SOME DEGENERATE CAUCHY PROBLEMS WITH OPERATOR COEFFICIENTS

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1. Motivated in part by connections with problems in transonic gas dynamics there has been considerable interest in equations of the form

$$u_{tt} - K(t)u_{xx} + bu_x + eu_t + du - h = 0 \tag{1.1}$$

where d, b, e and h are functions of (x, t) (see here Bers [4] for a bibliography and discussion). In particular there arises the Cauchy problem for (1.1) in the hyperbolic region with data given on the parabolic line $t = 0$ (see in particular Protter [20], Conti [9], Bers [3], Berezin [2], Hellwig [12; 13], Frankl [10], Weinstein [25], Krasnov [15; 16], Carroll [8], Germain and Bader [11], and Barancev [1]). Protter assumes that $K(t)$ is a monotone increasing function of t, $K(0) = 0$, and shows that the Cauchy problem for (1.1) with initial data $u(x, 0)$ and $u_t(x, 0)$ prescribed on a finite x-interval, is correctly set (under suitable regularity assumptions) if $tb(x, t)/\sqrt{K(t)} \to 0$ as $t \to 0$. Thus in particular if $b \equiv 0$ the condition is automatically true. Krasnov considers generalized solutions and the equation

$$u_{tt} - \Sigma \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial u}{\partial x_k}\right) + \Sigma b_i \frac{\partial u}{\partial x_i} + e\frac{\partial u}{\partial t} + du = h\,. \tag{1.2}$$

Again the presence of first order terms b_i complicates the matter and (as with Protter for $K(t) \sim t^{\alpha}$) it is assumed that $b_i = O(t^{\alpha/2-1}\beta(t))$ where $\beta(t) \to 0$ (additional assumptions are also made). Krasnov supposes $\Sigma a_{ik}\xi_i\xi_k \geqq ct^{\alpha}\Sigma\xi_i^2$ with $h/t^{\frac{\alpha-1+\delta_0}{2}} \in L^2$ ($\delta_0 > 0$ is a number for which bounds are determined in the proof) and finds solutions u such that $u_t/t^{\frac{\alpha+1+\delta_0}{2}} \in L^2$ and $u_{x_i}/t^{\frac{1+\delta_0}{2}} \in L^2$. Thus the growth of h appears to play an important role in determining a solution in this more general equation (1.2). Slightly more general degeneracies for $\Sigma a_{ik}\xi_i\xi_k$ are mentioned by Krasnov but always in some comparison to a power of t.

It is one of the aims of the present paper to give a more precise estimate of the allowable degeneracy in relation to the growth of h and to give estimates for the solution. In particular we will not require that $K(t)$ be monotone. For simplicity we omit here first order terms in $\partial u/\partial x_i$; this will be dealt with, in an abstract framework, in a subsequent article. A summary of some of the present work was

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given in [8]. We remark that an operational treatment of the type of degenerate problems considered by Tersenov [24] and Hu Hsien Sun [14] is also contemplated (this involves an equation of the form $K(t)u_{tt} - u_{xx} + bu_x + eu_t + du - h = 0$ with data given for $t = 0$). As indicated above our results generalize in certain respects those of Krasnov, however the methods employed here are quite different; for example Krasnov relies heavily on a Galerkin type method for existence whereas we employ an energy method based on work of Lions [17]. Further generalizations in our framework are clearly possible (see [16]).

2. Following Lions (see [18] for an extensive bibliography and treatment of operational differential equations) we reformulate (1.2) as follows. Let V and H, $V \subset H$, be Hilbert spaces, V dense in H, with the topology of V being finer than that induced by H.* The norms in V and H are denoted by $\|\ \|$ and $|\ |$ respectively. Let $(u, v) \to a(t, u, v)$ be a continuous sesquilinear form on $V \times V$ for t fixed, $0 \leqq t \leqq b < \infty$, with $a(t, u, v) = \overline{a(t, v, u)}$. Assume that $t \to a(t, u, v) \in C^1[0, b]$ for (u, v) fixed. We recall (see [18]) that the form $a(t, u, v)$ defines an unbounded operator $A(t)$: $D(A(t)) \to H$ by defining $D(A(t))$ to be the set of $u \in V$ such that $v \to a(t,u,v)$ is continuous on V in the topology of H. Then we can write for $u \in D(A(t))$, $(A(t)u, v) = a(t, u, v)$ for $v \in V$. Now let $\{B(t)\}$ be a family of bounded Hermitian operators in H with $t \to B(t) \in \mathscr{E}^1(\mathscr{L}_s(H, H))$ (here $\mathscr{E}^m(G)$ is the space of m-times continuously differentiable functions of t with values in G and $\mathscr{L}_s(H, H)$ is the space of continuous linear maps $H \to H$ with the topology of simple convergence—see [5]).

