LEVEL SETS ON SPHERES

L. M. SONNEBORN

The purpose of this paper is to prove that corresponding to any continuous real-valued function whose domain is the n-dimensional sphere ($n \geqq 2$), there is a connected set on the sphere which contains a pair of antipodal points and on which the function is constant. While this constant need not be unique, a stronger property is found which ensures uniqueness and gives continuity to the constant over homotopies of the function.

The weaker theorem was stated in abstract by R. D. Johnson, Jr. [2]. The proof which follows constitutes a portion of the author's dissertation, [4].

Throughout this paper, n will be used to denote any integer not less than 2. The usual n-dimensional measure on the n-sphere will be taken to be normalized so that the total measure of the sphere is one. Each time the measure of a set is mentioned, the set will be either open or closed, and therefore measurable. Everytime the components of a set are listed, the set will be open, and will therefore have a countable number of components. A subset of S^n ($n \geqq 2$) will be said to be "too big" if it has measure greater than one-half. A subset of the sphere is said to "cut up" the sphere if no component of its complement is too big.

The fundamental tool to be used here is the following:

THEOREM. *If O is an open set on the n-sphere, then either O or its complement $S^n - O$ has a component which cuts up the sphere.*

The method of proof is to assume that O has no such component and to prove that then its complement does.

LEMMA 1. *If A is a connected subset of S^n ($n > 1$), and if B is a component of $S^n - A$, then $S^n - B$ is connected, and $F(B)$, the boundary of B, is also connected.*

Proof. S^n is connected. The connectedness of $S^n - B$ follows from [3], page 78. $F(B)$ is connected since $F(B) = \bar{B} \cap \overline{S^n - B}$ and since S^n is unicoherent. See [5] pages 47–60.

Henceforth, let O denote an open subset of S^n, no component of which cuts up S^n. Corresponding to any component, O_i, of O, there is, then, a (unique-consider the measure) component, T_i, of its component, $S^n - O_i$, which is too big.

Received January 11, 1960, and in revised form July 18, 1960.

LEMMA 2. *For $i \neq j$, either*
(i) $T_i \subseteqq T_j$
(ii) $T_j \subseteqq T_i$
(iii) $T_i \cup T_j = S^n$.

Proof. By Lemma 1, T_i, T_j, $S^n - T_j$, and $S^n - T_i$ are all connected as is the boundary of each. O_i and O_j are connected and disjoint. Each lies in a single component of the complement of the component of the other. Hence, either $O_i \subseteqq T_j$ or $O_i \subseteqq S^n - T_j$. In the first case, $S^n - O_i$ contains the connected set $S^n - T_j$, and $S^n - T_j$ is contained in a component of $S^n - O_i$. Either this component is T_i or not. If it is T_i, then $S^n - T_j \subseteqq T_i$ and $T_i \cup T_j = S^n$. If not, then $S_n - T_j \cap T_i = \phi$, and $T_i \subseteqq T_j$. In the second case $O_i \subseteqq S^n - T_j$ and $T_j \subseteqq S^n - O_i$, so that T_j being connected lies in a single component of $S^n - O_i$. But this component must be T_i, for it is the only one big enough to contain T_j which is also too big. In this case, then $T_j \subseteqq T_i$.

COROLLARY. *For $i \neq j$, either*
(i) $S^n - T_i \supseteqq S^n - T_j$
(ii) $S^n - T_j \supseteqq S^n - T_i$
(iii) $(S^n - T_i) \cap (S^n - T_j) = \phi$.

Now, let $O' = \bigcup_j (S^n - T_j)$. Clearly $O' \supseteqq O$ since for each $i, O_i \subseteqq S^n - T_i$, and the O_i are the components of O. O' is the union of open sets and is, therefore, open. Let $X_j, j = 1, 2, \cdots$ (possibly finite) be the components of O'. Since for any i, $S^n - T_i$ is connected, it must lie entirely in one of the X_j's, and any X_j is the union of all the $S^n - T_i$'s contained in it.

