POWERS OF A CONTRACTION IN HILBERT SPACE

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Introduction. Let H be a Hilbert space and P an operator with $\|P\| = 1$. Our main problem is to find the weak limits of $P^n x$ as $n \to \infty$. This is applied to Markov Processes and to Measure Preserving Transformations.

Markov Processes. Let (Ω, Σ, μ) be a measure space. Let x_n be a sequence of real valued measurable functions on Ω and:

1. $\mu(x_{n+\alpha} \in A \cap x_{m+\alpha} \in B) = \mu(x_n \in A \cap x_m \in B)$.
2. *Conditional probability that* $x_k \in A$ *given* x_i *and* x_j, $i < j < k$, *is equal to conditional probability that* $x_k \in A$ *given* x_j.

Let $I(\sigma)$ denote the characteristic function of σ. Define $P(n)$ by linear extension of:

$P(n)\, I(x_0 \in A) =$ *Conditional probability that* $x_n \in A$ *given* x_0.

Then:

1′. $\|P(1)\| = 1$

2′. $P(n) = P(1)^n$.

For details see [1] and [2].

We will study limits of

$$(P(1)^n\, I(x_0 \in A), I(x_0 \in B)) = \mu(x_n \in A \cap x_0 \in B)\ .$$

Many of the results here appear in particular cases in [1,] [2] and [3].

1. Reduction to unitary operators. For every $x \in H$

a. $\|P^{*k}P^kP^nx - P^nx\|^2 \leqq 2\|P^nx\|^2 - 2\,Re(P^{*k}P^kP^nxP^nx)$
$= 2(\|P^nx\|^2 - \|P^{n+k}x\|^2) \underset{n\to\infty}{\longrightarrow} 0$

b. $\|P^kP^{*k}P^nx - P^nx\|^2 \leqq \|P^{*k}P^kP^{n-k}x - P^{n-k}x\|^2 \underset{n\to\infty}{\longrightarrow} 0.$

Therefore:

If weak lim $P^{n_i}x = y$ then $P^{*k}P^ky = P^kP^{*k}y = y$ (here and elsewhere n_i or m_i will denote a subsequence of the integers). This means $\|y\| = \|P^ky\| = \|P^{*k}y\|$. Notice that if $P^*Px = x$ then $\|Px\|^2 = (P^*Px, x) = \|x\|^2$. On the other hand

$$\|Px\|^2 = (P^*Px, x) \leqq \|P^*Px\|\,\|x\| \leqq \|x\|^2 \text{ since } \|P\| = 1.$$

Hence if $\|Px\| = \|x\|$ then $(P^*Px, x) = \|P^*Px\|\,\|x\|$ and thus $P^*Px = x$.

THEOREM 1.1. *Let* $K = \{x \mid \|P^kx\| = \|P^{*k}x\| = \|x\|\ k = 1, 2, \cdots\}$

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then K is a subspace of H, invariant under P and P^*. On K the operator P is unitary. If $x \perp K$ then

$$\text{weak} \lim_{n\to\infty} P^n x = \text{weak} \lim_{n\to\infty} P^{*n} x = 0 .$$

Proof. It is only necessary to prove the last part. If $x \perp K$ and $y = \text{weak} \lim P^{n_i} x$ then by the preceding remark $y \in K$ hence $y = 0$. Now from the weakly sequentially compactness follows: weak lim $P^n x = 0$.

This theorem is a consequence of Theorem 2 of [9] and was reproduced here only because of the elementary proof.

If F is the selfadjoint projection on K and H is finite dimensional, then F is the spectral measure of the circumference of the unit circle in the sence of Dunford's spectral theory, with respect to P. This is no longer true when H is infinite dimensional and P a spectral operator (even a scalar type operator) in the sense of Dunford. These remarks are proved in [4].

LEMMA 2.1. *Let $y = \text{weak} \lim P^{n_i} x$. Then $\|y\|^2 \leqq \limsup |(P^n x, x)|$.*

Proof. Let $x = u + v$ where $u \in K$ and $v \perp K$. Then $y = \text{weak} \lim P^{n_i} u$, $\limsup |(P^n x, x)| = \limsup |(P^n u, u)|$. Now

$$|(y, P^k u)| = \lim_{i\to\infty} |(P^{n_i} u, P^k u)| = \lim |(P^{n_i - k} u, u)|$$

since $u \in K$. Thus

$$\|y\|^2 = \lim |(y, P^{n_i} u)| \leqq \limsup |(P^n u, u)| .$$

This could also be written in the form

$$\limsup |(P^n x, z)| \leqq \|z\| \limsup |(P^n x, x)|^{1/2} .$$

DEFINITION A. Let $H_0 = \{x \mid \lim (P^n x, x) = 0\}$.