Let now $\psi > 0$ be a numerical function with $\psi \uparrow$ as $t \to 0$, $\psi \in C^0(0, b]$. Here ψ does not necessarily approach ∞. We assume q is another numerical function such that $q > 0$ on $(0, b]$ with $q \to 0$ as $t \to 0$ (in what follows all limits such as $q \to 0$ will refer to $t \to 0$). Let f be given such that $\psi f \in L^2(H)$ (for the spaces $L^p(H)$ and the integration of vector valued functions see [6; 7]). We assume $q \in C^1(0, b]$. Let $\mathscr{F}_s$ be the Hilbert space of functions u on $[0, s]$ such that $u(0) = 0$, $\psi u' \in L^2(H)$, and $\omega u \in L^2(V)$ with

$$\|u\|^2_{\mathscr{F}_s} = \int_0^s \{\|\omega u\|^2_V + |\psi u'|^2_H\}\, dt \tag{2.1}$$

(ω is a numerical function to be determined, $\omega > 0$, $\omega \to \infty$). Here all derivatives are taken in the sense of vector valued distributions in $\mathscr{D}'(H)$(see [23]) and $\mathscr{F}_s$ may be proved complete by standard arguments. Let now $\mathscr{H}_s$ be the space of functions h which satisfy $h(s) = 0$, $h/\psi \in L^2(H)$, $h'/\psi \in L^2(H)$, and $qh/\omega \in L^2(V)$. Set

* H is also assumed to be separable for simplicity in a later argument; this condition is not necessary however.

$$\tilde{E}_s(u,h) = \int_0^s \{qa(t, u, h) + (B(t)u', h) - (u', h')\}dt \tag{2.2}$$

and define

$$\tilde{L}_s(h) = \int_0^s (f, h)dt \ . \tag{2.3}$$

We note that (2.2) and (2.3) are well defined for $u \in \mathscr{F}_s, h \in \mathscr{H}_s$, and f as described. Thus assume ω as indicated has been given; then we pose

Problem 1. Find s and $u \in \mathscr{F}_s$ such that for all $h \in \mathscr{H}_s$

$$\tilde{E}_s(u, h) = \tilde{L}_s(h) \ . \tag{2.4}$$

Naturally we wish to find the best ω in some sense when posing problem 1. Here best will be left vague for the present in remarking only that ω furnishes a measure of how rapidly the solution u tends to 0 as $t \to 0$. We define now $\mathscr{K}_s$ to be the space of functions k such that $k = \int_0^t \varphi h d\xi$ for $h \in \mathscr{H}_s$ where φ is a numerical function to be determined (in general $\varphi \in C'[0, s]$, $\varphi > 0$ on $(0, s]$, and $\varphi \to 0$ as $t \to 0$). Clearly $k' = \varphi h$ and thus $k'/\varphi\psi = h/\psi \in L^2(H)$. For suitable choice of the numerical function $\delta > 0, \delta \to \infty$, we define $\mathscr{K}_s$ as a prehilbert space with norm

$$\|k\|^2_{\mathscr{K}_s} = \int_0^s \left\{ \|\delta k\|^2_V + \left|\frac{k'}{\varphi\psi}\right|^2_H \right\} dt \tag{2.5}$$

LEMMA 1. *Define* $v = \varphi/q$ *and assume*

(i) $\varphi\psi^2 \in L^\infty$

(ii) $\omega \leqq \delta$

(iii) $\omega^2 v^2 \in L^1$

(iv) $\delta^2 \int_0^t \omega^2 v^2 d\xi \in L^1$ *with* $\varphi, q, \omega, \psi, \delta \in C^0(0, s]$ *all positive on* $(0, s]$.

Then $\mathscr{K}_s \subset \mathscr{F}_s$ algebraically and topologically.

Proof. The following estimates are straightforward

$$|\psi k'| = \left|\frac{\varphi\psi^2 k'}{\varphi\psi}\right| \leqq c\left|\frac{k'}{\varphi\psi}\right| \tag{2.6}$$

$$\|\delta k\|^2 = \left\| \delta \int_0^t \frac{q}{\omega}\, \omega v h d\xi \right\|^2 \leqq \delta^2 \int_0^t \omega^2 v^2 d\xi \int_0^t \left\| \frac{qh}{\omega} \right\|^2 d\xi \ . \tag{2.7}$$

Thus by (2.7) for $k \in \mathscr{K}_s$ and δ satisfying the hypotheses we have $\int_0^s \|\delta k\|^2 d\xi < \infty$; also by (2.6) and the fact $\omega \leqq \delta$ it follows that $\|k\|_{\mathscr{F}_s} \leqq \tilde{c}\|k\|_{\mathscr{K}_s}$. From (2.7) we obtain also the result that $\|k\|^2 \to 0$ as $t \to 0$ which proves that in fact $\mathscr{K}_s \subset \mathscr{F}_s$.

LEMMA 2. *Assume* (i)–(iv) *and*

(v) $1/v\int_0^t \omega^2 v^2 d\xi \in L^\infty$

(vi) $\varphi'\psi^2 \in L^\infty$

(vii) $1/v\delta^2 \in L^\infty$

(viii) $-(1/v)'\,1/\delta^2 \in L^\infty$, $v' \geqq 0$. *Assume also that* $a(t, u, u) \geqq \alpha\,\|u\|^2$, *then*

$$2ReE_s(k, k) \geqq \int_0^s \|\delta k\|^2 \left\{-\alpha\left(\frac{1}{v}\right)' \frac{1}{\delta^2} - \frac{c_1}{v\delta^2}\right\} dt + \int_0^s \left|\frac{k'}{\varphi\psi}\right|^2 \{\varphi'\psi^2 - 2\beta\varphi\psi^2\}\, dt \tag{2.8}$$

where, for $k = \int_0^t \varphi h d\xi$, $E_s(u, k) = \tilde{E}_s(u, h)$.

