ON LOCALLY MEROMORPHIC FUNCTIONS WITH SINGLE-VALUED MODULI

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1. A meromorphic function of bounded characteristic in a disk is the quotient of two bounded analytic functions. This classical theorem can be extended to open Riemann surfaces W as follows. Consider the class MB of meromorphic functions w of bounded characteristic on W, defined in terms of capacity functions on subregions. Let L be the class of harmonic functions on W, regular except for logarithmic singularities with integral coefficients. Then $w \in MB$ if and only if $\log |w|$ is the difference of two positive functions in L. We shall construct these functions directly on W, without making use of uniformization.

The proof offers no essential difficulties. If $\log |w|$ is regular at the singularity of the capacity functions, then the classical reasoning carries over almost verbatim. In the general case we introduce the extended class M_e of locally meromorphic functions e^{u+iu^*}, $u \in L$, with single-valued moduli. This class seems to offer some interest in its own right.

2. The class O_{M_eB} of Riemann surfaces not admitting nonconstant M_eB-functions coincides with the class O_G of parabolic surfaces. Regarding the subclass $MB \subset M_eB$ and the strict inclusion relations $O_{HB} < O_{MB} < O_{AB}$, we refer to the pioneering work on Lindelöfian maps by M. Heins [2, 3] and M. Parreau [4], and the doctoral dissertation of K. V. R. Rao [5].

§ 1. Definitions.

3. Let W be an arbitrary open Riemann surface. Given $\zeta \in W$ let Ω, $\zeta \in \Omega$, be a relatively compact subregion of W whose boundary β_Ω consists of a finite number of analytic Jordan curves. The Green's function on Ω with pole at ζ is denoted by $g_\Omega(z, \zeta)$. For $\Omega_0 \subset \Omega$ we have $g_{\Omega_0} \leqq g_\Omega$ in Ω_0 and $\lim_{\Omega \to W} g_\Omega(z, \zeta)$ either $\equiv \infty$ or else $=$ the Green's function $g(z, \zeta)$ of W. By definition, the class O_G of parabolic Riemann surfaces consists of those W on which no $g(z, \zeta)$ exists. An equivalent definition of O_G is that there are no nonconstant nonnegative superharmonic functions on W.

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4. The capacity function $p_\Omega(z, \zeta)$ on Ω with pole at ζ is defined as the harmonic function with singularity

$$p_\Omega(z, \zeta) - \log|z - \zeta| \to 0$$

as $z \to \zeta$ and such that

$$p_\Omega(z, \zeta) = k_\Omega = \text{const. on } \beta_\Omega .$$

It is known [1] that $k_{\Omega_0} \leqq k_\Omega$ and the limit $k_\beta = \lim k_\Omega$ is thus well-defined. A necessary and sufficient condition for $W \in O_G$ is $k_\beta = \infty$.

5. Let M be the class of meromorphic functions w on W. The proximity function of w is defined [7] as

$$m(\Omega, w) = m(\Omega, \infty) = \frac{1}{2\pi}\int_{\beta_\Omega} \overset{+}{\log}|w|\, dp_\Omega^* . \tag{1}$$

If β_h is the level line $p_\Omega = h$, $-\infty \leqq h \leqq k_\Omega$, and $n(h, \infty)$ signifies the number of poles of w in $\bar{\Omega}_h$: $p_\Omega \leqq h$, counted with multiplicities, then the counting function is defined as

$$\begin{aligned} N(\Omega, w) &= N(\Omega, \infty) \\ &= \int_{-\infty}^{k_\Omega} (n(h, \infty) - n(-\infty, \infty))dh + n(-\infty, \infty)k_\Omega . \end{aligned} \tag{2}$$

The characteristic function is, by definition,

$$T(\Omega) = T(\Omega, w) = m(\Omega, w) + N(\Omega, w) .$$

The function w has at ζ the Laurent expansion

$$w(z) = c_\lambda(z - \zeta)^\lambda + c_{\lambda+1}(z - \zeta)^{\lambda+1} + \cdots , \tag{3}$$

$c_\lambda \neq 0$, and the Jensen formula reads [7, 8]

$$T(\Omega, w) = T(\Omega, w^{-1}) + \log|c_\lambda| . \tag{4}$$

6. We shall need a class M_e more comprehensive than M. We introduce:

DEFINITIONS. *The class L consists of functions u on W, harmonic except for logarithmic singularities $\lambda_i \log|z - z_i|$ at z_i, $i = 1, 2, \cdots$, with integral coefficients λ_i. The subclass of nonnegative functions in L will be denoted by LP.*

The class M_e is defined to consist of (multiple-valued) functions of the form

$$w = e^{u+iu^*} , \qquad u \in L . \tag{5}$$

The conjugate function u^* has periods around z_i and along some cycles in W. Every branch of w is locally meromorphic, the branches differing by multiplicative constants c with $|c| = 1$. The modulus $|w|$ is single-valued throughout W.