THEOREM 3.1. *$x \in H_0$ if and only if weak $\lim P^n x = 0$, if and only if weak $\lim P^{*n} x = 0$. The set H_0 is a closed subspace of H containing $K^\perp$. If T commutes with P or with P^* and $x \in H_0$ then $Tx \in H_0$.*

Proof. The first parts of the theorem follow from Lemma 2.1 and Theorem 1.1. Now if $TP = PT$ and $P^n x \xrightarrow{w} 0$ then $P^n T x = TP^n x \xrightarrow{w} 0$.

Applications.

1. Markov processes.

a. If $\lim_{n\to\infty} \mu(x_n \in A \cap x_0 \in A) = 0$ then $\lim_{n\to\infty} \mu(x_n \in A \cap x_0 \in B) = 0$ and $\lim_{n\to\infty} \mu(x_0 \in A \cap x_n \in B) = 0$ for every set B.

b. Let $\lim \mu(x_n \in A \cap x_0 \in A) = \mu(x_0 \in A)^2$. Put $x = I(x_0 \in A) - \mu(x_0 \in A)$. (Provided that $\mu(\Omega) < \infty$ so that $1 \in L_2$).
Then

$$\begin{aligned}(P(1)^n x, x) &= (I(x_n \in A) - \mu(x_0 \in A), I(x_0 \in A) - \mu(x_0 \in A)) \\ &= \mu(x_n \in A \cap x_0 \in A) - \mu(x_0 \in A)^2 \to 0 .\end{aligned}$$

Thus for every Borel set B:

$$\lim (I(x_n \in A) - \mu(x_0 \in A), I(x_0 B)) = 0$$

or

$$\mu(x_n \in A \cap x_0 \in B) \to \mu(x_0 \in A)\, \mu(x_0 \in B) .$$

Similarly

$$\mu(x_0 \in A \cap x_n \in B) \to \mu(x_0 \in A)\, \mu(x_0 \in B) .$$

2. **Measure preserving transformations.** Let φ be a M.P.T. on (Ω, Σ, μ). If $\mu(\varphi^{-n}(A) \cap A) \to 0$ then

$$\lim_{n\to\infty} \mu(\varphi^{-n}(A) \cap B) = \lim_{n\to\infty} \mu(A \cap \varphi^{-n}(B)) = 0 .$$

if $\lim \mu(\varphi^{-n}(A) \cap A) = \mu(A)^2$ and $\mu(\Omega) < \infty$ then

$$\begin{aligned}&\mu(\varphi^{-n}(A) \cap B) \to \mu(A)\mu(B) \\ &\mu(A \cap \varphi^{-n}(B)) \to \mu(A)\mu(B) .\end{aligned}$$

3. **Measure theory.** Let μ be a positive finite measure on Borel subsets of $(0, 2\pi)$. Define the operator P by $Pf(\vartheta) = e^{i\vartheta} f(\vartheta)$. Then H_0 is the set of all functions f such that

$$\int_0^{2\pi} e^{in\vartheta} |f(\theta)|^2 \mu(d\vartheta) \to 0 .$$

Let $f \in H_0$ and $A_\varepsilon = \{\vartheta \,|\, |f(\vartheta)| \geqq \varepsilon\}$. Define $g_\varepsilon = 1/f$ on A_ε and zero elsewhere. Finally let

$$T_\varepsilon h(\vartheta) = g_\varepsilon(\vartheta) h(\vartheta) .$$

Then T_ε commutes with P and by Theorem 3.1

$$\int_A e^{in\vartheta} \mu(d\vartheta) \to 0$$

where $A = \cup A_\varepsilon$.

By taking unions of such sets one can prove:
There exists a set B such that for every h whose support is contained in B a.e.

$$\int e^{in\vartheta} |h(\vartheta)|^2 \mu(d\vartheta) \rightarrow 0$$

and this holds only for such functions.

