SECONDARY COHOMOLOGY OPERATIONS
WHICH EXTEND THE TRIPLE PRODUCT

Donald W. Kahn

The triple product of Massey [5] is a secondary cohomology operation in three variables which is defined when the three classes satisfy two relations. The value of the operation is a certain natural coset of cohomology. In this note, I discuss certain secondary operations which generalize the triple product, when the values are taken in cohomology with real coefficients. Roughly speaking, for each pair of integers (M, N), there is such an operation; when one takes the pair $(1, 1)$, the operation is the triple product.

There are four sections. The first section, which is primarily motivation, discusses operations in real cohomology. In this section, explicit cochain formulas are given. In the remaining sections, we are concerned with operations whose domain is integral cohomology, and whose range is real cohomology. The second section studies the universal examples. In the third section, I define the operations and determine their ranges. Finally, in the fourth section, I prove a classification theorem for these operations, which asserts that in certain cases these operations constitute all such operations.

I would like to thank W. S. Massey and N. Stein for their help and encouragement during the preparation of this paper.

1. Consider now operations in real cohomology. We restrict attention to spaces which are 1-connected, countable simplicial complexes. In [4] the author showed that for such spaces, there is an anti-commutative cochain algebra over the real numbers, which gives the correct cohomology. If X is a space in our category, write this anti-commutative cochain algebra as $\mathscr{G}(X)$.

The Massey triple product may be defined as follows: consider classes $u, v, w \in H^*(X, R)$[1] such that $u \cdot v = v \cdot w = 0$. Choose

$$u', v', w' \in \mathscr{G}(X)$$

which represent these classes, and $a, b \in \mathscr{G}(X)$ such that

$$u' \cdot v' = \delta a, \quad v'w' = \delta b$$

Set

$$Z' = a \cdot w' - (-1)^{\mathrm{Dim}(u)} u' \cdot b$$

[1] Real cohomology is to be understood in this section.

One easily sees that $\delta Z' = 0$. Depending on the different choices, Z' represents a set of classes in $H^*(X)$ (in fact a coset).

Now, using the anti-commutativity of $\mathscr{G}(X)$, we make the following definition.

DEFINITION 1.1. (Double Product). Let $u \in H^p(X)$, $v \in H^q(X)$. Suppose q is odd, and $u \cdot v = 0$. Choose $u', v' \in \mathscr{G}(X)$, representing u and v. Let $u' \cdot v' = \delta a$.
Define

$$Z' = a \cdot v'$$

$$\delta(a \cdot v') = u' \cdot v' \cdot v' = u' \cdot (v')^2 = 0 \,, \text{ as } (v')^2 = 0 \,.$$

Set $Z = \{Z'\}$, and $\langle u, v \rangle = \{Z\}$, where the latter is the set of all classes represented by such Z. $\langle u, v \rangle$ is the double product of u and v.

DEFINITION 1.2. (Generalized Double Product). With the same conventions as Definition 1.1, we add the assumption that p is even. Define $Z' = a^n \cdot v'$, n being positive integer.

$$\delta Z' = (n \cdot a^{n-1} \cdot u'v')v' = 0 \,.$$

Set $Z = \{Z'\}$, and $\langle u, v \rangle_n = \{Z\}$, $\langle u, v \rangle_n$ is the generalized double product.

DEFINITION 1.3. (Generalized Triple Product). Consider classes $u, v, w \in H^*(X)$ in dimensions p, q, r (resp.). Suppose p and r are odd. (If one is even, one may choose k or l below to be 1). Suppose $u \cdot v = v \cdot w = 0$. Choose $u', v', w' \in \mathscr{G}(X)$ representing u, v, w, and choose $a, b \in \mathscr{G}(X)$ so that

$$u' \cdot v' = \delta a \qquad v' \cdot w' = \delta b \,.$$

Let k and l be positive integers. Set

$$Z' = l \cdot a^k w' b^{l-1} - (-1)^{k(p+q-1)+(q-1)} k \cdot a^{k-1} u' \cdot b^l \,.$$

An elementary computation shows that $\delta Z' = 0$. Let $Z = \{Z'\}$, and $\langle u, v, w \rangle_{k,l} = \{Z\}$, the generalized triple product.

REMARKS. One easily computes the dimensions for these operations. For example,

$$\dim \langle u, v, w \rangle_{K,l} = k \cdot p + (k + l - 1)(q - 1) + l \cdot r \,.$$

The double product $\langle u, v \rangle$ is quite similar to the triple product $\langle u, v, v \rangle$. However, the denominator modulo which it is defined is

actually smaller than the corresponding denominator for the triple product.