LEMMA 3. *If $S^n - T_i$ and $S^n - T_j$ are disjoint but are both contained in the same component, Y_k, of O', then there is an integer l such that $S^n - T_i \subseteqq S^n - T_l$ and $S^n - T_j \subseteqq S^n - T_l$.*

Proof. Assume there is no such integer l. Let T be the union of all $S^n - T_m$ which contain $S^n - T_i$. Clearly none of these intersects $S^n - T_j$ by the corollary to Lemma 2. Let S be an arc in X_k connecting $x \in S^n - T_i$ to $y \in S^n - T_j$. (X_k is open and connected and hence arcwise connected). S must intersect $F(T)$. Let $p \in S \cap F(T)$. $p \in S^n - T_q$ for some q such that $S^n - T_q \in X_k$. Some neighborhood of p also is in $S^n - T_q$ which is open. But this neighborhood of p contains a point $z \in T$ since $p \in F(T)$. Hence $z \in S^n - T_m$ for some m such that $S^n - T_m \supseteqq S^n - T_i$. Since $S^n - T_m$ and $S^n - T_q$ intersect, one contains the other by the corollary to Lemma 2. In either case, however, $S^n - T_q \subseteqq T$. But then $p \in T, p$ being a boundary point of the open set T.

This contradiction establishes the lemma.

LEMMA 4. *Each Y_k can be written as a countable expanding union of sets $S^n - T_j$ (i.e. a union in which each set contains the previous).*

Proof. If X_k contains only a finite number of $S^n - T_i$, it contains a biggest one (repeated application of Lemma 3.) Suppose then, that $X_k = \bigcup_{i=1}^{\infty} (S^n - T_i)$. We choose a subunion of this union as follows: Let $I_1 = S^n - T_1$; for $m > 1$, let $L_m = S^n - T_{i(m)}$ where $i(m)$ is the smallest number for which $S^n - T_{i(m)} \supsetneqq S^n - T_{i(m-1)}$ if there is such a number $i(m)$. If, at some stage, there is no such number, the union will be finite; otherwise it will be countably infinite. It remains to be shown that $\bigcup_m (I_m) = X_k$. Let $x \in X_k$. If $x \in I_1$, $x \in \bigcup_m (I_m)$, so suppose $x \notin I_1$. $x \in S^n - T_p$ for some p. There is, therefore, a smallest integer q for which $x \in S^n - T_q$ and $S^n - T_q \supsetneqq I_1$ (Lemma 3). There is a largest integer s for which $s = i(h)$ for some h, and $s < q$. It follows that $q = i(h+1)$ for $(S^n - T_q) \cap (S^n - T_s) \neq \phi$ and $x \in S^n - T_q$ while $x \notin S^n - T_s$. Hence $x \in \bigcup_m (I_m)$.

LEMMA 5. *For each k, the measure of X_k does not exceed one-half.*

Proof. Each T_i has measure greater than one-half, so that each $S^n - T_i$ has measure less than one-half. The expanding union of open sets measuring less than one-half cannot have measure greater than one-half. [1].

LEMMA 6. *$S^n - O'$ is connected.*

Proof. Each $S^n - X_i$ is either one of the T_j, or is expressible as the decreasing intersection of a countable number which are closed and connected. By Lemma 3.8 of page 80 of Wilder [5], $S^n - X_i$ is connected. Since X_i is also connected, it follows from Lemma 1 that $F(X_i)$ is connected. Now suppose $S^n - O'$ is not connected. Then $S^n - O' = A \cup B$ where A and B are disjoint, nonempty, and relatively closed in $S^n - O'$ and hence closed in S^n. Since each X_i has a connected boundary, each X_i has its boundary entirely in A or entirely in B. Then consider $S^n = A' \cup B'$ where $A' = A \cup (\bigcup_{i \in I} X_i)$, $I = \{i \mid F(X_i) \subseteqq A\}$ and $B' = B \cup (\bigcup_{j \in J} X_j)$, $J = \{j \mid F(X_j) \subseteqq B\}$. A' and B' are easily seen to be closed, nonempty and disjoint. Hence S^n is not connected. This contradiction establishes the lemma.

THEOREM 1. *If O is an open set of S^n $(n > 1)$, then either O or $S^n - O$ has a component which cuts up the sphere (i,e. a component* *whose complement consists of components with measures no than greater one-half*).

Proof. All the previous lemmas except the first were based on the assumption that O had no such component. Since $O' \supseteqq O$ it follows that $S^n - O' \subseteqq S^n - O$. But $S^n - O'$ is connected and lies in a component, A, of $S^n - O$. Since $A \supseteqq S^n - O'$, $S^n - A \subseteqq O'$, and every component of $S^n - A$ is contained in a component of O'. But the components of O' all have measure no greater than one-half, and so also do the components of $S^n - A$.

LEMMA 7. *If A and B are both connected, closed sets on S^n which cut up S^n, then $A \cap B$ is not empty.*

Proof. Suppose A and B are disjoint. A being connected, lies in a single component, say B_1, of $S^n - B$. $S^n - B_1$ is connected (Lemma 1) and lies in a single component, say A_1 of $S^n - A$. Now the measure of the open set A_1 is strictly greater than the measure of the closed set $S^n - B_1$ contained in it. However, $M(B_1) < 1/2$ by assumption, so that $M(S^n - B_1) \geqq 1/2$ and $M(A_1) > 1/2$ contrary to the assumption that A cuts up S^n. This contradiction establishes the theorem.

