homeomorphic to the product of an open interval and K. Hence Y is an orientable connected integral cohomology $(n-2)$-manifold and consequently an orientable connected rational cohomology $(n-2)$-manifold.

Every $h \in N-T$ preserves the orientation of Y. In fact, let $f\colon [0, 1] \to G$ be a map such that

(i) $f(0)$ is the identity,

(ii) $f(1) = h$ and

(iii) whenever $0 \leqq t < t' < 1$, $f(t)N \cap f(t')N = \phi$. Then whenever $0 \leqq t < t' < 1$, $f(t)Y \cap f(t')Y = \phi$. Since $f(0)Y = Y = hY = f(1)Y$, $f([0, 1])Y$ is a connected integral cohomology $(n-1)$-manifold. We have seen that $\bar{Y} - Y = F(Z_p) \cap B$ is of integral cohomology dimension $\leqq n-4$. Therefore $f([0, 1])(\bar{Y} - Y)$ is of integral cohomology dimension $\leqq n-3$ so that $X - f([0, 1])(\bar{Y} - Y)$ is an orientable cohomology n-manifold with $H_c^{n-1}(X - f([0, 1])(\bar{Y} - Y); Z) = 0$. Since $f([0, 1])Y$ is closed in $X - f([0, 1])(\bar{Y} - Y)$, it follows that $f([0, 1])Y$ is orientable. Hence h preserves the orientation of Y.

Now we may apply Lemma 1 to (N, Y) and conclude that Y/N is a nonorientable connected rational cohomology $(n-3)$-manifold. Since $F(Z_p)/N = X/G$,

$$Y/N = (X - B)/G.$$

However, $X - B$ is an orientable connected rational cohomology n-manifold and the sheaf $\mathscr{H}^3(Gx; R)$ on $(X - B)/G$ is constant; we infer that $(X - B)/G$ is orientable. Hence we have arrived at a contradiction. The proof of Theorem 1 is thus completed.

THEOREM 2. *Let X be an orientable connected integral cohomology n-manifold with $H_c^k(X; Z) = 0$ for $k = n-1$, $n-2$, $n-3$ and let $G = \mathrm{SO}(3)$ act on X such that the principal isotropy group is finite. Then one of the following must hold.*

(1) *The principal isotropy group is trivial.*

(2) *The principal isotropy group is the dihedral group of order 4. There exists a 2-dimensional singular orbit and all the 2-dimensional singular orbits are projective planes. Moreover, the union of all the singular orbits is of integral cohomology dimension $n-2$.*

(3) *The principal isotropy group is the icosahedral group. Every singular orbit is a stationary point of G and the stationary point set of G is an integral cohomology $(n-4)$-manifold.*

Proof. Assume that the principal isotropy group is not trivial. Then, by Theorem 1, it contains a dihedral subgroup of order 4. Let Z_2 be the cyclic subgroup of T of order 2 and let D_2 be a dihedral subgroup of N of order 4. We have shown in Lemma 2 that the stationary point set $F(Z_2)$ of Z_2 is a connected mod 2 cohomology

$(n-2)$-manifold and that the stationary point set $F(D_2)$ of D_2 is a mod 2 cohomology $(n-3)$-manifold. Since $H_c^n(X; Z_2) = Z_2$ and $H_c^k(X; Z_2) = 0$ for $k > n$ and for $k = n-1, n-2, n-3$, we infer that $H_c^{n-3}(F(Z_2); Z_2) = 0$ and that $H_c^{n-3}(F(D_2); Z_2) = Z_2$ so that $F(D_2)$ is connected.

The transformation group $(N, F(Z_2))$ satisfies the hypothesis of Lemma 3. In fact, $F(Z_2)$ is a connected mod 2 cohomology $(n-2)$-manifold with $H_c^{n-3}(F(Z_2); Z_2) = 0$, the principal orbits of $(N, F(Z_2))$ are 1-dimensional and the fixed point set of $h \in N - T$ in $F(Z_2)$ is the stationary point set of the dihedral group generated by Z_2 and h so that it is a mod 2 cohomology $(n-3)$-manifold. By Lemma 3, the stationary point set $F(N)$ of N is a mod 2 cohomology $(n-4)$-manifold.

Suppose first that there exists a 2-dimensional singular orbit Gz. Since the isotropy group G_z at z contains the principal isotropy group which has been shown to contain a dihedral subgroup of order 4, it follows that G_z is isomorphic to N so that Gz is a projective plane. Therefore $F(N)$ is the stationary point set of T and hence is an integral cohomology $(n-4)$-manifold. The union of all the singular orbits is $GF(N)$ which is clearly of integral cohomology dimension $n-2$.

The principal isotropy group is a subgroup of G_z so that it is a dihedral group containing D_2 As in the proof of [3; (3.6)], one can easily show that if y is a point of $F(D_2)$ such that Gy is a principal orbit, then the isotropy group G_y leaves every point of $F(D_2)$ fixed. If G_y is not of order 4, then G_y leaves only one point of Gz fixed, contrary to the fact that $F(D_2) \cap G_z$ contains three points.

Suppose next that no singular orbits is 2-dimensional. Then every singular orbit is a stationary point of G. Hence the union B of all the singular orbits is the stationary point set of T which is clearly an integral cohomology $(n-4)$-manifold.

If $n = 3$, then X is an integral cohomology 3-sphere and G acts transitively on X. (Here $H_c^0(X; Z)$ means the reduced group.) Hence the principal isotropy group is the icosahedral group.

If $n > 3$, then there is a point z of B. As in [1; Chapter XV], the principal orbits are integral cohomology 3-spheres. In fact, there is a neighborhood V of z in $F(D_2)$ invariant under the normalizer C of D_2. Since V is a mod 2 cohomology $(n-3)$-manifold and the stationary point set of C in V is a mod 2 cohomology $(n-4)$-manifold, we may choose V such that $V - B$ contains exactly two components each of which is a cross-section of $(G, G(V-B))$. Hence we may follow the argument of [1; p. 213] to show that the principal orbits are integral cohomology 3-spheres. Consequently the principal isotropy group is the icosahedral group.

REMARK. It is not hard to see that all three cases in Theorem 2 actually occur. In fact, we can have linear actions of SO(3) on spheres as examples for the first two cases as seen in [3] and a typical example for the third case is seen in [2]. In the third case, the stationary point set of the icosahedral group is not a cohomology manifold for $n > 3$; in fact, it is an integral cohomology $(n-3)$-manifold with the stationary point set of G as its boundary. Hence the third case never occurs when (G, X) is a differentiable transformation group and $n > 3$.

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