Finally,

DEFINITION 1.4. With the notations of Definition 1.3, suppose dim. v is odd.
Let

$$Z' = v' \cdot a^m \cdot b^n , \qquad \{Z'\} = Z .$$

Let

$$_{m,n}\langle u, \, v, \, w \rangle = \{Z\} .$$

REMARKS. It is clear that these operations are generalizations of the triple product (in real cohomology). However, they do not lend themselves to computation, because of the difficulty of computing in $\mathscr{G}(X)$. Therefore, in order to obtain a fuller theory, the remainder of this paper will study similar operations which have integral cohomology as domain and real cohomology as range. The technique of universal examples can then be applied.

We shall assume that all spaces have the homotopy type of a 1-connected countable simplicial complex.

2. We now give the basic definitions and theorems for universal examples. A primary operation in one variable is a natural function

$$\varphi : H^p(X, G) \longrightarrow H^q(X, H)$$

where p and q are positive integers, and G and H are Abelian groups. Such an operation is said to be of type $\{p, G_{jq}, H\}$. Serre [9] and others observed that the cohomology of Eilenberg-MacLane spaces described all such operations. Precisely, there is a natural one to one correspondence between operations of type $\{p, G ; q, H\}$ and classes of $H^q(G, p ; H)$. If φ is of type $\{p, G ; q, H\}$ then this correspondence assigns to φ the class $\varphi(i)$, where i is the fundamental class of $H^p(G, p; G)$. Hence, a knowledge of operations is equivalent, in principle, to a knowledge of the operations in a " universal example ", here a space $K(G, p)$.

The theory of universal examples may be extended to cover operations in several variables, and higher order operations, which are defined when certain operations vanish, and which take on a values certain sets of elements in the cohomology of a space. In the definition of such operations, one imposes only one condition, naturality. For a general discussion of such operations, and the corresponding universal examples, we refer to [1], [3], and [7]. The information

which we shall need is contained in the following two theorems.

THEOREM 2.1. Let X be a space in our category. The universal example for a primary operation φ, in the variables $u_i \in H^{p_i}(X, G_i)$ is a space

$$K(G_1, p_1) \times \cdots \times K(G_n, p_n) \, .$$

That is, the operations $\varphi(u_1, \cdots, u_n) \in H^q(X, H)$ are in one-to-one correspondence with the classes of

$$H^q(K(G_1, p_1) \times \cdots \times K(G_n, p_n) \, ; H) \, .$$

THEOREM 2.2. Consider X and u_i as above. Suppose that the classes satisfy the homogeneous distinct relations

$$P_1(u_1, \cdots, u_n) = 0, \cdots, P_m(u_1, \cdots, u_n) = 0$$

where we have associated to each relation, a pairing of the groups $G_1, \cdots, G_n$ to a group A_i.

Then, the universal example for operations in these variables, defined when these relations are satisfied (i.e., on the kernel of the operations $P(u_1, \cdots, u_n), \cdots, P_m(u_1, \cdots, u_n)$), is a fibre space $(E, F, B; P)$, with

$$B = K(G_1 \, ; p_1) \times \cdots \times K(G_n \, ; p_n)$$
$$F = K(A_1 \, ; \deg. P_1 - 1) \times \cdots \times K(A_m \, ; \deg. P_m - 1) \, .$$

The structure of the fibre space is determined by a map

$$f : B \longrightarrow K(A_1, \deg. P_1) \times \cdots \times K(A_m \, ; \deg. P_m)$$

which is given by the product of the relations. That is, such operations $\varphi(u_1, \cdots, u_n) \in H^q(X, H)$ are in one to one correspondence with the classes of $H^q(E, H)$.
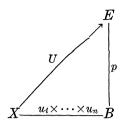
The correspondence is as follows: Write $u_1, \cdots, u_n$ as maps

$$X \xrightarrow{u_1} K(G_1, p_1), \cdots, X \xrightarrow{u_n} K(G_n, p_n) \, .$$

If the classes $u_1, \cdots, u_n$ satisfy the relations, then the composition

$$X \xrightarrow{u_1 \times \cdots \times u_n} K(G_1, p_1) \times \cdots \times K(G_n, p_n) \xrightarrow{f} K(A_i, \deg. P_1)$$
$$\times \cdots \times K(A_m, \deg. P_m)$$

is homotopic to zero. Hence, the map $u_1 \times \cdots \times u_n$ may be lifted to a map $U : X \longrightarrow E$, so that the following diagram is commutative.