Proof. Formally we have

$$2ReE_s(k, k) = \frac{q}{\varphi} a(t, k, k)\Big|_0^s - \int_0^s \left\{\left(\frac{q}{\varphi}\right)' a(t, k, k) - \left(\frac{q}{\varphi}\right) a'(t, k, k)\right\} dt\Big\} + 2Re\int_0^s \frac{1}{\varphi}(Bk', k')dt - \varphi\,|h|^2\Big|_0^s + \int_0^s \varphi'\,|h|^2\, dt \,. \tag{2.9}$$

Noting that $\lim \varphi\,|h|^2 = \lim 1/\varphi\,|k'|^2 = \theta^2 \geqq 0$ will exist if all the other terms make sense we have

$$\frac{q}{\varphi}\, a(t, k, k) \leqq \frac{c}{v}\,\|k\|^2 \leqq \frac{c}{v}\int_0^t \omega^2 v^2 d\xi \int_0^t \left\|\frac{qh}{\omega}\right\|^2 d\xi \tag{2.10}$$

which vanishes as $t \to 0$. Note by the Banach Steinhaus theorem it follows that (see [18])

$$|a(t, u, h)| \leqq c\|u\|\,\|h\| \tag{2.11}$$

$$|a'(t, u, h)| \leqq c_1\|u\|\,\|h\| \tag{2.12}$$

$$\left|\int_0^s \frac{1}{\varphi}\,(Bk', k')dt\right| \leqq \beta\int_0^s \left|\frac{k'}{\varphi\psi}\right|^2 \varphi\psi^2 dt < \infty \,. \tag{2.13}$$

Moreover under the hypotheses above

$$\int_0^s \frac{\varphi'}{\varphi^2}\,|k'|^2 dt = \int_0^s \varphi'\psi^2 \left|\frac{k'}{\varphi\psi}\right|^2 dt < \infty \tag{2.14}$$

$$\left|\int_0^s \frac{q}{\varphi}\, a'(t, k, k)dt\right| \leqq c_1\int_0^s \frac{1}{v\delta^2}\,\|\delta k\|^2 dt < \infty \tag{2.15}$$

$$-\int_0^s \left(\frac{q}{\varphi}\right)' a(t, k, k)dt \leqq c\int_0^s -\left(\frac{1}{v}\right)' \frac{1}{\delta^2}\,\|\delta k\|^2 dt < \infty \tag{2.16}$$

Thus (2.9) is valid and (2.8) follows.

The formula (2.8) indicates the properties desired of δ and φ in order to obtain an estimate $ReE_s(k, k) \geqq \Omega \|k\|^2_{\mathscr{H}_s}$ thus enabling us to apply the Lions projection theorem (see [18]). We will give here a natural choice for δ, φ etc. without seeking the best possible result. To this end set

$$\varphi = \hat{c}\int_0^t \frac{d\xi}{\psi^2}\,. \tag{2.17}$$

Then $\varphi \in C^1[0, b]$, $\varphi \to 0$, and since ψ is monotone $\varphi/\varphi' = \psi^2\int_0^t d\xi/\psi^2 \leqq Nt$. Hence $\varphi\psi^2 = \hat{c}\varphi/\varphi' \to 0$ also and thus $1/\varphi\psi \to \infty$. Next let $R \neq 0$ be a constant and

$$-\left(\frac{1}{v}\right)'\frac{1}{\delta^2} = R;\ v = \frac{1}{\left[\delta_1 + \int_t^s R\delta^2 d\xi\right]} \tag{2.18}$$

where $\delta_1 > 0$ is determined by $v(s)$. Thus $v \to 0$ corresponds to $\delta \notin L^2$ and in any case, noting $v' = Rv^2\delta^2$,

$$\frac{1}{v}\int_0^t \omega^2 v^2 d\xi \leqq \frac{1}{v}\int_0^t \delta^2 v^2 d\xi = \frac{1}{R}\left[1 - \frac{v(0)}{v(t)}\right] = \frac{1}{R}\left\{1 - \frac{\delta_1 + \int_t^s R\delta^2 d\xi}{\delta_1 + \int_0^s R\delta^2 d\xi}\right\}. \tag{2.19}$$

(This shows that $\int_0^t \omega^2 v^2 d\xi < \infty$ and that $1/v\int_0^t \omega^2 v^2 d\xi \leqq M$. The last term in (2.19) is taken to be zero if $\delta \notin L^2$ or $v(0) = 0$, and $v(0)/v(t)$ is seen to be bounded by one in all other cases.) Thus (i), (ii) (by assumption), (iii), (v), (vi), and (viii) hold. Also the $\varphi'\psi^2$ term dominates in the second integral of (2.8) for s small. Now for (vii) we note that $1/v\delta^2 = (v/v')R$ and $v' = (\varphi/q)'$; thus

$$\frac{v'}{v} = \frac{\varphi'}{\varphi} - \frac{q'}{q} = \frac{\varphi'}{\varphi}\left[1 - \frac{q'\psi^2}{q}\int_0^t \frac{d\xi}{\psi^2}\right]. \tag{2.20}$$