The quantities $m(\Omega, w)$, $N(\Omega, w)$, $T(\Omega, w)$, and the Jensen formula carry over to M_e without modifications [7]. We further introduce:

DEFINITION. *The class* MB *(or* M_eB*) consists of functions* w *in* M *(or* M_e*) with bounded characteristics,*

$$T(\Omega) = O(1) . \tag{6}$$

Explicitly, one requires the existence of a bound $C < \infty$ independent of Ω such that $T(\Omega) < C$ for all $\Omega \subset W$. That (6) is independent of ζ will be a consequence of a decomposition theorem which we proceed to establish.

§ 2. The decomposition theorem.

7. We continue considering arbitrary open Riemann surfaces W.

THEOREM. *A necessary and sufficient condition for* $w \in M_eB$ *on* W *is that*

$$\log |w| = u - v , \tag{7}$$

where $u, v \in LP$.

The proof will be given in nos. 8–18. As a corollary we observe that $w \in MB$ on W if and only if (7) holds.

8. First we shall discuss in nos. 8–11 the case $w(\zeta) = 0$ or ∞.

Suppose $w \in M_eB$. We begin by showing that $W \notin O_G$. If $w(\zeta) = \infty$, then

$$T(\Omega) \geqq N(\Omega, w) \geqq n(-\infty, \infty)k_\Omega \geqq k_\Omega .$$

From $W \in O_G$ it would follow that $k_\Omega \to \infty$ as $\Omega \to W$ and consequently $T(\Omega) \to \infty$, a contradiction. We conclude that $W \notin O_G$. If $w(\zeta) = 0$, then in Jensen's formula

$$T(\Omega, w) = T\left(\Omega, \frac{1}{w}\right) + O(1)$$

we have

$$T\left(\Omega, \frac{1}{w}\right) \geqq N\left(\Omega, \frac{1}{w}\right) \geqq n(-\infty, 0)k_\Omega \geqq k_\Omega$$

and arrive at the same conclusion $W \notin O_G$.

On the other hand, if condition (7) is true, the existence of nonnegative superharmonic functions u, v implies $W \notin O_G$. Thus either condition of the theorem gives the hyperbolicity of W, and we may henceforth assume the existence of $g(z, \zeta)$ on W if $w(\zeta) = 0$ or ∞.

9. The functions

$$\varphi(z) = e^{\lambda(g(z\ \zeta)+ig^*(z\ \zeta))}\ , \tag{8}$$

$$w_1(z) = w(z)\varphi(z) \tag{9}$$

belong to M_e. We shall show:

LEMMA. *A necessary and sufficient condition for $w \in M_eB$ is that $w_1 \in M_eB$.*

Proof. By definition,

$$T(\Omega, \varphi) = N(\Omega, \varphi) + m(\Omega, \varphi)\ . \tag{10}$$

For $\lambda > 0$ we have trivially $N(\Omega, \varphi^{-1}) \equiv 0, m(\Omega, \varphi^{-1}) \equiv 0$, hence $T(\Omega, \varphi^{-1}) \equiv 0$, and it follows from Jensen's formula that $T(\Omega, \varphi) = O(1)$. If $\lambda < 0$, then $N(\Omega, \varphi) \equiv m(\Omega, \varphi) \equiv 0$, and $T(\Omega, \varphi) \equiv 0$, hence $T(\Omega, \varphi^{-1}) = O(1)$. In both cases

$$T(\Omega, \varphi) = O(1),\ T(\Omega, \varphi^{-1}) = O(1)\ . \tag{11}$$

The inequalities

$$T(\Omega, w) \leqq T(\Omega, w_1) + T(\Omega, \varphi^{-1}) = T(\Omega, w_1) + O(1)\ ,$$
$$T(\Omega, w_1) \leqq T(\Omega, w) + T(\Omega, \varphi) = T(\Omega, w) + O(1)$$

yield

$$T(\Omega, w) = T(\Omega, w_1) + O(1) \tag{12}$$

and the lemma follows.

10. The following intermediate result can now be established:

LEMMA. *A necessary and sufficient condition for*

$$\log |w| = u - v \tag{13}$$

with $u, v \in LP$ is that

$$\log |w_1| = u_1 - v_1 \tag{14}$$

with $u_1, v_1 \in LP$.