$$\begin{array}{ccc}
& & E \\
& \nearrow & | \\
U & & | \, p \\
& & | \\
X & \xrightarrow{\;u_1 \times \cdots \times u_n\;} & B
\end{array}$$

If $\varphi \in H^q(E, H)$, we define $\varphi(u_1, \cdots, u_n)$ to be the collection $\{U^*(\varphi)\}$ in $H^q(X, H)$ where U ranges over all lifting maps which make the above diagram commutative.

Proof. See the references mentioned above. The proofs of these theorems follow immediately from the techniques of those paper.

Conventions. For the remainder of this paper, assume that all operations go from integral cohomology. That is, the G_i and A_i are always Z, while H is R. The pairings will always be the natural multiplication in Z. The cohomology of a space will be real singular cohomology.

I will now indicate how the above theorems apply to the cup-product, and to the triple product [5].

(a) If we were to choose as coefficients any ring with identity, the cup-product is an operation in two variables, given by the composition

$$H^p(X) \otimes H^q(X) \longrightarrow H^{p+q}(X \times X) \xrightarrow{\;\Delta^*\;} H^{p+q}(X)$$

where Δ is the diagonal. We shall consider the particular cup-product

(*) $$H^p(X; Z) \otimes H^q(X; Z) \longrightarrow H^{p+q}(X; R)$$

This operation is described in terms of the universal example as follows. The space involved is $K(Z, p) \times K(Z, q)$. By the Kunneth theorem:

$$H^{p+q}(K(Z, p) \times K(Z, q); R) \approx \sum_{0 \le i \le p+q} H^i(Z, p; R) \otimes H^{p+q-i}(Z, q; R) \,.$$

Let i_p and i_q be the generators of $H^p(Z, p; R)$ and $H^q(Z, q; R)$, i.e., the images, under the coefficient homomorphism $Z \to R$, of the fundamental classes. Then, our cup-product (*) corresponds to the class $i_p \otimes i_q$.

More generally, we have the following.

PROPOSITION 2.1. Consider classes

$$u_1 \in H^{p_1}(X, Z), \cdots, u_n \in H^{p_n}(X, Z) \,,$$

where X is some space in our category. Then, all primary operations

in these classes, which take on values in real cohomology, are given by (formal) polynomials in these classes, with respect to cup-product, that is linear combinations of monomials with respect to cup-product multiplication.

Proof. One simply computes the real cohomology of the universal example, which is

$$K(Z, p_1) \times \cdots \times K(Z, p_n) .$$

(b) We define the triple product. Consider three cohomology classes $u \in H^p(X; Z)$, $v \in H^q(X, Z)$, $w \in H^r(X, Z)$. Suppose that these classes satisfy the relations

$$u \cdot v = 0 \qquad v \cdot w = 0 \qquad \text{(pairing } Z \otimes Z \to R\text{)}.$$

Choose representatives u', v', w' and cochains a and b so that

$$u' \cdot v' = \delta a \qquad v' \cdot w' = \delta b .$$

We set

$$J^n(u, w) = u \cdot H^{n-p}(X) + H^{n-r}(X) \cdot w \qquad \text{(real cohomology)}$$

where

$$n = p + q + r - 1 .$$

Define

$$Z' = a \cdot w' - (-1)^p u' \cdot b \qquad \text{(pairing } Z \otimes Z \to R\text{)}.$$

One easily checks that $\delta Z' = 0$, and that $\{Z'\}$ represents a coset of $H^n(X)/J^n(u, w)$. This coset is defined to be the triple product $\langle u, v, w \rangle$. When no confusion is possible, we shall not distinguish between $\langle u, v, w \rangle$ and $\{Z'\}$.

Using Theorem 2.2, we construct a universal example in the case which we are considering. It is a fibre space $(E, F, B; p)$, with

$$B = K(Z, p) \times K(Z, q) \times K(Z, r)$$
$$F = K(Z, p + q - 1) \times K(Z, q + r - 1) .$$

The structure of the fibre space is given by a map

$$B \longrightarrow K(Z, p + q) \times K(Z, q + r)$$

representing the two cup-products.