If we assume for example that $(q'\psi^2/q)\int_0^t d\xi/\psi^2 \leqq 1 - \varepsilon_1$ for t small then $v'/v \geqq \varepsilon_1\varphi'/\varphi \to \infty$ since $\varphi, \varphi' > 0$ on $(0, b]$ and $\varphi/\varphi' \to 0$. In any case if $v'/v \to \infty$ then $v/v' \to 0$ and $1/v\delta^2 \to 0$ which means not only that (vii) holds but that the $-\alpha(1/v)'1/\delta^2$ term dominates in the first integral of (2.8) for s small. Note here that φ and hence v are defined on $[0, b]$ independently of s by say (2.17) whereas (2.18) determines δ^2 on any interval $(0, s]$ for v given. Finally with regard to (iv) there are various hypotheses on ω and v which would work but we assume simply that

$$\omega^2 = \frac{v'}{v^{2-\varepsilon}},\ 0 < \varepsilon < 1 \tag{2.21}$$

Then if say $v \in C^0[0, b]$

$$(2.22)\qquad \begin{aligned}\int_0^s \delta^2\left(\int_0^t \omega^2 v^2 d\xi\right) dt &= \int_0^s \frac{v'}{Rv^2}\left(\int_0^t v' v^{\varepsilon} d\xi\right) dt \\ &= \frac{1}{R(1+\varepsilon)}\int_0^s \frac{v'}{v^{1-\varepsilon}} dt = \frac{1}{R\varepsilon(1+\varepsilon)} v^{\varepsilon}(t)\Big|_0^s .\end{aligned}$$

It should be noted that $v \in C^0[0, b]$ now implies that $\omega \leqq c\delta$ since $\omega^2/\delta^2 = Rv^{\varepsilon}$ and this would be a condition equivalent to (ii). We remark that $v \to 0$ implies $\omega \notin L^2$ since $\int_t^s \omega^2 d\xi = \int_t^s v'/v^{2-\varepsilon} d\xi = 0(1/v^{1-\varepsilon})$. This proves

LEMMA 3. *Assume* $a(t, u, u) \geqq \alpha\|u\|^2$, $v'/v \to \infty$, $v \in C^0[0, b]$, $\omega^2 = v'/v^{2-\varepsilon}$, $\varphi = \hat{c}\int_0^t d\xi/\psi^2$, *and* $v = 1/\delta_1 + \int_t^s R\delta^2 d\xi$. *Then* $\omega \leqq c\delta$ *and* (i), (iii)–(viii) *hold with* $ReE_s(k, k) \geqq \Omega\|k\|^2_{\mathscr{K}_s}$ *for s sufficiently small.*

Using the above lemmas and the Lions projection theorem (see [18]) there results

THEOREM 1. *Under the hypotheses of Lemma* 3 *and the conditions on* $a(t, u, v)$, $B(t)$ *stipulated above there exist functions* ω ($\omega \notin L^2$ *if* $v \to 0$) *such that for s small problem* 1 *has a solution.*

Proof. We need only check that the map $u \to E_s(u, k)$: $\mathscr{F}_s \to \boldsymbol{C}$ is continuous for $k \in \mathscr{K}_s$ fixed and that the map $k \to L_s(k) = \tilde{L}_s(h)$: $\mathscr{K}_s \to \boldsymbol{C}$ is continuous. This verification is immediate.

Now since $q > 0$ on $(0, b]$ we can treat $qa(t, u, v)$ as a nondegenerate form on say $[s/2, b]$ and apply Lions' results for such problems (see [17; 18]). We want to solve

Problem 2. Find $u \in \mathscr{F}_b$ such that $\tilde{E}_b(u, h) = \tilde{L}_b(h)$ for all $h \in \mathscr{H}_b$.

Thus suppose the problem has been solved for $[0, s]$, that is suppose problem 1 has been solved with solution u_1. Then following [17] let $p \in C^1$ with $p = 1$ on $[0, 2/3\,s]$ and $p = 0$ in a neighborhood of s. Set $u_2 = u - pu_1$; then $u_2 = 0$ on $[0, 2/3\,s]$ and $u_2 = u$ for $t \geqq s$. The problem 2 for u becomes

$$(2.23)\qquad \begin{aligned}\tilde{E}_b(u_2, h) &= \int_0^b (f, h)dt - \int_0^b p'[(Bu_1, h) + (u_1', h)]dt \\ &\quad - \int_0^b \{qa(t, u_1, ph) + (Bu_1', ph) - (u_1', (ph)')\}dt .\end{aligned}$$

Now if $h \in \mathscr{H}_b$ we see that $ph \in \mathscr{H}_s$; hence

$$(2.24)\qquad \tilde{E}_b(u_2, h) = \int_0^b (f, h - ph)dt - \int_0^b p'[(Bu_1, h) + (u_1', h)]dt .$$