COROLLARY. *If $g: S^n \rightarrow S^n$ is a measure-preserving homeomorphism, and if A is a connected, closed subset of S^n which cuts up S^n, then there is a point $x \in A$ for which $g(x) \in A$. In particular, any such set A, contains a pair of antipodal points of S^n.*

THEOREM 2. *Let $F: S^n \times I \rightarrow E^1$, be continuous, $(n > 1)$, and define $f_t: S^n \rightarrow E^1$ for each t, $O \leqq t \leqq 1$, by $f_t(x) = F(x, t)$ for each $x \in S^n$. Then for each t, $O \leqq t \leqq 1$, there exists an unique real number k_t such that $f_t^{-1}(k_t)$ contains a closed connected subset which cuts up S^n. This subset contains a pair of antipodal points of S^n. Further, k_t is a continuous function of t on $O \leqq t \leqq 1$.*

Proof. The uniqueness of k_t and the fact that the subset contains a pair of antipodal points follow from Lemma 7 and its corollary. The continuity of k_t follows in the usual way from the compactness of $S^n \times I$ and the resulting uniform continuity of F. The existence of k_t remains to be proved, that is it must be shown that for every function $f: S^n \rightarrow E^1$, there exists a real number k, such that $f^{-1}(k)$ contains a closed connected subset which cuts up S^n. For each positive integer m, there exists an open subset, O_m of E^1 with the property that all components of both O_m and of $E^1 - O_m$ have diameter less than $1/m$. For each m, $f^{-1}(O_m)$ is an open subset of S^n, so that according to Theorem 1, there is a component of either $f^{-1}(O_m)$ or of $S^n - f^{-1}(O_m) = f^{-1}(E^1 - O_m)$ which cuts up S^n. Denote by A_m one such component. Then the diameter of $f(A_m)$ which is connected and which is either in O_m or in $E^1 - O_m$ is less than $1/m$. For each m, pick a point $x_m \in A_m$. Since S^n is compact, the sequence $\{x_m\}$ has a limit point. Let x be such a limit point, and set $k = f(x)$. Also let $B_r = \{s \mid k - 1/r \leqq s \leqq k + 1/r\}$ and let C_r be that component of $f^{-1}(B_r)$ which contains x. Then each of the sets C_r contains at least one of the sets A_m. For, there is a number $\delta > 0$ for which $|y - x| < \delta$ implies $|f(y) - k| < 1/2r$, and there exists $m(\delta) > 2r$ for which $|x_{m(\delta)} - x| < \delta$. Now $A_{m(\delta)} \subseteqq C_r$; for, since $|x_{m(\delta)} - x| < \delta$, the segment of great circle connecting x to $x_{m(\delta)}$ also satisfies this property so that for every point y on this segment $|f(y) - k| < 1/2r$ and $x_{m(\delta)} \in C_r$. Also for any point $z \in A_{m(\delta)}$, $|f(z) - k| \leqq |f(z) - f(x_{m(\delta)})| + |f(x_{m(\delta)})) - k| < 1/2r + 1/2r = 1/r$. Thus the connected set consisting of the segment and $A_{m(\delta)}$ is all mapped into B_r, so that C_r contains $A_{m(\delta)}$ and hence C_r cuts up S^n for each r.

Now let $C = \bigcap_{r=1}^{\infty} C_r$. C is then the intersection of a decreasing sequence of closed, connected sets in a compact space and is thus closed, connected and nonempty. ([3] page 81.) Quite clearly, $x \in C$ and $f(C) = k$. Suppose now that C does not cut up S^n. Then there is a component, say D, of $S^n - C$, with measure more than one-half. Let $w \in D$. For all sufficiently large r, $w \notin C_r$. Let D_r be that component of $S^n - C_r$ which contains w. $\{D_r\}$ is an increasing sequence of open connected sets. $D = \bigcup_{r=1}^{\infty} D_r$ for otherwise there would be a point $v \in D$ not in any D_r. D being open and connected contains an arc joining w to v. If $v \notin \cup D_r$, there is a first point u along this arc such that $u \notin \cup D_r$. But since $u \in D$, $u \notin \cap C_r$ so for some r, $u \notin C_r$. For this value of r, u and some neighborhood of it are $S^n - C_r$. Also for some $i > r$, points of this neighborhood are in D_i, and so must u be. Thus $u \in \cup D_r$ and this contradiction establishes that $D = \cup D_r$. But now each D_r is a component of the complement of C_r and each C_r contains some A_m. Hence, since each A_m cuts up S^n, each D_r has measure not greater than one-half. However, the expanding union of sets with measure not greater than one-half cannot have measure greater than one-half, so that D has measure no greater than one-half contrary to the hypothesis above, and C does cut up S^n. This concludes the proof of Theorem 2.

