MODULE CLASSES OF FINITE TYPE

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1. Finite type. In this paper we consider only rings with minimum condition on left and on right ideals. Also, we only consider finitely generated modules over these rings (such modules always possess a composition series of submodules).

There have been several papers [3, 4, 5, 10, 11, 12] on the problem of constructing indecomposable modules over such rings. Most of these papers are devoted to showing that certain rings have an infinite number of non-isomorphic modules of a given composition length for each of an infinite number of composition lengths. In this paper we shall consider a finiteness condition, not on the class of all finitely generated modules but on certain subclassses of that class.

DEFINITION. If C is a class of modules over the ring R we shall say that C is of finite type if for each integer n there are only a finite number of non-isomorphic modules in C of composition length less than n.

We shall study conditions under which the following classes of modules are of finite type:

1. LT the class of left modules which are submodules of projectives. From the results of [1], it is clear that these are the torsionless modules.
2. LW the class of left W-modules, these modules A for which $\mathrm{Ext}^1_R(A, R) = 0$
3. LN the non-torsionless left modules
4. LQ the torsionless left modules which are not duals of right modules.
5. LD the class of duals of right modules.
6. LR the class of reflexive left modules [1].
7. LTW the class of torsionless W-modules.

In the above definitions the dual of a module A is $\mathrm{Hom}_R(A, R)$ denoted by A^*. Also, A is reflexive if the natural homomorphism $A \rightarrow A^{**}$ is an isomorphism. See [7].

The corresponding classes of right modules (RT, RW, etc.) are defined analogously. All the theorems we prove go through with left and right interchanged.

A useful tool in our study is the following theorem proved by Morita and Tachikawa in [9] and also mentioned by Brauer in [2].

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THEOREM A. *If P is projective and if the diagram*

$$\begin{array}{c} 0 \to X \to P \to A \to 0 \\ \qquad\qquad\quad \downarrow \theta \\ 0 \to Y \to P \to B \to 0 \end{array} \tag{1}$$

has exact rows and if θ is an isomorphism, then the diagram can be embedded in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X & \to & P & \to & A & \to & 0 \\ & & \downarrow \mu & & \downarrow \rho & & \downarrow \theta & & \\ 0 & \to & Y & \to & P & \to & B & \to & 0 \end{array} \tag{1'}$$

where ρ and μ are also isomorphisms.

It should be noted that the proof of Theorem A requires our standing hypothesis that every module under consideration has a composition series.

Before we deduce some corollaries from Theorem A, we need some additional information. Let l be the left composition length of the ring R and let r be the right composition length. Note that l and r need not be equal. Let $C(A)$ be the composition length of the module A.

LEMMA 1.1. *If the left module A has $C(A) = n$ then there exists a free module F_n, the direct sum of n copies of R considered as a left module, of composition length ln such that $F_n \to A \to 0$ is exact.*

The proof, an in induction on n, is essentially the same as the proof of Lemma 2.6 of [6]. By dualizing the above sequence we obtain

LEMMA 1.2. *If the left module A has $C(A) = n$ then A^* has composition length $\leqq nr$.*

Proof. The sequence of Lemma 1.1 induces $0 \to A^* \to F_n^*$ exact. The module F_n^* is a direct sum of copies of R considered as a right module [7] and hence $C(F_n^*) = nr$. Since A^* is a submodule of F_n^*, $C(A) \leqq nr$.

LEMMA 1.3. *If the left module A is torsionless with $C(A) = n$, then A can be embedded in a free module F such that $C(F) \leqq nrl$.*

Proof. By Lemma 1.2 $C(A^*) \leqq nr$ and by Lemma 1.1 there exists a free right module F_0 (the direct sum of nr copies of R) of composition length nr^2 such that $F_0 \to A^* \to 0$ is exact. This dualizes to

$$0 \to A^{**} \to F_0^* \qquad \text{exact}\,,$$

whereby the proof of Lemma 1.2 $C(F_0^*) = nrl$. But since $A \to A^{**}$ is a monomorphism, this can be used to embed A in F_0^*. The idea of the above proof is due to Bass [1], although, being in a more general situation he was not concerned there with composition length.

It should be noted that the inequalities of Lemma 1.2 and 1.3 are, for most rings, quite crude. Using the above lemmas, and Theorem A have the following results.

THEOREM 1.4. *If LN is of finite type then so is LQ.*

Proof. Suppose that for some n there were an infinite number of non-isomorphic modules $\{T_\alpha\}$ in LQ all of composition length n. Then by Lemma 1.3 we can embed them *all* as submodules of a free module F of composition length nlr. Consider the infinite collection of factors $\{F/T_\alpha\}$. By [1, 7] these are modules in LN all having composition length $n(lr - 1)$.