In particular we see that everything vanishes on say $[0, s/2]$; hence we pose the Cauchy problem with initial data given at $s/2$ as follows. Let $\mathscr{F}_{s/2\ s_1}$ be the space of u such that $\omega u \in L^2(v)$ and $\psi u' \in L^2(H)$ on $[s/2, s/2+s_1]$ with $u(s/2)=0$. The space $\mathscr{H}_{s/2\ s_1}$ corresponding to $\mathscr{H}_s$ is defined similarly on $[s/2, s/2+s_1]$. We extend ω and δ to be constant on $[s, b]$; then since ψ, ω, δ etc. are positive and continuous we may define say $\mathscr{F}_{s/2, s_1}$ in terms of $u \in L_2(V)$ and $u' \in L^2(H)$. Let $\tilde{E}_{s/2\ s_1}$ denote the terms in $\tilde{E}_b$ integrated over $[s/2, s/2+s_1]$, and denote the right side of (2.24) integrated from $s/2$ to $s/2+s_1$ by $\tilde{\tilde{L}}_{s/2\ s_1}(h)$. Then consider

Problem 3. Find $u_2 \in \mathscr{F}_{s/2\ s_1}$ such that $\tilde{E}_{s/2, s_1}(u_2, h) = \tilde{\tilde{L}}_{s/2\ s_1}(h)$ for all $h \in \mathscr{H}_{s/2, s_1}$.

Problem 3 has a (unique) solution for s_1 sufficiently small by [17] and the above extension procedure may be repeated in steps of length $s_1/2$. Thus u will eventually be determined on $[0, b]$ satisfying problem 2. Hence

THEOREM 2. *Under the hypotheses of Theorem 1 there exists a solution of problem 2.*

3. Suppose now that $\tilde{E}_s(u, h) = 0$ for all $h \in \mathscr{H}_s$. Let $h = -\int_t^s J u d\xi$, $h' = Ju$, $J \to \infty$. Then

LEMMA 4. *Assume*
(a) $J^2/\omega^2 \int_0^t d\xi/\psi^2 \in L^1$
(b) $J/\omega\psi \in L^\infty$
(c) $J^2/\omega^2 \int_0^t (q^2/\omega^2)\, d\xi \in L^1$. *Then* $h \in \mathscr{H}_s$ *if* $u \in \mathscr{F}_s$ *and* $h = -\int_t^s J u d\xi$.

Proof. Clearly $h'/\psi = (J/\omega\psi)\omega u \in L^2(V)$ (hence certainly $h'/\psi \in L^2(H)$) and $h(s) = 0$; also

$$(3.1) \qquad \left|\frac{h}{\psi}\right|^2 \leqq c\left|\left|\frac{h}{\psi}\right|\right|^2 \leqq \left(\frac{1}{\psi}\int_t^s \frac{J}{\omega}||\omega u||\, d\xi\right)^2 \leqq \frac{1}{\psi^2}\int_t^s \frac{J^2}{\omega^2}\, d\xi \int_t^s ||\omega u||^2 d\xi$$

$$(3.2) \qquad \int_0^s \left|\left|\frac{q}{\omega} h\right|\right|^2 d\xi \leqq \int_0^s \frac{q^2}{\omega^2}\left(\int_t^s \frac{J^2}{\omega^2}\, d\xi\right) dt \int_0^s ||\omega u||^2\, d\xi\ .$$

Using the Fubini and Tonelli theorems (see e.g. [19]) the lemma follows.

We note now explicitly the fact that if $u \in L^2(H)$ and $u' \in L^2(H)$ (u' taken in $\mathscr{D}'(H)$ on $(0, s)$) then u may be identified with a continuous function and $u(0) = 0$ makes sense. Indeed for u, determined almost everywhere, we see that $u' \in L^1(H)$ on $[0, s]$ and clearly $D\tilde{u} = u'$ in $\mathscr{D}'(H)$ where $\tilde{u} = \int_0^t u' d\xi \in \mathscr{E}^0(H)$ (see [23]). Thus $D(\tilde{u} - u) = 0$ and by [21] for any $h \in H$, $(\tilde{u} - u, h) = c_h$ in $\mathscr{D}'$. Hence $(\tilde{u} - u, h) = c_h$

almost everywhere as a function and thus u may be identified scalarly with the continuous function $\tilde{u}$. Since H is separable we may then identify u with a continuous function and $u(0) = 0$ is meaningful (see [23], [22]). Hence $u = \tilde{u}$ follows. Thus setting $u = \int_0^t u' d\xi$, $h = -\int_t^s h' d\xi$

$$(3.3) \quad |(u, h)| = \left| -\int_0^t \int_t^s (u'(\xi), h'(\eta)) d\eta d\xi \right|$$
$$\leqq \sup \left| \frac{\psi(\eta)}{\psi(\xi)} \right| \int_0^t \int_t^s |\psi u'| \left| \frac{h'}{\psi} \right| d\eta d\xi \leqq \frac{N}{2} \int_0^t \int_t^s \left\{ |\psi u'|^2 + \left| \frac{h'}{\psi} \right|^2 \right\} d\eta d\xi$$
$$\leqq \frac{N}{2} \left\{ \int_0^t (s - t) |\psi u'|^2 d\xi + t \int_t^s \left| \frac{h'}{\psi} \right|^2 d\eta \right\} .$$

Thus $(u, h) = 0$ at $t = 0$ and we note that $\int_0^s (Bu', h) dt = -\int_0^s (B'u, h) dt - \int_0^s (Bu, h') \, dt$. Hence $\tilde{E}_s(u, h) = 0$ becomes, with h as above

$$(3.4) \qquad \int_0^s \left\{ \frac{q}{J} a(t, h', h) - (B'u, h) - J(Bu, u) - J(u', u) \right\} dt = 0 .$$