2. **Positive contractions.** In this section we assume that H is the real Hilbert space $L_2(\Omega, \Sigma, \mu)$ where $\mu \geqq 0$ and $\mu(\Omega) = 1$. An operator S will be called positive if:

a. If $f \geqq 0$ a.e. than $Sf \geqq 0$ a.e.

b. $S1 = 1$.

c. $||S|| = 1$.

We will assume that P is positive. It is easily seen that so are P^*, P^nP^{*n} and $P^{*n}P^n$.

LEMMA 1.2. *Let S be a positive operator on $L_2(\Omega, \Sigma, \mu)$. The space*

$$L = \{f | Sf = f\}$$

is generate by characteristic functions of a σ subfield, Σ', of Σ: $f \in L$ if and only if f is Σ' measurable.

Proof. Let Σ' contain all $\sigma \in \Sigma$ such that $SI(\sigma) = I(\sigma)$. If $Sf = f$ then

$$||f||^2 \geqq (S|f|, |f|) \geqq |(Sf, f)| = ||f||^2$$

hence $S|f| = |f|$ therefore if $f, g \in L$ so do $\max(f, g)$ and $\min(f, g)$. This shows in particular that Σ' is a field and since L is closed it is a σ field.

Now if $f \in L$ so does $f - c$ for any constant, thus it is enough to show that

$$\{\omega | f(\omega) > 0\} \in \Sigma' :$$

Let f_+ be the positive part of f, $2f_+ = |f| + f \in L$. Thus $\varepsilon^{-1} \min(\varepsilon, f^+) \in L$ but as $\varepsilon \rightarrow 0$ this converges to $I\{\omega | f(\omega) > 0\}$.

This Lemma was proved in [8].

THEOREM 2.2. *The space K is generated by characteristic functions of a σ subfield Σ_1 of Σ. If $\sigma \in \Sigma_1$ then $PI(\sigma) = I(\tau)$ where $\tau \in \Sigma_1$, similarly for P^*.*

Proof. The space K is the intersection of the space

$$\{f \mid \|P^n f\| = \|f\|\}, \quad \{f \mid \|P^{*n} f\| = \|f\|\} \quad n = 1, 2, \cdots$$

By Lemma 1 each of this is generated by a σ subfield of Σ. Thus K is generated by the intersection of these subfields.

Now if $\sigma \in \Sigma_1$ then $\sigma' = \Omega - \sigma \in \Sigma_1$ too. The functions $P(I(\sigma))$ and $P(I(\sigma'))$ are positive, bounded by 1 and $(P(I(\sigma)), P(I(\sigma'))) = (P^*P(I(\sigma)), I(\sigma')) = (I(\sigma), I(\sigma')) = 0$. Moreover $P(I(\sigma)) + P(I(\sigma')) = 1$, therefore, both functions are characteristic functions. As K is invariant under P these are characteristic functions of sets in Σ_1.

Let $I(A)$ and $I(B)$ belong to K. Then

$$P(I(A)\cdot I(B)) \leqq \min\{P(I(A)), P(I(B))\} = P(I(A))\cdot P(I(B)) .$$

On the other hand

$$P^*[(P(I(A))\cdot P(I(B))] \leqq I(A)\cdot I(B)$$

or

$$P(I(A))\cdot P(I(B)) \leqq P(I(A)\cdot I(B)) .$$

Therefore

$$P(I(A)\cdot I(B)) = P(I(A))\cdot P(I(B)) .$$

It could be shown that if $f, g \in K$ and $f\cdot g \in L_2$ then $P(fg) = Pf\cdot Pg$.

Thus if $Pf = \alpha f$ and $Pg = \beta g$ where $|\alpha| = |\beta| = 1$ then $f, g \in K$ and if $f\cdot g\, L_2$ then $P(fg) = \alpha\beta fg$.

If $Pf = \alpha f$ where $|\alpha| = 1$ let $f = |f|\, h$ then:

$$\|f\|^2 \geqq (P|f|, |f|) \geqq |(Pf, f)| = \|f\|^2 .$$

Therefore, $P|f| = |f|$ necessarily $Ph = \alpha h$. It follows that

$$P(|f|\, h^2) = \alpha^2 |f|\, h^2 .$$

This is a Theorem of [8].