Proof. We know that

$$\log|w_1| = \log|w| + \lambda g = \log|w| + (n_0 - n_\infty)g\ , \tag{15}$$

where n_0, n_∞ are the multiplicities of the zero or pole of $w(z)$ at ζ. If (13) is true, then

$$\log|w_1| = (u + n_0 g) - (v + n_\infty g) \tag{16}$$

and (14) follows. Conversely, (14) implies

$$\log|w| = (u_1 + n_\infty g) - (v_1 + n_0 g)\ . \tag{17}$$

This proves the lemma.

11. We conclude that Theorem 7 will be proved for w with $w(\zeta) = 0$ or ∞ if we establish it for w_1. Since $w_1(\zeta) \neq 0, \infty$, the proof for w_1 will also apply to w with this property. Explicitly, we are to show that $w_1 \in M_e B$ if and only if $\log|w_1| = u_1 - v_1$, $u_1, v_1 \in LP$.

12. Let $p_{\Omega z}$ be the capacity function in Ω with pole at z. For a harmonic function h on $\bar{\Omega}$ it is known [7] that

$$h(z) = \frac{1}{2\pi}\int_{\beta_\Omega} h\, dp^*_{\Omega z}\ . \tag{18}$$

Denote by a_μ, b_ν the zeros and poles of w in W. Those in $W - \zeta$ are the zeros and poles of w_1 in W. Suppose first there is no a_μ, b_ν on β_Ω. Then the function

$$h(z) = \log|w_1(z)| + \sum_{a_\mu \in \Omega - \zeta} g_\Omega(z, a_\mu) - \sum_{b_\nu \in \Omega - \zeta} g_\Omega(z, b_\nu) \tag{19}$$

is harmonic on $\bar{\Omega}$. Throughout this paper the zeros and poles are counted with their multiplicities. We set

$$x_\Omega(z, w_1) = \frac{1}{2\pi}\int_{\beta_\Omega} \overset{+}{\log}|w_1|\, dp^*_{\Omega z}\ , \tag{20}$$

$$y_\Omega(z, w_1) = \sum_{b_\nu \in \Omega - \zeta} g_\Omega(z, b_\nu)\ , \tag{21}$$

and

$$u_\Omega(z, w_1) = x_\Omega(z, w_1) + y_\Omega(z, w_1)\ . \tag{22}$$

Then

$$\log|w_1(z)| = u_\Omega(z, w_1) - u_\Omega(z, w_1^{-1})\ . \tag{23}$$

Since all terms are continuous in a_μ, b_ν, the equation remains valid if there are zeros or poles of w on β_Ω.

We observe that

$$x_\Omega(\zeta, w_1) = m(\Omega, w_1) , \tag{24}$$

$$y_\Omega(\zeta, w_1) = N(\Omega, w_1) . \tag{25}$$

Here we shall only make use of the consequence

$$u_\Omega(\zeta, w_1) = T(\Omega, w_1) . \tag{26}$$

13. We next show:

LEMMA. *For $\Omega_0 \subset \Omega$,*

$$u_{\Omega_0}(z, w_1) \leqq u_\Omega(z, w_1) , \tag{27}$$

$$u_{\Omega_0}(z, w_1^{-1}) \leqq u_\Omega(z, w_1^{-1}) . \tag{27$'$}$$

Proof. By (23),

$$\overset{+}{\log} | w_1(z) | \leqq u_\Omega(z, w_1) \tag{28}$$

for every Ω. It follows that

$$\begin{aligned} x_{\Omega_0}(z, w_1) &\leqq \frac{1}{2\pi} \int_{\beta_{\Omega_0}} u_\Omega(t, w_1) dp^*_{\Omega_0 z} \\ &= \frac{1}{2\pi} \int_{\beta_{\Omega_0}} (u_\Omega(t, w_1) - y_{\Omega_0}(t, w_1)) dp^*_{\Omega_0 z} \\ &= u_\Omega(z, w_1) - y_{\Omega_0}(z, w_1) , \end{aligned}$$

because this difference is regular harmonic in Ω_0. We have reached statement (27),

$$x_{\Omega_0}(z, w_1) + y_{\Omega_0}(z, w_1) \leqq u_\Omega(z, w_1) ,$$

and inequality (27)$'$ follows in the same fashion.

14. From (26) and (27) we infer that $T(\Omega, w_1)$ increases with Ω. We can set

$$T(W, w_1) = \lim_{\Omega \to W} T(\Omega, w_1) \tag{29}$$

and use alternatively the notations $T(\Omega) = 0(1)$ and $T(W) < \infty$.