PROPOSITION 2.2. Consider the universal example for the triple product $(E, F, B; p)$ Suppose $p < r$. Let $u \in H^p(Z, p)$ be a generator

in the base, and $y \in H^{q+r-1}(Z, q + r - 1)$ be a generator in the fibre[2]. Then $u \otimes y \in E_2$, in the Serre spectral sequence, remains until E_p, and represents the triple product $\langle u, v, w \rangle$ up to a sign.

If $p = r$, let $w \in H^r(Z, r)$ and $x \in H^{p+q-1}(Z, p + q - 1)$ be generators for the base and fibre respectively. Then
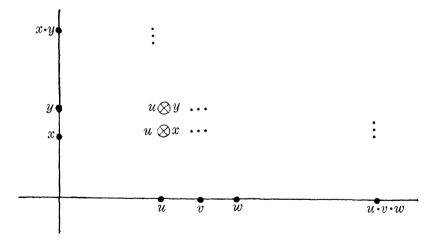
$$u \otimes y - (-1)^{r \cdot (p+q)} w \otimes x$$

remains and represents the triple product up to a sign.

Proof. Suppose $p < r$. Consider the universal fibre space

$$K(Z, p + q - 1) \times K(Z, q + r - 1) \longrightarrow \text{paths}$$
$$\downarrow$$
$$K(Z, p + q) \times K(Z, q + r) .$$

We use x and y to denote the generators in the fibre, also. It is well-known that $\tau(x)$ and $\tau(y)$ generate the cohomology of the base, and it follows from the naturality of the transgression that in the universal example $(E, F, B ; p)$, x and y are transgressive and

$$\tau(x) = u \cdot v \qquad \tau(y) = v \cdot w$$

where u, v, w generate $H^p(Z, p)$, $H^q(Z, q)$, $H^r(Z, r)$ (resp.). Now consider the E_2-term of the spectral sequence of $(E, F, B ; p)$ (see [10]).

Let $n = p + q + r - 1$. As y is transgressive,

$$d_i(u \otimes y) = 0, \ i \leq n - p, \ d_{n-p+1}(u \otimes y) = \{u \cdot v \cdot w\} ,$$

where the brackets indicate the representation of an element in a

[2] We take as generators the images of the generators in integral cohomology, under the map induced by the inclusion $Z \to R$.

higher term of the spectral sequence. However, already

$$d_{n-r+1}(\pm\, w \otimes x) = \{u \cdot v \cdot w\}\,.$$

Hence $\{u \otimes y\}$ is a cocycle for all di. One easily sees that $\{u \otimes y\}$ is not a coboundary for any d_i.

It is easy to see that $\{u \otimes y\}$ is the only element of dim. n in E_∞ so that it must represent some multiple of the triple product. We now make this determination more precise. Consider the classes p^*u, p^*v, and p^*w in $H^*(E)$. Since x and y are transgressive, with $\tau(x) = u \cdot v$, $\tau(y) = v \cdot w$, we can choose cochains C_x and C_y in $C^*(E)$ such that

$$\delta C_x = p^*u' \cdot p^*v' \qquad \{i^*(C_x)\} = x$$
$$\delta C_y = p^*v' \cdot p^*w' \qquad \{i^*(C_y)\} = y$$

with $u' \cdot v' \cdot w'$ representing u, v, w.
We have, by definition

$$\langle p^*u,\, p^*v,\, p^*w \rangle = \{C_x \cdot p^*w' - (-1)^p p^*u' \cdot C_y\}\,.$$

In the Serre spectral sequence, C_x and C_y represent elements in $E_2^{0,*}$ corresponding to x and y in the fibre. It follows that the cochain $C_x \cdot p^*w' - (-1)^p p^*u' \cdot C_y$ represents in E_∞ that element which comes from the term

$$(1 \otimes x)(w \otimes 1) - (-1)^p(u \otimes y)$$

in E_2. $w \otimes x$ disappears by E_{n-r+1}. Up to a sign, the element $\{u \otimes y\}$ then determines the triple product.

The case where $p \geqq r$ is handled in a similar manner, and is left to the reader.

3. We now define the double product and generalized double product in terms of the universal example[4].

DEFINITION 3.1.

(a) *Double Product*: Consider classes u, v; dim. $u = p > 1$ dim. $v = q > 1$, q odd. Suppose that $u \cdot v = 0$. The universal example for operations defined under these circumstances is a fibre space $(E, F, B\,; p)$, with

$$B = K(Z, p) \times K(Z, q)$$
$$F = K(Z, p + q - 1)$$

---

[3] Concerning groups, real cohomology is to be understood, for the remainder of the paper. Operations will go from integral to real cohomology.

with classifying map

$$f : K(Z, p) \times K(Z, q) \longrightarrow K(Z, p + q)$$

representing the cup-product, $u \cdot v$.