Following [1] let us define:

Doeblin's Condition. There exists a positive finite measure ν define on Σ, and a positive ε such that: If $\nu(\sigma) < \varepsilon$ then for some n either

$$\|P^{n}(I(\sigma))\| < \mu(\sigma)^{1/2}$$

or

$$\|P^{*n}(I(\sigma))\| < \mu(\sigma)^{1/2} .$$

Using the same arguments as in Theorem 3.11 and its corollaries of [1] we conclude.

THEOREM 3.2. *If Doeblin's condition holds then $\Sigma_1 = \{\sigma_1, \cdots, \sigma_n\}$ where σ_i are disjoint sets such that*

1. $\bigcup_{i=1}^{n} \sigma_i = \Omega$
2. $P^n(I(\sigma_i)) = I(\sigma_i) = P^{*n}(I(\sigma_i))$.
3. *The operator $P(P^*)$ acts as a permutation on the σ_i sets.*
4. *For each $f, g, \in L_2$*

$$\lim_{k\to\infty} (P^{nk+d}f, g) = \sum_{i=1}^{n} \mu(\sigma_i)^{-1} \int_{\sigma_i} f(\omega)\mu(d\omega) \int_{P^d\sigma_i} g(\omega)\mu(d\omega)$$

where $P^d\sigma_i$ denotes the set whose characteristic function is $P^d(I(\sigma_i))$.

Thus if x_n is a Markov process and $\mu(\Omega) = 1$ then

$$\lim \mu(x_{kn+d} \in A \cap x_0 \in B) = \sum_{i=1}^{n} \mu(\sigma_i)^{-1}\mu(x_0 \in A \cap \sigma_i)\mu(x_0 \in B \cap P^d\sigma_i) .$$

For detailed proves of these results and treatment of the case $\mu(\Omega) = \infty$ in the case of Markov processes see [1] and [3].

Measure Preserving Transformations. Let φ be a measure preserving transformation on (Ω, Σ, μ). The operator P is defined on $L_2(\Omega, \Sigma, \mu)$ by $Pf = g$ where $g(\omega) = f(\varphi(\omega))$. It is a positive contraction. Thus the space K is generated by all characteristic functions f that satisfy $\|P^{*n}f\| = \|f\|$, for P is an isometry. Let the restriction of P to K be denoted by U and let Σ_1 be the Boolean algebra that generates K. On Σ_1 φ acts like a measure preserving invertable transformation. (It maps Σ_1 onto itself).

We will use here the terminology of [5]

THEOREM 4.2. *The transformation φ on Σ is ergodic, weakly mixing or strongly mixing, if and only if, φ on Σ_1 is ergodic, weakly mixing or strongly mixing, respectively.*

Proof. It is clear that if P satisfies any of the requirements so does U. Conversely:

a. Let U be ergodic. If P was not then for some nonconstant function f, $Pf = f$. But then $P^nf = P^{*n}f = f$ and $f \in K$, so U is not ergodic.

b. Let U be weakly mixing. Given $f = f_1 + f_2$ where $f_1 \in K f_2 \perp K$ then for every g

$$\frac{1}{n}\sum_{j=0}^{n-1} |(P^jf, g) - (f, 1)(1, g)| \leqq \frac{1}{n}\sum_{j=0}^{n-1} |(P^jf_1, g) - (f_1, 1)(1, g)|$$
$$+ \frac{1}{n}\sum_{j=0}^{n-1} |(P^jf_2, g) - (f_2, 1)(1, g)| .$$

The first term tends to zero because U is weakly mixing and g can be replaced by the projection of g on K. The second term is equal to

$$\frac{1}{n}\sum_{j=0}^{n-1}|(P^j f_2, g)|$$

for $(f_2, 1) = 0$. Thus it tends to zero with $(P^n f_2, g)$.

c. Let U be strongly mixing. Put again $f = f_1 + f_2$ $P^n f_1$ tends weakly to $(f_1, 1)1 = (f, 1)1$ and $P^n f_2$ tends weakly to zero.

COROLLARY. *The transformation φ is weakly mixing, if and only if, P has on the unit circle no eigenvalue except for* 1 *which is a simple eigenvalue.*

This generalizes the 'Mixing Theorem' in [5] page 39.

Proof. The operator U satisfies the same condition and by the 'Mixing Theorem' is weakly mixing. By the previous theorem so is P.