15. The convergence of u_Ω can now be established:

LEMMA. *If $T(W, w_1) < \infty$, then the functions*

$$u(z, w_1) = \lim_{\Omega \to W} u_\Omega(z, w_1) , \tag{30}$$

$$u(z, w_1^{-1}) = \lim_{\Omega \to W} u_\Omega(z, w_1^{-1}) \tag{30}$$

are positive harmonic on W except for logarithmic poles of $u(z, w_1)$ at the $b_\nu \in W - \zeta$ and those of $u(z, w_1^{-1})$ at the $a_\mu \in W - \zeta$.

Proof. By Harnack's principle the limit in (30) is either identically infinite or else harmonic on $W - \{b_\nu\}$. That the latter alternative occurs is a consequence of

$$\lim_{\Omega \to W} u_\Omega(\zeta, w_1) = T(W, w_1) .$$

The statement for $u_\Omega(z, w_1^{-1})$ follows similarly from $u_\Omega(\zeta, w_1^{-1}) = T(\Omega, w_1^{-1}) = T(\Omega, w_1) + O(1)$.

16. On combining the lemma with (23) we see that $w_1 \in M_e B$ has the asserted representation

$$\log | w_1(z) | = u(z, w_1) - u(z, w_1^{-1}) \tag{31}$$

with the u-functions in LP. It remains to establish the converse.

17. Suppose

$$\log | w_1(z) | = u_1(z) - v_1(z) \tag{32}$$

where $u_1, v_1 \in LP$. The positive logarithmic poles of $u_\Omega(z, w_1)$ are those of $\log | w_1(z) |$ in Ω, hence among those of $u_1(z)$. Consequently $u_1(z) - u_\Omega(z, w_1)$ is superharmonic in Ω and its minimum on $\bar{\Omega}$ is reached on β_Ω, where $u_1(z) - u_\Omega(z, w_1) = u_1(z) - \overset{+}{\log} | w_1(z) | \geqq 0$. One infers that $u_1(z) \geqq u_\Omega(z, w_1)$ in $\bar{\Omega}$. At ζ this means

$$T(\Omega, w_1) = u_\Omega(\zeta, w_1) \leqq u_1(\zeta) . \tag{33}$$

If $u_1(\zeta) < \infty$, the proof is complete.

18. If $u_1(\zeta) = \infty$, then

$$u_1(z) + \lambda_1 \log | z - \zeta | \tag{34}$$

is harmonic at ζ for some positive integer λ_1. We set

$$w_2 = w_1 \cdot e^{-\lambda_1 (g + i g^*)} \in M_e , \tag{35}$$

where $g = g(z, \zeta)$, and obtain

$$\log | w_2 | = \log | w_1 | - \lambda_1 g = (u_1 - \lambda_1 g) - v_1 . \tag{36}$$

The function $u_1 - \lambda_1 g_\Omega$ with $g_\Omega = g_\Omega(z, \zeta)$ is superharmonic on Ω, hence its minimum on $\bar{\Omega}$ is taken on β_Ω, where

$$(37) \qquad u_1 - \lambda_1 g_\Omega = u_1 \geqq 0 .$$

From $u_1 \geqq \lambda_1 g_\Omega$ on Ω it follows that

$$(38) \qquad u_1 - \lambda_1 g = \lim_{\Omega \to W} (u_1 - \lambda_1 g_\Omega) \geqq 0$$

on W. On setting

$$(39) \qquad u_2 = u_1 - \lambda_1 g,\ v_2 = v_1$$

one gets

$$(40) \qquad \log | w_2 | = u_2 - v_2$$

with $u_2, v_2 \in LP$.

The positive logarithmic poles of $u_\Omega(z, w_2)$ are those of $\log | w_2 |$ on Ω, hence among those of u_2. The minimum of the superharmonic function $u_2(z) - u_\Omega(z, w_2)$ on $\bar{\Omega}$ is taken on β_Ω, where it is

$$\min_{\beta_\Omega} (u_2 - \overset{+}{\log} | w_2 |) \geqq 0 .$$

One infers that

$$(41) \qquad T(\Omega, w_2) = u_\Omega(\zeta, w_2) \leqq u_2(\zeta) < \infty ,$$

that is, $T(\Omega, w_2) = O(1)$. The reasoning leading to (12) yields

$$(42) \qquad T(\Omega, w_1) = T(\Omega, w_2) + O(1) ,$$

and consequently $T(\Omega, w_1) = O(1)$.

We have shown that (32) implies $T(W, w_1) < \infty$. The proof of Theorem 7 is complete.

19. As an immediate consequence we see that the property $T(\Omega, w) = O(1)$ and thus the class $M_e B$ is independent of ζ.