Consider the E_2-term in the spectral sequence for this fibre space. Let u and v generate $H^p(Z, p)$ and $H^q(Z, q)$ in the base, and x generate $H^{p+q-1}(Z, p + q - 1)$ in the fibre. As in Prop. 2.2, $\tau(x) = u \cdot v$.
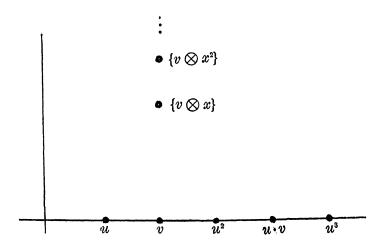
Clearly $d_i \equiv 0$ for $2 \le i < n = p + q$. We look at d_n.

$$d_n(x) = u \cdot v. \quad d_n(u \otimes x) = u^2 v. \quad d_n(v \otimes x) = 0.$$

$\{v \otimes x\}$ is not a coboundary under d_n, as $d_n(E_{n-1})$ is contained in the ideal generated by $\{u \cdot v \otimes 1\}$. Hence, $\{v \otimes x\}$ remains until E_∞, and represents an element in $H^*(E)$. We define a class in $H^*(E)$, the double product, to be this element, and write, by abuse of language,

$$\langle u, v \rangle = \{v \otimes x\}.$$

(b) *Generalized Double Product*: We make the same assumptions as in (a), but also assume that p is even. Then $p + q - 1$ is even. We consider the universal example and note that if $n = p + q$, then $d_i \equiv 0$, for $2 \le i < n$. We compute ker $d_n : d_n(x) = u \cdot v$, $d_n(u \otimes x) = u^2 \cdot v$, $d_n(v \otimes x) = 0$, $d_n(x^k) = k \cdot uv \otimes x^{k-1}$, $d_n(u \otimes x^k) = \pm k \cdot u^2 v \otimes x^{k-1}$, $d_n(v \otimes x^k) = 0$. We may represent E_{n+1} by the following picture, in which ● indicates the presence of an element:

We now show that $\{v \otimes x^k\}$, $k \ge 1$ remains until E_∞. The difference in dimension between u^a and u^{a+1} is p. The difference in dimension between $v \otimes x^k$ and $v \otimes x^{k+1}$ is $p + q - 1 > p$. Hence, if u^a (or $v \cdot u^a$) is killed off by d_i of $(v \otimes x^k)$ for some i and k, u^{a+1} (or $v \cdot u^{a+1}$) cannot

be killed.   Hence, if we had $d_i(v \otimes x^k) = u^a$, then in E_∞, we would have $u^a = 0$, $u^{a+1} \neq 0$, which is absurd.   Therefore, $\{v \otimes x^k\}$ represents a non-zero element in E_∞, and we define, as before,

$$\langle u, v \rangle_k = \{v \otimes x^k\} \;.$$

It is clear that $\langle u, v \rangle_1 = \langle u, v \rangle$.

REMARK.   The operations $\langle u, v \rangle_k$ are clearly non-trivial.   One simply computes them in E.

Next, I want to discuss an extension of the double product, which amounts to forming a double product in which each variable is actually a (formal) polynomial, with respect to the cup-product, of certain classes.

Consider classes $u_1, \cdots, u_n$, which satisfy a homogeneous relation $P[u_1, \cdots, u_n] = 0$, where P is a homogeneous-degree (formal) polynomial. The universal example for secondary operations, which are then defined, is a fibre space $(E, F, B \,; p)$, with

$$B = K(Z, \text{dim. } u_1) \times \cdots \times K(Z, \text{dim. } u_n)$$

and

$$F = K(Z, \text{dim. } P - 1) \;.$$

Let $u_1, \cdots, u_n$ denote generators of $H^*(B)$, and x a generator of $H^*(F)$.   Then $\tau(x) = P[u_1, \cdots, u_n]$.   Let Q be a polynomial in the u_i, such that $P \cdot Q = 0$, i.e., $P \cdot Q$ is forced to vanish by the anti-commutativity.   Consider