But the hypotheses of the theorem require that there are only a finite number of non-isomorphic modules in LN of each composition length. Thus for some $\alpha \neq \beta$ $F/T_\alpha \cong F/T_\beta$ and by Theorem A we have $T_\alpha \cong T_\beta$. This contradicts the assumption that the collection $\{T_\alpha\}$ consists of non-isomorphic modules.

The following theorem is modeled on the duality Theorem 1.1 of [7].

THEOREM 1.5. *LT is of finite type if and only if RT is of finite type.*

Proof. By right-left symmetry it is sufficient to prove the statement in one direction only.

Suppose that RT is of finite type and $\{T_\alpha\}$ is an infinite collection of non-isomorphic torsionless left modules all of composition length n. By Lemma 1.1 there is a free module F of composition length ln and an infinite collection of short exact sequences

$$F \xrightarrow{\mu_\alpha} T_\alpha \longrightarrow 0\,.$$

Now form the dual exact sequences.

$$0 \longrightarrow T_\alpha^* \xrightarrow{\mu_\alpha^*} F^* \longrightarrow F^*/T_\alpha^* \longrightarrow 0\,.$$

The right modules F/T^*_α are torsionless right modules [1; statements 4.2 and 4.4] and each of these modules has composition length less than rn. Since RT is of finite type there exist two indices α and β such that $F^*/T^*_\alpha \xrightarrow{\theta} F^*/T^*_\beta$ is an isomorphism. Using Theorem A we construct the exact commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T^*_\alpha & \xrightarrow{\mu_\alpha} & F^* & \longrightarrow & F^*/T^*_\alpha & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \mu} & & \big\downarrow{\scriptstyle \rho} & & \big\downarrow{\scriptstyle \theta} & & \\
0 & \longrightarrow & T^*_\beta & \xrightarrow[\mu_\beta]{} & F^* & \longrightarrow & F^*/T^*_\beta & \longrightarrow & 0
\end{array}$$

with vertical isomorphisms. This gives the commutative diagram

$$\begin{array}{ccc}
F^{**} & \xrightarrow{\mu_\alpha^{**}} & T^{**}_\beta \\
\big\downarrow{\scriptstyle \rho^*} & & \big\downarrow{\scriptstyle \mu^*} \\
F^{**} & \xrightarrow[\mu_\beta^{**}]{} & T^{**}_\alpha .
\end{array}$$

In this situation $Im\mu_\beta{}^{**}$ coincides with the natural image of T_β in T^{**}_β and the similar situation holds for the subscript α. Then commutativity then implies that T_α is isomorphic with T_β via the isomorphism μ^*. This contradicts the assumption that the collection $\{T_\alpha\}$ consisted of non-isomorphic modules.

2. A dual to Theorem A. A dual to Theorem A would state that if two submodules of a free module F were isomorphic, then the isomorphism can be extended to an automorphism of F. This is not, in general, true as we shall show by an example. However, by assuming enough extra conditions we can obtain the desired conclusion. Recall that X is a W-module if $\mathrm{Ext}^1_R(X, R) = 0$; see [8].

THEOREM 2.1. *If in the diagram*

$$\begin{array}{ccccccccc}
0 & \to & A & \to & F & \to & F/A & \to & 0 \\
 & & \big\downarrow{\scriptstyle \theta} & & & & & & \\
0 & \to & B & \to & F & \to & F/B & \to & 0
\end{array}$$

θ is an isomorphism, F is a free module and F/A and F/B are W modules, then the diagram can be embedded in a commutative diagram.

$$\begin{array}{ccccccccc}
0 & \to & A & \to & F & \to & F/A & \to & 0 \\
 & & \big\downarrow{\scriptstyle \theta} & & \big\downarrow{\scriptstyle \rho^*} & & \big\downarrow{\scriptstyle \mu} & & \\
0 & \to & B & \to & F & \to & F/B & \to & 0
\end{array}$$

with all the vertical maps isomorphisms.

Proof. Consider the dual sequences

$$(*)\qquad \begin{array}{ccccccccc} 0 & \to & (F/A)^* & \to & F^* & \to & A^* & \to & 0 \\ & & & & & & \uparrow{\scriptstyle \theta^*} & & \\ 0 & \to & (F/B)^* & \to & F^* & \to & B^* & \to & 0 \end{array}$$

The exactness at A^* and B^* comes from the fact that F/A and F/B are W-modules. Also θ^* is an isomorphism because θ is one. By Theorem A there exists an automorphism ρ of F^* so that the diagram

$$\begin{array}{ccccc} F^* & \to & A^* & \to & 0 \\ \uparrow{\scriptstyle \rho} & & \uparrow{\scriptstyle \theta^*} & & \\ F^* & \to & B^* & \to & 0 \end{array}$$

is commutative.