Set now $\tilde{\theta}^2 = \lim q/J \, a(t, h, h)$ which will exist if everything else makes sense in the following. Then we have

LEMMA 5. *Assume* (a)–(c) *from Lemma* 4 *and*
(d) $J \int_0^t d\xi / \psi^2 \in L^\infty$
(e) $-J'/\omega^2 \in L^\infty$; $J' < 0$
(f) $J \to \infty$; $J/J' \to 0$
(g) $(q/J)'/(q/J) \to \infty$. *Then if* $h = -\int_t^s J u d\xi$, $u \in \mathscr{F}_s$, *and if* $a(t, h, h) \geqq \alpha \|h\|^2$ *it follows that*

$$(3.5) \qquad \int_0^s \left\{ \alpha \left(\frac{q}{J} \right)' \frac{\omega^2}{q^2} - c_1 \left(\frac{q}{J} \right) \frac{\omega^2}{q^2} \right\} \left\| \frac{qh}{\omega} \right\|^2 dt$$
$$+ \int_0^s \left\{ -\frac{J'}{\omega^2} - \frac{2\beta J}{\omega^2} - \frac{\hat{\beta}}{\omega^2} \int_t^s J d\xi - \frac{\hat{\beta} t J}{\omega^2} \right\} |\omega u|^2 \, dt \leqq 0$$

Proof. By (d) we have

$$J |u|^2 \leqq J \left(\int_0^t |\psi u'| \frac{d\xi}{\psi} \right)^2 \leqq J \int_0^t \frac{d\xi}{\psi^2} \int_0^t |\psi u'|^2 d\xi \to 0$$

whereas from (e) there results $-J' |u|^2 = -J'/\omega^2 |\omega u|^2 \in L^1$. Next by (f) and (e) it follows that $\lim Jq/\omega^2 = \lim (J/-J')(-J'q/\omega^2) = 0$; hence $Jq/\omega^2 \in L^\infty$ and

$$(3.6) \quad \int_0^s \left(\frac{q}{J} \right)' \|h\|^2 d\xi \leqq \int_0^s \left(\frac{q}{J} \right)' \left(\int_t^s J \|u\| d\xi \right)^2 dt$$
$$\leqq \int_0^s \left(\frac{q}{J} \right)' \left(\int_t^s \frac{J^2}{\omega^2} d\xi \int_t^s \|\omega u\|^2 d\xi \right) dt \leqq \left(\int_0^s \|\omega u\|^2 d\xi \right) \int_0^s \frac{Jq}{\omega^2} d\xi .$$

Note here $q/J \to 0$ and $q/J = \int_0^t (q/J)'\, d\xi$; also by (g) surely $\int_0^s q/J \,\|h\|^2\, d\xi < \infty$. Now by (f) it follows that $J|u|^2 = (J/J')\, J'|u|^2 \in L^1$ and finally we remark that

$$\left|2Re\int_0^s (B'u, h)\, d\xi\right| \leqq \hat{\beta}\int_0^s\int_t^s J(\xi)\,\{|u(t)|^2 + |u(\xi)|^2\}\, d\xi dt \tag{3.7}$$

$$\leqq \hat{\beta}\left\{\int_0^s |\omega u|^2 \left(\frac{1}{\omega^2}\int_t^s J d\xi\right) dt + \int_0^s \frac{Jt}{\omega^2}\,|\omega u|^2\, dt\right\}.$$

Here the Jt/ω^2 term makes sense since $Jt/\omega^2 = (J/-J')(-J't/\omega^2) \to 0$ by (e) and (f). Then we note that

$$\frac{1}{\omega^2}\int_t^s J d\xi = \left(\frac{-J'}{\omega^2}\right)\left(\frac{J}{-J'}\right)\left(\frac{1}{J}\int_t^s J d\xi\right);$$

but by 1' Hospital's rule $\lim 1/J \int_t^s J d\xi = \lim J/-J^1 = 0$ (here note that $J' \neq 0$, $J \neq 0$ for $t > 0$). Hence we may write

$$\tilde{\theta}^2 + \int_0^s \left\{\left(\frac{q}{J}\right)' a(t, h, h) + \left(\frac{q}{J}\right) a'(t, h, h)\right\} dt \tag{3.8}$$

$$+ 2Re\int_0^s (B'u, h)dt + 2Re\int_0^s J(Bu, u)dt$$

$$- \int_0^s J'\,|u|^2\, dt + J|u(s)|^2 = 0\,.$$

The lemma follows immediately.