3. The space H_c.

DEFINITION. $H_c = \{x \mid x \in K$ *and the set* $P^n x$ $n = 1, 2, \cdots$ *is conditionally compact*$\}$.

The set H_c is a subspace of H, invariant under P and P^*. $P^{n_i}x$ converges for $x \in K$ iff $(P^{n_i}x, P^{n_j}x) \to_{n_i, n_j \to \infty} ||x||^2$. This is equivalent to $(P^{*n_i}x, P^{*n_j}x) \to ||x||^2$ because P is unitary. Thus P could be replaced by P^* in the definition.

THEOREM 1.3. *The following conditions are equivalent:*
a. $x \in K$ *and* $P^n x$ *contains a convergent subsequence.*
b. *There exists a subsequence* m_i *such that* $x = \lim P^{m_i}x$.
c. $\limsup |(P^n x, x)| = ||x||^2$.

Proof.
$a \Rightarrow b$: Let $P^{n_i}x \to y$ then

$$||x||^2 = ||y||^2 = \lim (P^{n_i}x, P^{n_{i-1}}x) = \lim (P^{n_i - n_{i-1}}x, x)$$

because $x \in K$.
Hence $||x - P^{n_i - n_{i-1}}x|| \to 0$.
$b \Rightarrow c$: obvious.
$c \Rightarrow a$: Let $\lim |(P^{n_i}x, x)| = ||x||^2$ and weak $\lim P^{n_i}x = y$. Then $|(y, x)| = ||x||^2$ while $||y|| \leqq ||x||$ hence $y = \alpha x$ where $|\alpha| = 1$. From [7] page 79 $P^{n_i}x$ converges strongly to αx. Finall if $Z \in H_0$ then:

$$(Z, x) = \lim \alpha^{-1}(Z, P^{n_i}x) = \lim \alpha^{-1}(P^{*n_i}Z, x) = 0 .$$

It is clear that if $x \in H_c$ then condition (a) is satisfied hence the other conditions. In particular $H_c \perp H_0$.

THEOREM 2.3. *If $x \in H_c$ and $y = \lim_{i\to\infty} P^{n_i}x$ then there exists a subsequence k_i so that*

$$x = \lim P^{k_i}y .$$

Proof Let k_i be chosen so that

$$x = \lim P^{n_i+k_i}x .$$

Then

$$\lim ||x - P^{k_i}y|| = \lim ||P^{n_i}x - y|| = 0 .$$

4. Finitely many limits. Let x be such that the sequence $(P^n x, x)$ has finitely many limits. Let these be $c_1, c_2, \cdots, c_r$ where $|c_i| \leqq |c_{i+1}|$.

DEFINITION C. $L = \{z \mid P^n z = z$ for some $n\}$. If $z \in L$ then $az \in L$. If $z \in L$ and $y \in L$ then:

$$P^n z = z , \quad P^m y = y \Rightarrow P^{nm}(z + y) = z + y .$$

Thus L is a linear manifold, also $\bar{L} \subset H_c$.

If $z \in H$ let $\{z\}^0$ be the set consisting of z alone and $\{z\}^n$ be the set of all weak limits of $P^m y$ where $y \in \{z\}^{n-1}$.

Let $x = x_0 + x_1$ where $x_0 \in H_0\, x_1 \perp H_0$. Then

$$(P^n x, x) = (P^n x_0, x_0) + (P^n x_1, x_1), \lim (P^n x_0, x_0) = 0 .$$

Thus we will assume that $x \perp H_0$.

LEMMA 1.4. *For some k $\{x\}^k \cap L \neq 0$.*

Proof. Let $0 \neq y \in \{x\}^1$ then for every n $(y, P^n x)$ is equal to one of the values c_i and:

a. For every $n \geqq 0$ $(P^n y, y)$ can assume only the values c_i $1 \leqq i \leqq r$.

Let $(y, y) = |c_i|$. If for some k $|(P^k y, y)| = (y, y)$ then $P^k y = \lambda y$ with $|\lambda| = 1$. Thus λ must be a root of one for $(P^{nk}y, y) = \lambda^n(y, y)$ assumes finitely many values. Therefore in this case $y \in L$.