Now dualize again to obtain the commutative diagram

$$\begin{array}{ccccc} 0 & \to & A^{**} & \to & F^{**} \\ & & \downarrow{\scriptstyle \theta^{**}} & & \downarrow{\scriptstyle \rho^*} \\ 0 & \to & B^{**} & \to & F^{**} \end{array}$$

Since both A, B are torsionless and F is reflexive [1, 7] we can identify A and B with their images in A^{**} and B^{**}. Also the mappings with two stars on them, when restricted to these images, coincide with the original maps. Thus, identifying F with F^{**}, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \to & F & \to & F/A & \to & 0 \\ & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \rho^*} & & \downarrow{\scriptstyle \mu} & & \\ 0 & \to & B & \to & F & \to & F/B & \to & 0 \end{array}$$

where ρ^* induces μ on F/A to F/B. All the vertical maps are isomorphisms.

COROLLARY 2.2. *If LT is of finite type then so is LW.*

Proof. Suppose $\{W_\alpha\}$ is an infinite collection of nonisomorphic W-modules such that $C(W_\alpha) = n$. By Lemma 1.1 they are all epimorphic images of a free module F, $F \xrightarrow{\pi_\alpha} W_\alpha \to 0$ and $C(F) = ln$. The submodules Ker π_α of F all satisfy $C(\ker \pi_\alpha) = (l-1)n$ and by the assumption that LT is of finite type there exist two indices $\alpha \neq \beta$

such that $\operatorname{Ker} \pi_\alpha \cong \operatorname{Ker} \pi_\beta$. Now Theorem 2.1 implies that $W_\alpha \cong W_\beta$ contradicting the assumption that the elements in the collection $\{W_\alpha\}$ were non-isomorphic.

COROLLARY 2.3. *LTW is of finite type if and only if LR is of finite type.*

Proof. For the "if" part of the proof we proceed exactly as in the proof of Corollary 2.2. We use this fact, proved in [3], that if W is a torsionless W-module and

$$0 \to \operatorname{Ker} \pi \to F \to W \to 0$$

is exact with F free then $\operatorname{Ker} \pi$ is reflexive. Then the proof of 2.2 with the class LR replacing LT works here.

Conversely, if LTW is of finite type and if $\{Q_\alpha\}$ is an infinite collection of reflexives with $C(Q_\alpha) = n$, then by Lemma 1.3 they can all be embedded in a free module F with $C(F) \leqq lnr$,

$$0 \to Q_\alpha \to F .$$

But by [8] this embedding of the reflexive Q_α results in F/Q_α being a torsionless W-module. Hence by assumption there exists $\alpha \neq \beta$ such that $F/Q_\alpha \cong F/Q_\beta$. Then Theorem A implies $Q_\alpha \cong Q_\beta$ contradicting the assumption that the collection $\{Q_\alpha\}$ consists of non-isomorphic modules.

We conclude with an example which shows that Theorem 2.1. does not hold without the hypothesis that F/A and F/B are W-modules. Let R be the ring of matrices

$$\begin{pmatrix} x & 0 & 0 \\ y & x & 0 \\ z & 0 & x \end{pmatrix}$$

with x, y, z in a field K having more than 2 elements. R is commutative and is an indecomposable free module over itself. The radical N of R is the direct sum of two simple modules, $N = S_1 \oplus S_2$. If α, β are two distinct nonzero elements of K there is an automorphism θ of N which is "multiplication by α on S_1 and multiplication by β on S_2". Any extension of θ to a K-linear transformation on R will have two distinct eigenvalues. However, since R is indecomposable every module endomorphism (or automorphism) has only one eigenvalue, therefor θ cannot be extended to R.

REFERENCES

1. H. Bass, *Finitistic dimension and a homological generalization of semi primary rings*, Trans. Amer. Math. Soc., **95** (1960), 466-488.
2. R. Brauer, *Some remarks on associative rings and algebras*, National Research Council Publication **502** (1957), 4-11.
3. A. Heller, and I. Reiner, *Indecomposable representations*, Illinois J. of Math., **5** (1961), 314-323.
4. J. P. Jans, *On the indecomposable representations of algebras*, Annals of Math., **66** (1957), 418-429.
5. ———, *The representation type of algebras and subalgebras*, Canadian J. of Math., **X** (1958), 39-49.
6. ———, *Some generalizations of finite projective dimension*, Illinois J. of Math., **5** (1961) 334-344.
7. ———, *Duality in Noetherian rings*, Proc. Amer. Math. Soc., **12** (1961), 829-835.
8. ———, *On finitely generated modules over Noetherian rings*, To appear in Trans. Amer. Math. Soc.
9. K. Morita, and H. Tachikawa, *Character modules, submodules of a free module and quasi-Frobenius rings*, Math. Zeitschr, **65** (1956), 414-428.
10. H. Tachikawa, *A note on algebras of unbounded representation type*, Proc. Japan Acad., **36** (1960), 59-61.
11. T. Yoshii, *Note on algebras of strongly unbounded representation type*, Proc. Japan Acad., **32** (1956), 383-387.
12. ———, *On algebras of unbounded representation type*, Osaka Math. J., **8** (1956), 51-105.

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