Now let $\omega^2 = v'/v^{2-\varepsilon}$ as before and consider the following choice for the function J

$$J = j + \check{c}\int_t^s \omega^2 d\xi;\ -\frac{J'}{\omega^2} = \check{c}\,. \tag{3.9}$$

It follows that (e) holds (we assume ω, v etc. are as before) and since $v = \varphi/q$ (d) is a consequence of the fact that

$$\check{c}\int_t^s \omega^2 d\xi \int_0^t \frac{d\eta}{\psi^2} \leqq \check{c}\varphi\int_t^s \delta^2 d\xi = \check{c}\varphi\int_t^s -\left(\frac{1}{v}\right)'\frac{d\xi}{R} \tag{3.10}$$

$$= \check{c}\,\frac{\varphi}{R}\left[\frac{1}{v(t)} - \frac{1}{v(s)}\right] = \frac{\check{c}}{R}\left[q(t) - \varphi(t)\,\frac{q(s)}{\varphi(s)}\right].$$

Note now that with the above choice of ω we can write J in the form $J = j + \check{c}\int_t^s v'/v^{2-\varepsilon}\, d\xi = j - (\check{c}/1-\varepsilon)\,(1/v(s))^{1-\varepsilon} + (\check{c}/1-\varepsilon)\,(1/v(t))^{1-\varepsilon}$. If j is taken to be $j = (\check{c}/1-\varepsilon)\,(1/v(s))^{1-\varepsilon}$ then

$$J = \frac{\check{c}}{1-\varepsilon}\left(\frac{1}{v}\right)^{1-\varepsilon};\ \frac{J}{J'} = \frac{-1}{1-\varepsilon}\left(\frac{v}{v'}\right). \tag{3.11}$$

Thus if $v/v' \to 0$ then $J/-J' \to 0$. Moreover since $\omega^2 = (v'/v)\,(1/v)^{1-\varepsilon}$ it follows that $\omega \to \infty$ if $v \to 0$ and $v/v' \to \infty$ and also by (3.11) $J \to \infty$ if $v \to 0$. Hence if $v'/v \to \infty$ and $v \to 0$ then (f) holds and $\omega \to \infty$.

Consider now condition (a); using (d) we have $J^2/\omega^2 \int_0^t d\xi/\psi^2 \leqq c\, J/\omega^2 = -\check{c}c\, J/J' \to 0$ which implies (a). For (c) we note

$$\int_0^s \frac{J^2}{\omega^2}\left(\int_0^t \frac{q^2}{\omega^2} d\xi\right) dt \tag{3.12}$$

$$\leqq \int_0^s \left\{ \frac{j^2 + 2j\check{c}\int_t^s \omega^2 d\xi + \left(\check{c}\int_t^s \omega^2 d\xi\right)^2}{\omega^2} \right\} \left(\int_0^t \frac{q^2}{\omega^2} d\xi\right) dt\ .$$

However $1/\omega^2 \int_t^s \omega^2 d\xi = v^{2-\varepsilon}/v' \int_t^s v'/v^{2-\varepsilon}\, d\xi = (1/1-\varepsilon)\,\{v/v' - c/\omega^2\}$ and if $v/v' \to 0$ and $\omega \to \infty$ it follows that the first two integrals in (3.12) exist. The last integral in (3.12) is bounded by

$$c\int_0^s \left[\frac{1}{\omega^2}\int_t^s \omega^2 d\xi\right]\left[\int_t^s \omega^2 d\xi \int_0^t \frac{d\eta}{\omega^2}\right] dt\ .$$

The first term in the integrand vanishes as $t \to 0$ by the above remarks and using 1' Hospital's rule on the second term we note that $\lim \int_t^s \omega^2 d\xi \int_0^t d\eta/\omega^2 = \lim \left(\int_t^s \omega^2 d\xi\right)^2 \Big/ \omega^4$ which is zero by the above (note here if $\omega \in L^2$ (3.12) is seen immediately to exist and no recourse to the preceding argument is intended). Thus if $v'/v \to \infty$ and $\omega \to \infty$ (c) surely holds.

Now since $J/\omega\psi = (\check{c}/1-\varepsilon)\, 1/\omega\psi v^{1-\varepsilon}$ it follows that (b) holds if $\omega^2 v^{2-2\varepsilon} > c/\psi^2$ or $(v'/v)\varepsilon > c/\psi^2$. It is not necessary that $\psi \uparrow \infty$ in general; when $v \to 0$ (b) will hold if $v' > c/\psi^2$. Thus (b) holds if $v \to 0$ and

$$1 - \left(\frac{\psi^2 q'}{q}\right)\int_0^t \frac{d\xi}{\psi^2} > \tilde{c}q \tag{3.13}$$

since $v' = \varphi'/q - \varphi q'/q^2$ and $\varphi = \hat{c}\int_0^t d\xi/\psi^2$. In particular (3.13) holds if for example $(\psi^2 q'/q)\int_0^t d\xi/\psi^2 \leqq 1 - \varepsilon_1$, since $q \to 0$ (see here also equation (2.20)). This proves

LEMMA 6. *Assume* (h) $(q'\psi^2/q)\int_0^t d\xi/\psi^2 \leqq 1 - \varepsilon_1$ *for t small. Then if* $J = (\check{c}/1-\varepsilon)\,1/v^{1-\varepsilon}$ $(J' = -\check{c}\omega^2)$ *and* $v \to 0$ *it follows that* $v'/v \to \infty$ *and* (a)–(f) *hold.*