§ 3. Extremal decompositions.

20. Consider an arbitrary $w \in M_e$. In contrast with no. 12 we now make no restrictive assumptions on $w(\zeta)$ and form

$$(43) \qquad x_\Omega(z, w) = \frac{1}{2\pi} \int_{\beta_\Omega} \overset{+}{\log} | w | \, dp^*_{\Omega z} ,$$

$$(44) \qquad y_\Omega(z, w) = \sum_{b_\nu \in \Omega} g_\Omega(z, b_\nu) ,$$

$$(45) \qquad u_\Omega(z, w) = x_\Omega(z, w) + y_\Omega(z, w) .$$

It is seen as in no. 13 that u_Ω increases with Ω and that

$$u(z, w) = \lim_{\Omega \to W} u_\Omega(z, w) \tag{46}$$

is either identically infinite or else positive harmonic on W except for logarithmic poles b_ν. The same is true of

$$u(z, w^{-1}) = \lim_{\Omega \to W} u_\Omega(z, w^{-1}) \tag{47}$$

with singularities a_μ.

The functions (46) and (47) will now be shown to be extremal in all decompositions (7):

THEOREM. *If there is a decomposition*

$$\log |w(z)| = u_1(z) - u_2(z) \tag{48}$$

with $u_1, u_2 \in LP$, *then also*

$$\log |w(z)| = u(z, w) - u(z, w^{-1}) \tag{49}$$

and

$$\begin{aligned} u(z, w) &\leqq u_1(z) \\ u(z, w^{-1}) &\leqq u_2(z)\ . \end{aligned} \tag{50}$$

Proof. One observes that the positive logarithmic poles of $u_\Omega(z, w)$ are those of $\log |w(z)|$ in Ω, hence among those of $u_1(z)$ in Ω. The superharmonic function $u_1(z) - u_\Omega(z, w)$ in Ω dominates

$$\min_{\beta\Omega} (u_1(z) - \overset{+}{\log} |w(z)|) \geqq 0$$

and we find that $u_1(z) - u(z, w) = \lim_{\Omega \to W}(u_1(z) - u_\Omega(z, w)) \geqq 0$ in W. Similarly, the superharmonic function $u_2(z) - u_\Omega(z, w^{-1}) \geqq 0$ on Ω, and $u_2(z) \geqq u(z, w^{-1})$ on W. By virtue of Harnack's principle, equality (49) then follows on letting $\Omega \to W$ in

$$\log |w(z)| = u_\Omega(z, w) - u_\Omega(z, w^{-1})\ . \tag{51}$$

21. The extremal functions $u(z, w), u(z, w^{-1})$ can in turn be decomposed:

THEOREM. *A function w on W belongs to $M_e B$ if and only if*

$$\log |w| = (x(z, w) + y(z, w)) - (x(z, w^{-1}) + y(z, w^{-1}))\ , \tag{52}$$

where the functions $x \geqq 0$ are regular harmonic and the functions $y \geqq 0$ have the representations

$$
\begin{aligned}
y(z, w) &= \Sigma\, g(z, b_\nu) \\
y(z, w^{-1}) &= \Sigma\, g(z, a_\mu)\ .
\end{aligned}
\tag{53}
$$

Here the sums are extended over all poles b_ν and all zeros a_μ of w on W respectively, each counted with its multiplicity.

22. Suppose indeed that $w \in M_e B$. It is evident from the maximum principle that

$$y_{\Omega_0}(z, w) \leqq y_\Omega(z, w) \tag{54}$$

for $\Omega_0 \subset \Omega$. We know that

$$\log |w| = u_1 - u_2\ , \tag{55}$$

$u_1, u_2 \in LP$, and the superharmonic function $u_1(z) - y_\Omega(z, w)$ on Ω cannot exceed $\min_{\beta_\Omega} u_1 \geqq 0$. Hence $y_\Omega(z, w) \leqq u_1(z)$ on Ω and, by Harnack's principle,

$$y(z, w) = \lim_{\Omega \to W} y_\Omega(z, w) \tag{56}$$

is positive harmonic on W except for logarithmic poles b_ν. Analogous reasoning shows that

$$y(z, w^{-1}) = \lim_{\Omega \to W} y_\Omega(z, w^{-1}) \tag{57}$$

is positive harmonic on $W - \{a_\mu\}$.