Extensions and related topics. The only property of the real numbers used in the foregoing is that fact that for every $\varepsilon > 0$, there exists an open subset of them with the property that every component of the open set and of its complement has diameter less than ε. Thus the reals could be replaced by any (metric) space with this property. Hence, since E^1 cannot be replaced by E^2 in Theorem 2, we conclude that E^2 does not have this property. The theorems which follow are easily deducible from this fact.

THEOREM. *If O is an open subset of the unit square $I \times I$, then either some component of O or some component of $(I \times I) - O$ contains a pair of points belonging to opposite faces of $I \times I$.*

THEOREM. *If $f: I \times I \rightarrow E^1$ is continuous, there is a connected subset of $I \times I$ which contains a pair of points on opposite faces of $I \times I$ and on which f is a constant.*

THEOREM. *If f: $S^1 \times S^1 \rightarrow E^1$ is continuous, and if p: $E^2 \rightarrow S^1 \times S^1$ is the usual projection map of E^2 as the universal covering space of $S^1 \times S^1$, then there is a connected subset A of E^2 such that diam $A = \infty$ and $fp|A$ is a constant.*

THEOREM. *If f: $S^1 \times S^1 \rightarrow E^1$ is continuous, there is a connected subset B of $S^1 \times S^1$ such that $f|B$ is a constant and such that B carries a nontrivial one-dimensional Čech cycle of $S^1 \times S^1$.*

The proofs of all these theorems are straightforward and are given in the author's dissertation [4].

A different extension is given by the following theorems.

THEOREM. *If $n \geqq 2m + 1$ and f: $S^n \rightarrow E^m$ is continuous, there exists a connected subset of S^n which contains a pair of antipodal points and on which f is constant. (This theorem follows easily from Yang* [6].)

THEOREM. *If $n \leqq 2m-1$, $m \geqq 1$, there exists a continuous function f: $S^n \rightarrow E^m$ such that on no connected subset of S^n containing a pair of antipodal points is f a constant.*

Proof. Consider the case $n = 2m - 1$. $S^{2m-1} = \{\bar{x} = (x_1, x_2, \cdots, x_{2m}) | \sum (x_i)^2 = 1\}$. For $1 \leqq i \leqq m$, define A_i, B_i and C_i, by $A_i = \{\bar{x} | x_{2i-1} = O, x_{2i} \geqq O\}$, $B_i = \{\bar{x} | x_{2i} = -x_{2i-1}, x_{2i-1} \geqq O\}$ and $C_i = \{\bar{x} | x_{2i} = x_{2i-1}, x_{2-1} \leqq O\}$. Let $D_i = A_i \cup B_i \cup C_i$, $1 \leqq i \leqq m$. Let f_i: $S^{2m-1} \rightarrow E^1$ be given by $f_i(y) = d(y, D_i)$. Since every closed connected set containing a pair of antipodal points of S^{2m-1} intersects D_i, the points which are connected to their antipodal points by a level set of f_i consist of those points for which $x_i = x_{i+1} = 0$. Thus f: $S^{2m-1} \rightarrow E^m$ given by $f = (f_1, f_2, \cdots, f_m)$ satisfies the conditions of the theorem. For $n < 2m - 1$ one can take a great n-sphere on the $2m - 1$ sphere and use the restriction of the above example.

I can give no information in the case f: $S^{2m} \to E^m$, $m \geqq 2$.

Bibliography

1. P. R. Halmos, *Measure Theory*, Van Nostrand (1954), page 38.
2. R. D. Johnson, Jr., *Bulletin of the American Mathematical Society*, v. **60**, page 36 (1954), abstract.
3. M. H. A. Newman, *Elements of the Topology of Plane Sets of Points* (Second edition), Cambridge, 1951.
4. L. M. Sonneborn, *Level Sets on Spheres*, Ph. D. thesis, California Institute of Technology, 1956.
5. R. L. Wilder, *Topology of Manifolds*, American Mathematical Society, 1949.
6. C. T. Yang, *On Theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson II*, Annals of Mathematics, **62** (1955), 273.

THE UNIVERSITY OF KANSAS