$$Q \otimes x^k \in E_2 \;.$$

Clearly, $d_i(Q \otimes x^k) = 0$, $2 \leq i < n = \text{dim. } P$.   But, also, $d_n(Q \otimes x^k) = k \cdot Q \cdot P \otimes x^{k-1} = 0$.   Suppose that $Q \otimes x^k$ remains until E_∞, and hence represents an element in $H^*(E)$.   Then, we say that $Q \otimes x^k$ determines a generalized double product, and write

$$\langle P/Q \,; Q \rangle_k = \{Q \otimes x^k\} \;.$$

REMARK.   If there are l independent generalized double products in some fixed dimension, then we can conclude that there are l independent secondary operations which are defined on the kernal, $P = 0$, and which take on values in the fixed dimension.   In general, we cannot identify particular elements in E_∞ with particular classes in $H^*(E)$.

The analogous statements also apply to the generalized triple products.   Details are left to the reader.

We now determine the denominators, modulo which the double

and generalized double products are defined.

PROPOSITION 3.1. Consider a space X in our category, and classes u and z in dimensions p and q. Suppose q is odd. Then

$$\langle u, z \rangle \in H^n(X)/z \cdot H^{n-p}(X), \ n = p + 2q - 1 \ ,$$

and if p is even,

$$\langle u, z \rangle_k \in H^n(X)/z \cdot H^{n-p}(X) + \langle u, z \rangle \cdot H^{n-p-2q+1} + \cdots$$
$$+ \langle u, z \rangle_{k-1} \cdot H^{n-p-(k-1)q+1}(X) \ ,$$
$$n = q + k \cdot (p + q - 1)$$

as well-defined elements of these groups.

Proof. We make full use of the theory of principal fibre spaces, for which we refer to [7] [8]. The universal example, $(E, F, B ; p)$ may be considered a principal fibre space, with maps:

$$\mu : E \times F \longrightarrow E$$
$$m : F \times F \longrightarrow F \ .$$

As in section 20 of [2], $E \times F$ is a fibre space over B, with projection map $\bar{p} : E \times F \longrightarrow B$ given by $\bar{p}(e, f) = p(e)$. Denote the spectral sequence of this fibre space by $\{\bar{E}_r ; \bar{d}_r\}$.
Then, we have

$$\bar{E}_r = E_r \otimes H^*(F)^4$$

$\bar{d}_r$ is d_r on the first factor, and null on the second.

$$\mu^* : E_2 \longrightarrow \bar{E}_2$$

is given by $1 \otimes m^*$.

$H^*(F)$ is either an exterior algebra on an odd-dimensional generator, or a polynomial algebra on an even-dimensional generator.
In the first case,

$$m^*(x) = x \otimes 1 + 1 \otimes x \ ,$$

while in the second

$$m^*(x^k) = \sum_{\substack{0 \le i, \ j \le k \\ i + j = k}} \binom{k}{i} x^i \otimes x^j$$

where no repeats are taken in the sum. These facts follow immediately from the knowledge of $H^*(F)$, see [3], and duality. Hence, in E_2,

---
[4] We take the spectral sequence with real coefficients.

$$\mu^*(z \otimes x^k) = z \otimes m^*(x^k) = z \otimes \left(\sum_{\substack{0 \le i,\, j \le k \\ i+j=k}} \binom{k}{i} x^i \otimes x^j \right) \in \bar{E}_2 \, .$$

We will now show that there are non-zero real numbers $\alpha_{i,j}$, so that if $\{z \otimes x^k\}$ represents the class in $H^*(E)$,

$$\mu^*(\{z \otimes x^k\}) = \sum_{\substack{0 \le i,\, j \le k \\ i+j=k}} \alpha_{i,j}(\{z \otimes x^i\} \otimes x^j) \in H^*(E \times F) \, .$$

First, notice that this relation with $\alpha_{i,j} \equiv 1$ is valid in E_∞, because of the properties of the $\bar{d}_r$. Now E_∞ and $\bar{E}_\infty$ are graded groups of $H^*(E)$ and $H^*(E \times F)$, with respect to certain filtrations. A homogeneous element of $\bar{E}_\infty$ has a degree with respect to the fibre, the dimension of the second part in $\bar{E}_\infty = E_\infty \otimes H^*(F)$. Furthermore, a homogeneous element in $\bar{E}_\infty$ and a representative for this element in $H^*(E \times F)$ have the same degree with respect to the fibre.