23. To prove (53) we must show that

$$\lim_{\Omega \to W} \sum_{b_\nu \in \Omega} g_\Omega(z, b_\nu) = \sum_{b_\nu \in W} g(z, b_\nu) \tag{58}$$

and similarly for $\Sigma\, g(z, a_\mu)$. First,

$$\sum_{b_\nu \in \Omega} g_\Omega(z, b_\nu) \leqq \sum_{b_\nu \in \Omega} g(z, b_\nu) \leqq \sum_{b_\nu \in W} g(z, b_\nu)\ , \tag{59}$$

and we have

$$\overline{\lim_{\Omega \to W}} \sum_{b_\nu \in \Omega} g_\Omega(z, b_\nu) \leqq \sum_{b_\nu \in W} g(z, b_\nu)\ . \tag{60}$$

Second, for $\Omega_0 \subset \Omega$,

$$\sum_{b_\nu \in \Omega_0} g(z, b_\nu) = \lim_{\Omega \to W} \sum_{b_\nu \in \Omega_0} g_\Omega(z, b_\nu) \leqq \underline{\lim}_{\Omega \to W} \sum_{b_\nu \in \Omega} g_\Omega(z, b_\nu) \tag{61}$$

and a fortiori

$$\sum_{b_\nu \in W} g(z, b_\nu) = \lim_{\Omega_0 \to W} \sum_{b_\nu \in \Omega_0} g(z, b_\nu) \leqq \underline{\lim}_{\Omega \to W} \sum_{b_\nu \in \Omega} g_\Omega(z, b_\nu)\ . \tag{62}$$

Statement (58) follows.

24. The convergence of $x_\Omega(z, w)$ is obtained at once from

$$x_\Omega(z, w) = u_\Omega(z, w) - y_\Omega(z, w) , \tag{63}$$

and the limiting function is

$$x(z, w) = u(z, w) - y(z, w) . \tag{64}$$

The limit $x(z, w^{-1})$ of $x_\Omega(z, w^{-1})$ is obtained in the same way. Both limits are obviously positive and regular harmonic on W.

Necessity of (52) for $w \in M_eB$ has thus been established. Sufficiency is a corollary of the main Theorem 7.

§ 4. Consequences.

25. If only the x-terms in (52) are considered, the following corollary of Theorem 21 is obtained:

THEOREM. *If $w \in M_eB$ on W, then*

$$\lim_{\Omega \to W} \int_{\beta_\Omega} | \log | w || \, dp_\Omega^* < \infty \tag{65}$$

for any ζ.

Here p_Ω signifies, as before, the capaity function on Ω with pole at ζ. For the proof we have

$$\begin{aligned} \int_{\beta_\Omega} | \log | w || \, dp_\Omega^* &= \int_{\beta_\Omega} \overset{+}{\log} | w | \, dp_\Omega^* + \int_{\beta_\Omega} \overset{+}{\log} \left| \frac{1}{w} \right| dp_\Omega^* \\ &= 2\pi(x_\Omega(\zeta, w) + x_\Omega(\zeta, w^{-1})) , \end{aligned} \tag{66}$$

and this quantity tends to

$$2\pi(x(\zeta, w) + x(\zeta, w^{-1})) < \infty . \tag{67}$$

The limit (65) thus exists.

26. A consideration of the y-terms in (52) gives:

THEOREM. *Suppose $w \in M_eB$. Then the sum $\Sigma g(z, z_i)$, with z_i ranging over all poles and zeros of w, is harmonic on $W - \{a_\mu\} - \{b_\nu\}$.*

In fact,

$$\sum_{z_i \in W} g(z, z_i) = \lim_{\Omega \to W} \sum_{z_i \in \Omega} g(z, z_i) \tag{68}$$
$$= \lim_{\Omega \to W} \Big(\sum_{a_\mu \in \Omega} g(z, a_\mu) + \sum_{b_\nu \in \Omega} g(z, b_\nu)\Big)$$
$$= \sum_{a_\mu \in W} g(z, a_\mu) + \sum_{b_\nu \in W} g(z, b_\nu) \ .$$

27. For a sufficient condition the first terms of both x- and y-parts in (52) must be taken into account:

THEOREM. *If for some* $\zeta \in W$

$$\int_{\beta_\Omega} \overset{+}{\log} |w| \, dp_\Omega^* = O(1) \tag{69}$$

and

$$\sum_{b_\nu \in W} g(z, b_\nu) < \infty \ \ \textit{in} \ \ W - \{b_\nu\} \ , \tag{70}$$

then $w \in M_e B$ *and hence*

$$\lim_{\Omega \to W} \int_{\beta_\Omega} |\log |w|| \, dp_\Omega^* < \infty \tag{71}$$

and

$$\sum_{a_\mu \in W} g(z, a_\mu) < \infty \ \ \textit{on} \ \ W - \{a_\mu\} \tag{72}$$

as well.