If $|(P^n y, y)| < (y, y)$ for every n then

$$\limsup_{n\to\infty} |(P^n y, y)| < (y, y) .$$

Also $\limsup (P^n y, y,) \neq 0$ for $y \perp H_0$. Thus we may choose a subsequence n_i so that $P^{n_i} y$ will converge weakly to $z \neq 0$. Now z satisfies a and $\|z\| < \|y\|$ by Lemma 2.1.

This procedure cannot be continued more than r times thus at some stage we must get an element of L.

LEMMA 2.4. *If u is the projection of x on $\bar{L}$ then $u \in L$.*

Proof. Let $0 \neq y \in \{x\}^k \cap L$. Then $y \in \{u\}^k + \{x - u\}^k$. Now $y \in L$ and $x - u \perp L$. Also L is invariant under P and P^* hence $\{x - u\}^k \perp L$ and $y \in \{u\}^k$. By Theorem 2.3 $u \in \{\overline{P^n y}\}$ which is a finite set in L.

THEOREM 3.4. *If the sequence $(P^n x, x)$ has finitely many limits then $x = x_0 + x_1$ where $x_0 \in H_0$ and $x_1 \in L$.*

Proof. Let $x_1 = u + v$ where $u \in L$ (by Lemma 2.4.) and $v \perp L$. Now $(P^n v, v) = (P^n x_1, x_1) - (P^n u, u)$ has finitely many limits and by Lemma 1.4 cannot be orthogonal to L unless it is zero.

If limit $(P^n x, x)$ exists then $P x_1 = x_1$.

If L is one dimensional (for instance ergodic transformations) then the conditions of Theorem 3.4 imply that $P x_1 = x_1$.

THEOREM 4.4. *Let $A = \{x$ the sequence $(P^n x, x)$ has finitely many limits$\}$. If linear combinations of elements of A are dense in H, then the eigenvalues of P on the circumference of the unit circle, are roots of* 1.

Proof. Let $Px = \lambda x$ where $|\lambda| = 1$. Let $x_i \in A$ and $y = \Sigma a_i x_i$ where $\|x - y\| < 1/2 \|x\|$.
Since $x \perp H_0$ we may assume that for some integers k_i $P^{k_i} x_i = x_i$. Hence for $k = k_1 k_2 \cdots k_n$ we have $P^k y = y$. Thus

$$\lambda^{km} x = P^{km} x = y + P^{km}(x - y) .$$

Therefore

$$|\lambda^{km} - 1| \|x\| \leqq \|\lambda^{km} x - y\| + \|y - x\| < \|x\| .$$

This equation cannot be satisfied for all values of m unless λ^k is a root of 1.

5. Semi groups of contractions. Let $P(t)$ be a strongly continuous semi group of contractions $0 \leqq t$. For every $\delta > 0$ $P(\delta)$ defines the subspace $K(\delta)$ as in Theorem 1.1.

LEMMA 1.5. $x \in K(\delta)$ *if and only if*

$$\|P(t)x\| = \|P(t)^*x\| = \|x\| \quad 0 \leqq t < \infty \ .$$

Proof. Trivially the condition is sufficient. If $x \in K(\delta)$ and $t \leqq n\delta$ then

$$\|x\| = \|P(n\delta)x\| = \|P(n\delta - t)\,P(t)x\| \leqq \|P(t)x\| \leqq \|x\| \ .$$

Thus $\|P(t)x\| = \|x\|$ and similarly $\|P(t)^*x\| = \|x\|$.

Thus all the spaces $K(\delta)$ are the same and will be denoted by K.

THEOREM 2.5. *The space K is invariant under $P(t)$ and $P(t)^*$ for all t. On K $P(t)$ is unitary. If $x \perp K$ then*

$$\text{weak}\lim_{t\to\infty} P(t)x = 0$$

and by symmetry

$$\text{weak}\lim_{t\to\infty} P(t)^*x = 0 \ .$$

Proof It was shown that $K = K(t)$ hence by Theorem 1.1 K is invariant under $P(t)$ and $P(t)^*$ and $P(t)$ is unitary on K.