Corresponding to each degree in the fibre, in which an x^i, $i \le k$ lies, the term in $\bar{E}_\infty$ and the term in $H^*(E \times F)$ are both 1-dimensional vector spaces over R. In $\bar{E}_\infty$, each term in the sum which is

$$\mu^*(\{z \otimes x^k\})$$

has a different degree with respect to the fibre. Hence, in $H^*(E \times F)$, the corresponding sum consists of distinct non-zero elements, which each have different degrees with respect to the fibre. As each $\{z \otimes x^i\} \otimes x^j$ is a non zero element in the corresponding 1-dimensional vector space, there exist non-zero $\alpha_{i,j}$ such that the desired relation holds. It is easy to see that $\alpha_{k,1} = 1$.

To prove the proposition, consider two maps $v, v' : X \to E$ such that $p \cdot v = p \cdot v'$. By Lemma 1.2 of [8], $\exists\, w : X \to F$ such that $\mu_\sharp(v, w) = v'$. Then

$$\begin{aligned}
v'(\{z \otimes x^k\}) &= \mu_\sharp(v, w)^*(\{z \otimes x^k\}) \\
&= (v, w)^*(\mu^*(\{z \otimes x^k\}) \\
&= (v, w)^*(\sum_{i+j=k} \alpha_{i,j}\{z \otimes x^i\} \otimes x^j) \\
&= v^*(\alpha_{k,1}\{z \otimes x^k\} \cup 1) + \sum_{\substack{i+j=k \\ j<k}} \alpha_{i,j} v^*\{z \otimes x^i\} \cup w^*(x^j) \\
&= v^*(\{z \otimes x^k\}) + \sum_{\substack{i+j=k \\ j<k}} \alpha_{i,j} v^*(\{z \otimes x^i\}) \cup w^*(x^j) \, .
\end{aligned}$$

Hence, by choosing different maps v, v', $\langle u, z \rangle_k$ varies by at most an element of

$$z \cdot H^{n-p}(X) + \cdots + \langle u, z \rangle_{k-1} \cdot H^{n-p-(k-1)q+1}(X) \, .$$

where $n = q + k \cdot (p + q - 1)$. Therefore, $\langle u, z \rangle_k$ is a well-defined element of the stated group, completing the proof.

REMARK. Using the same technique, a similar result may be established for the generalized triple product.

4. We now give some classification theorems for operations which go from integral cohomology to real cohomology.

THEOREM 4.1. Let $u_1, \cdots, u_n$ be classes which satisfy one, non-trivial, homogeneous relation $P[u_1, \cdots, u_n] = 0$. Then, all the secondary operations which are defined under those circumstances are given by the generalized double products. Precisely, in each dimension, the term E_∞ in the spectral sequence for the universal example is spanned by $p^*(H^*(B))$ and the generalized double products.

Proof. The universal example here is a fibre space $(E, F, B\,;\,p)$ with
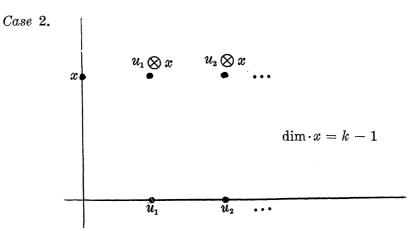
$$B = K(Z, \dim. u_1) \times \cdots \times K(Z, \dim. u_n)$$
$$F = K(Z, \dim. P - 1)\,.$$

As before, take $u_1, \cdots, u_n$ as generators of $H^*(B)$ and x as a generator of $H^*(F)$, with $\tau(x) = P[u_1, \cdots, u_n]$. We consider three cases.

Case 1. dim. u_i is even for all i. We will show that there are no non-trivial secondary operations. As dim x is odd, there are only 2 nonzero rows in the E_2-term of the spectral sequence of $(E, F, B\,;p)$. The elements are of the form $R \otimes 1$ or $R \otimes x$, where R is a polynomial in the u_i. Clearly $d_i(R \otimes 1) = 0$ for all i, and these terms respresent $P^*(H^*(B))$. Now,

$$d_i(R \otimes x) = 0,\ \text{if}$$
$$i < \dim. P = k,\ d_k(R \otimes x) = R \cdot P \otimes 1\,.$$

But $R \cdot P$ is not zero, unless $R = 0$.
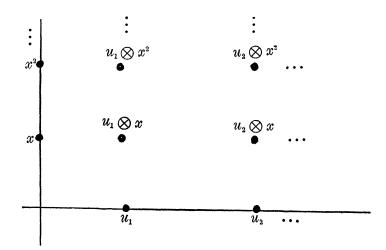
Case 2.

dim $\cdot x = k - 1$

Suppose dim. P is even, admitting the possibility of some u_i have odd dimension.   Again we have 2 non-zero rows in the E_2-term.