Indeed, the characteristic

$$T(\Omega) = u_\Omega(\zeta, w) = x_\Omega(\zeta, w) + y_\Omega(\zeta, w)$$
$$= \frac{1}{2\pi} \int_{\beta_\Omega} \overset{+}{\log} |w| \, dp_\Omega^* + \sum_{b_\nu \in \Omega} g_\Omega(\zeta, b_\nu)$$

is $O(1)$ if (69), (70) hold. Properties (71), (72) then follow from $w \in M_e B$.

Another sufficient condition for $w \in M_e B$ is, of course, that $\int_{\beta_\Omega} \overset{+}{\log} |w^{-1}| \, dp_\Omega$ is bounded and $\Sigma g(\zeta, a_\mu) < \infty$ in $W - \{a_\mu\}$.

28. For "entire" functions in $M_e B$ the conditions simplify. Let $E_e B$ be the class of such functions, characterized by $w(z) \neq \infty$ on W.

THEOREM. *A necessary and sufficient condition for* $w \in E_e B$ *on* W *is that*

$$\int_{\beta_\Omega} \overset{+}{\log} |w| \, dp_\Omega = O(1) \ . \tag{73}$$

The proof is evident.

29. Consider the class H of regular harmonic functions h on W and let HP be the subclass of nonnegative functions. Set $\overset{+}{h} = \max(0, h)$.

THEOREM. *A harmonic function h on W has a decomposition*

$$h = u_1 - u_2 , \qquad u_1, u_2 \in HP \tag{74}$$

if and only if, for some ζ,

$$\int_{\beta_\Omega} \overset{+}{h}\, dp_\Omega^* = O(1) , \tag{75}$$

or, equivalently,

$$\lim_{\Omega\to W} \int_{\beta_\Omega} |h|\, dp_\Omega^* < \infty . \tag{76}$$

Proof. The multiple-valued function $w = e^{h+ih^*}$ is in M_e, and $w \neq 0, \infty$ on W. If (74) is given, then $\log|w| = u_1 - u_2$ and $w \in M_eB$. This implies

$$\lim_{\Omega\to W} \int_{\beta_\Omega} |\log|w||\, dp_\Omega^* = \lim_{\Omega\to W} \int_{\beta_\Omega} |h|\, dp_\Omega^* < \infty$$

and consequently $\int_{\beta_\Omega} \overset{+}{h}\, dp_\Omega^* = O(1)$. Conversely, suppose the latter condition holds,

$$\int_{\beta_\Omega} \overset{+}{\log}|w|\, dp_\Omega^* = O(1) .$$

Then $w \in M_eB$ and

$$h = \log|w| = x(z, w) - x(z, w^{-1}) ,$$

the y-terms vanishing because of the absence of zeros and poles of w.

It is known that functions u harmonic in the interior W of a compact bordered Riemann surface and with property (76) have a Poisson-Stieltjes representation (e.g., Rodin [6]). For further interesting results see Rao [5].

30. It is clear that theorems on $\log|w|$ can also be expressed directly in terms of $|w|$. Theorem 7, e.g., takes the following form:

THEOREM. *$w \in M_eB$ if and only if*

$$|w| = \left| \frac{\eta(z, w)}{\eta(z, w^{-1})} \right| , \tag{77}$$

where $\eta \in M_e B$ *and* $|\eta| < 1$ *on* W.

Proof. Suppose $w \in M_e B$, hence

$$\log |w| = u(z, w) - u(z, w^{-1}) , \tag{78}$$

$u \in LP$. Set

$$\eta(z, w) = \exp[-u(z, w^{-1}) - iu(z, w^{-1})^*] , \tag{79}$$

and (77) follows. Conversely, if (77) is given, then

$$\log |w| = \log |\eta(z, w)| - \log |\eta(z, w^{-1})| \tag{80}$$

is a difference of two functions in LP, and we have $w \in M_e B$.

31. The counterpart of Theorem 21 is as follows:

THEOREM. $w \in M_e B$ *if and only if*

$$|w| = \left| \frac{\varphi(z, w)\psi(z, w)}{\varphi(z, w^{-1})\psi(z, w^{-1})} \right| , \tag{81}$$

where $\varphi, \psi \in M_e B$ *and* $\varphi \neq 0$ *on* W, $|\varphi| < 1, |\psi| < 1$.

If $w \in M_e B$, choose

$$\begin{aligned} \varphi(z, w) &= \exp[- x(z, w^{-1}) - ix(z, w^{-1})^*] , \\ \psi(z, w) &= \exp[- y(z, w^{-1}) - iy(z, w^{-1})^*] , \end{aligned} \tag{82}$$

and we have (81). Conversely, (81) gives $\log |w| = u_1 - u_2$ with u_1, $u_2 \in LP$, hence $w \in M_e B$.