Let $x \perp K$ and let $y \in H$ and $\varepsilon > 0$ be given. Choose η so that

$$\|P(s)x - x\| < \varepsilon. \quad \text{if } s \leqq \eta \ .$$

Choose n_0 so that

$$|(P(n\eta)x, y)| < \varepsilon \quad \text{if } n \geqq n_0 \ .$$

This is possible by Theorem 1.1. If

$$(n+1)\eta \geqq t \geqq n\eta > n_0\eta$$

then

$$|(P(t)x, y)| \leqq |(P(n\eta)x, y)| + |(P(t)x - P(n\eta)x, y)| \ .$$

The first term is less than ε because $n > n_0$. The second term is bounded by

$$\begin{aligned}\|y\|\,\|P(t)x - P(n\eta)x\| &= \|y\|\,\|P(n\eta)\,(P(t-n\eta)x - x)\| \\ &\leqq \|y\|\,\|P(t-n\eta)x - x\| \leqq \|y\|\varepsilon\end{aligned}$$

for $0 \leqq t - n\eta \leqq \eta$.

This is proved also in [9] Theorem 4.

Let us assume in this section:

(*) *For some $t_0 > 0$ the operator $P(t_0)\,P(t_0)^*$ is the sum of a compact operator and an operator of norm less then one.*

This is equivalent to:

(**) *For some* $0 < t_0$ *the point* 1 *is isolated in the spectrum of* $P(t_0)\,P(t_0)^*$ *and the space of eigenvectors corresponding to it is finite.*

It is clear that (**) implies (*). Now if 1 is not an isolated point of the spectrum, with finite eigenvectors space, there is a sequence of orthonormal vectors x_n such that

$$||P(t_0)\,P(t_0)^*x_n - x_n|| \to 0 \ .$$

(We use here the fact that $P(t_0)\,P(t_0)^*$ is self adjoint). Let

$$P(t_0)\,P(t_0)^* = A + B$$

where B is compact and $||A|| < 1$. Then

$$||Ax_n + Bx_n - x_n|| \to 0 \ .$$

But B is compact hence $Bx_n \to 0$ hence

$$||Ax_n - x_n|| \to 0$$

and 1 is the spectrum of A contrary to assumption.

It is easily seen that $P(t)\,P(t)^*$ satisfy, also, the condition if $t > t_0$: $P(t)\,P(t)^* = P(t - t_0)P(t_0)P(t_0)^*P(t - t_0)^*$. Let

$$K(t) = \{x \,|\, ||P(t)^*x|| = ||x||\} = \{x \,|\, P(t)P(t)^*x = x\} \ .$$

Then $K(t_1) \subset K(t_2)$ if $t_1 > t_2$ and $K(t)$ is finite dimensional when $t \geqq t_0$.

For some $s > 0$ $\dim K(s)$ is minimal hence $K(s) = K(s + h)$ for all $h \geqq 0$. Let us denote $K(s)$ by K.

LEMMA 3.5. *The space* K *is invariant under* $P(h)^*$ *and* $P(h)$ *for all* $h > 0$.

Proof. If $x \in K$ then $x \in K(s + h)$ hence

$$||P(s + h)^*x|| = ||x||$$

hence

$$||x|| = ||P(s)^*P(h)^*x|| \leqq ||P(h)^*x|| \leqq ||x||$$

or $P(h)^*x \in K$.

Now on the finite dimensional space K, the operator $P(h)^*$ is norm preserving and therefore onto.

If $x \in K$ then for some $y \in K$ $P(h)^*y = x$ and $||x|| = ||y||$. Thus $P(h)x = y \in K$.

We may assume that $s \geqq t_0$.

The subspace $K^\perp$ is also invariant under $P(t)$ and $P(t)^*$. Now

$P(s)\, P^*(s)$ is quasi compact on K and

$$(P(s)\, P^*(s)x, x) < 1 \quad x \in K^\perp .$$

Hence on $K^\perp$ $\|P(s)\| = c < 1$:
The operator $P(s)$ *is quasi compact on* H (in the sense of (*).

Let A be the infinitesimal generator of $P(t)$ then:

1. On K the operator $(1/i)A$ is self adjoint.
2. On $K^\perp$

$$\sigma(\mathrm{A}) \subset \{\lambda \mid Re\, \lambda \leqq \omega_0\}$$

where

$$\omega_0 = \lim_{t\to\infty} t^{-1} \log \|P(t)\| .$$

See [6] corollary to Theorem 11.5.1
Now

$$\omega_0 = \lim_{n\to\infty}(ns)^{-1} \log \|P(ns)\| \leqq \lim_{n\to\infty}(ns)^{-1} \log \|P(s)\|^n \leqq s^{-1} \log c < 0 .$$

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