If $d_k(R \otimes x) = R \cdot P \otimes 1 = 0$ (recall $k = \text{dim. } P$), then $\{R \otimes x\}$ remains until E_∞, and represents the operation $\langle P/R, R \rangle$.

Case 3.   dim. P is odd, so that dim. x is even.   In this case, the E_2-term has infinitely many, non-zero rows.
Suppose that we have, in E_2 a homogeneous element of dim. m,

$$u = R_0 + R_1 \otimes x + \cdots + R_n \otimes X^n ; \quad \text{dim. } R_i = m - i(k - 1) ,$$

which is a cocycle for each d_i, i.e., remains until E_∞

By assumption

$$d_i u = 0, \ 2 \leqq i < k$$

and

$$d_k u = R_1 \cdot P \otimes 1 + 2R_2 \cdot P \otimes x + \cdots + n \cdot R_n \cdot P \otimes x^{n-1} = 0 .$$

As each term has a different complementary degree, we have

$$R_1 \cdot P = 0, \ \cdots, \ R^n \cdot P = 0 .$$

Now, R_0 and $R_1 \otimes x$ represent cocycles for each d_i; we have

$$d_i(R_2 \otimes x^2 + \cdots + R_n \otimes x^n) = 0, \ \text{all } i ,$$

and in the same way.

$$d_i(R_2 \otimes x^2) = 0, \ \text{all } i .$$

Hence, proceeding in this way, we see that each term, $R_i \otimes x^i$, remains until E_∞. Therefore, $R_i \otimes x^i$ represents a double product $\langle p/R_i, R_i \rangle_i$, or

$$\{u\} = R_0 + \langle p/R_1 \rangle_1 + \cdots + \langle p/R_n, R_n \rangle_n .$$

We state the following similar theorem without proof; the proof is complicated.

THEOREM 4.2. Consider operations in 3 variables u, v, w, defined on the kernel $\{u, v, w/u \cdot v = 0, \ v \cdot w = 0\}$. All such operations are given (in the sense of the above theorem) by sums of products of (possibly generalized) double and triple products.

REMARK. In either of the above theorems, if there is exactly one element in a given dimension in E_∞, then the operation involved, as a class in $H^*(E)$ is determined exactly.

The following theorem of Massey may be proven either by these methods, or by using the Gysin sequence.

THEOREM 4.3. (Massey). Consider n even-dimensional cohomology classes which satisfy two distinct, non-trivial, polynomial relations.

$$f(u_1, \cdots, u_n) = 0$$

$$g(u_1, \cdots, u_n) = 0 .$$

Let h be the greatest common divisor of f and g, over the reals.

$$f = h \cdot f_1 \qquad g = h \cdot g_1 .$$

Then, $\langle f_1, h, g_1 \rangle$ generates all secondary operations which are then defined, as a module over the primary operations.

BIBLIOGRAPHY

1. J. F. Adams, *On the Non-Existense of Elements of Hope Invariant One*, Annals of Math., (72) 1960.

2. A. Borel, *Sur la Cohomologie des Espaces Fibrés Principeaux* $\cdots$ Annals of Math., (57) 1953.

3. S. Eilenberg and S. MacLane, *On the Groups $H(\pi, n)$ III*, Annals of Math., (60) 1954.

4. D. W. Kahn, *On the existence and applications of anti-commutative cochain algebras*, Illinois J. Math., (to appear).

5. W. S. Massey and H. Uehara, *The Jacobi Identity for Whitehead Products*, Alg. Geometry and Topology, Princeton University Press.

6. W. S. Massey, *On the Cohomology Ring of a Sphere Bundle*, Jour. of Math. and Mech. (7) 1958.

7. F. P. Peterson and N. Stein, *Secondary operations*; *Two formulas*, Amer. J. Math., (81) 1959.

8. F. P. Peterson and E. Thomas, *A Note on Non-Stable Cohomology Operations*, Bol. Soc. Mat. Mexican 1958.

9. J. P. Serre, *La Cohomologie Mod 2* $\cdots$ Comm. Math. Helv., 1953.

10. ————, *Homologie Singulière des Espaces Fibrés*, Annals of Math. (54) 1951.

COLUMBIA UNIVERSITY