32. We introduce the classes O_{MB} and $O_{M_e B}$ of Riemann surfaces on which there are no nonconstant functions in MB and $M_e B$ respectively. Similarly, let O_{EB} and $O_{E_e B}$ be the subclasses determined by entire functions $w(z) \neq \infty$ on W in MB and $M_e B$. The problem here is to arrange these four classes in the general classification scheme of Riemann surfaces [1].

The inclusion relations

$$\begin{aligned} O_{M_e B} \subset O_{MB} \subset O_{EB} , \\ O_{M_e B} \subset O_{E_e B} \subset O_{EB} \end{aligned} \tag{83}$$

are immediately verified.

33. The smallest class in (83) is easily identified:

THEOREM. *All functions in M_eB on W reduce to constants if and only if W is parabolic,*

$$O_G = O_{M_eB} \,. \tag{84}$$

Proof. If $W \notin O_G$, there is a Green's function $g(z, \zeta)$, and

$$w = e^{-g-ig^*} \in M_eB \,. \tag{85}$$

In fact, g is bounded above in any $W - \Omega$, hence $m(\Omega, w) = O(1)$, and $N(\Omega, w) = 0$ gives $T(\Omega) = O(1)$. Conversely, if there is a nonconstant $w \in M_eB$ on W, then $\log|w| = u_1 - u_2$ where at least one $u_i \in LP$ is nonconstant superharmonic. This means that $W \notin O_G$. The same proof gives $O_G = O_{E_eB}$.

34. By the preceding theorem, every M_e-function on a parabolic W has unbounded characteristic. Even more can be said of M-functions on the larger class O_{MB} by comparing $T(\Omega)$ with k_Ω (no. 4):

THEOREM. *On $W \in O_{MB}$, the characteristic $T(\Omega)$ of any $w \in M$ tends so rapidly to infinity that*

$$\varliminf_{\Omega \to W} \frac{T(\Omega)}{k_\Omega} \geqq 1 \,. \tag{86}$$

Proof. Let $w(\zeta) = a$. The counting function of w for a is, by denfinition,

$$N(\Omega, a) = \int_{-\infty}^{k_\Omega} (n(h, a) - n(-\infty, a))dh + n(-\infty, a)k_\Omega \,,$$

where $n(h, a)$ is the number of a-points of w in the set $\bar{\Omega}_h$: $p_\Omega \leqq h \leqq k_\Omega$. We obtain from the first fundamental theorem [7] that

$$T(\Omega) + O(1) \geqq N(\Omega, a) \geqq n(-\infty, a)k_\Omega \,, \tag{87}$$

and (86) follows.

Thus (86) is obviously a property of every $w \in M, w \notin MB$, on every W.

35. We also observe:

THEOREM. *A function $w \in M$ on $W \in O_{MB}$ cannot omit a set of values of positive capacity.*

More accurately, the counting function $N(\Omega, a)$ of $w \in M$ on O_{MB} is unbounded on any set E of positive capacity. To see this we distribute mass $d\mu(a) > 0$ at $a \in E$, with $\int_E d\mu = 1$, and integrate Jensen's formula

$$\log|w(\zeta) - a| = \frac{1}{2\pi}\int_{\beta_\Omega} \log|w - a|\, dp_\Omega^* + N(\Omega, \infty) - N(\Omega, a) \tag{88}$$

$(w(\zeta) \neq \infty)$ over E with respect to $d\mu(a)$. We obtain Frostman's formula on W:

$$N(\Omega, \infty) - \frac{1}{2\pi}\int_{\beta_\Omega} u(w)dp_\Omega^* = \int_E N(\Omega, a)d\mu(a) - u(w(\zeta)) , \tag{89}$$

where $u(w) = \int_E \log|w - a|^{-1} d\mu(a)$. For equilibrium distribution $d\mu$ it is known from the classical theory that $u(w) = -\overset{+}{\log}|w| + O(1)$, and a fortiori $\int_{\beta_\Omega} u(w)dp_\Omega^* = -2\pi\, m(\Omega, \infty) + O(1)$, where $O(1)$ depends on E only. Substitution into (89) gives

$$T(\Omega) = \int_E N(\Omega, a)d\mu(a) + O(1). \tag{90}$$

This proves our assertion.

36. A comprehensive study of the role played by O_{MB} in the classification theory of Riemann surfaces is contained in the doctoral dissertation of K. V. R. Rao [5].

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