We recall that φ and v are defined independently of s (see (2.17)) and our constructions and proofs have shown that for t small enough the $(q/J)'\,\omega^2/q^2$ and $-J'/\omega^2$ terms will dominate in the first and second integrals respectively of (3.5). It remains to check only a few terms in order to see whether by suitable choice of s this domination prevails over $[0, s]$. Now by (3.11) J/J' is independent of s as is J/ω^2 (indeed a priori ω^2 and δ^2 depend only on v). Now since $-J' = \check{c}\omega^2 > 0$ we have J monotone decreasing and clearly

$$\frac{1}{J(t)} \int_t^s J(\xi)d\xi \leqq s - t \leqq b \ .$$

Hence referring to the proof of Lemma 5 we can establish domination over an interval $[0, s]$ in the second integral of (3.5). There remains the $(q/J)'$ term for which we may write

$$\frac{\left(\dfrac{q}{J}\right)'}{\left(\dfrac{q}{J}\right)} = \frac{q'}{q} + (1-\varepsilon)\frac{v'}{v} = \frac{\varphi'}{\varphi}\left\{1 - \varepsilon\left[1 - \frac{q'\varphi}{q\varphi'}\right]\right\} , \tag{3.14}$$

Thus in particular the ratio in (3.14) is a priori independent of s and the desired domination may be obtained on an interval $[0, s]$ by choosing s sufficiently small. Thus we have proved

LEMMA 7. *If the hypotheses of Lemma* 6 *hold and* (g) *is true it follows that for suitably small* s, $\int_0^s |\omega u|^2 dt \leqq 0$.

Clearly the condition (h) in Lemma 6 is much stronger than is necessary but it gives a manageable criterion. We note now that if $q' \geqq 0$ then by (h) $\varepsilon_1 \leqq [1 - q'\varphi/q\varphi'] \leqq 1$ and from (3.14) it results that $(q/J)'/(q/J) \geqq (1-\varepsilon)\varphi'/\varphi \rightarrow \infty$. Thus if q is monotone, for any ε, $0 < \varepsilon < 1$, (g) is a consequence of (h). Another case of interest would be if $1 - q'\varphi/q\varphi' \leqq \tilde{Q}$; then if $\varepsilon \leqq 1/\tilde{Q}$ (g) holds. A somewhat better result may be obtained as follows. We note that

$$\frac{q'\varphi}{q\varphi'} = \frac{q'\psi^2}{q}\int_0^t \frac{d\xi}{\psi^2} = \frac{(\log q)'}{\left(\log \int_0^t \dfrac{d\xi}{\psi^2}\right)'} \ .$$

Then assume that $Q = \lim\, (q'\psi^2/q)\int_0^t d\xi/\psi^2$ exists as $t \rightarrow 0$. We note that the conditions needed to apply l'Hospital's rule hold and thus $Q = \lim \log q/\log \int_0^t d\xi/\psi^2$. Therefore for t small (h) implies that

$$\log q/\log \int_0^t \frac{d\xi}{\psi^2} \leqq 1 - \varepsilon_2\ , \quad 0 < \varepsilon_2 < \varepsilon_1\ .$$

But for t small the logarithms are negative and thus $\log q \geqq \log\left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2}$ or $q \geqq \left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2} = c\varphi^{1-\varepsilon_2}$. Conversely if $q \geqq c\varphi^{1-\varepsilon_2}$ and if $Q = \lim q'\varphi/q\varphi'$ exists then $Q \leqq 1 - \varepsilon_3$ for some ε_3, $0 < \varepsilon_3 < \varepsilon_2$.

Hence if Q exists as defined and $q \geqq c\varphi^{1-\varepsilon_2}$ then (h) holds and moreover $v = \varphi/q \leqq \varphi/c\varphi^{1-\varepsilon_2} = (1/c)\varphi^{\varepsilon_2} \to 0$. We note that by construction if Q exists then $Q = \lim \text{loq}\, q/\text{loq} \int_0^t d\xi/\psi^2 \geqq 0$; hence $\varepsilon[1 - q'\varphi/q\varphi'] < \varepsilon(1 + \varepsilon_4)$ for t small enough and $\varepsilon_4 > 0$ given. Choose now ε_4 such that $\varepsilon(1+\varepsilon_4) < 1$ or $\varepsilon_4 < (1-\varepsilon)/\varepsilon$ then from (3.14) $(q/J)'/(q/J) \geqq c\varphi'/\varphi$ for t small. This proves

THEOREM 3. *Assume* $Q = \lim (q'\psi^2/q) \int_0^t d\xi/\psi^2$ *exists and that* $q \geqq \left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2}$, $0 < \varepsilon_2 < 1$. *Then* (h) *holds*, $v \to 0$, *and* $(q/J)'/(q/J) \to \infty$ *for* $J = c/v^{1-\varepsilon}$ *as above. Hence for s small enough the solution of problem* 1 *is unique.*

Again using [17] we conclude

THEOREM 4. *Assume* $a(t, u, u) \geqq \alpha \|u\|^2$, $t \to a(t, u, v) \in C^1[0, b]$, $t \to B(t) \in \mathscr{E}^1(\mathscr{L}_s(H, H))$, $a(t, u, v) = \overline{a(t, v, u)}$, $q \in C^1(0, b]$, $q > 0$ *for* $t > 0$, $q \to 0$ *as* $t \to 0$, $\psi \in C^0(0, b]$, $\psi > 0$, $\psi \uparrow$ *as* $t \to 0$, $\psi f \in L^2(H)$, $q \geqq \left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2}$ $(0 < \varepsilon_2 < 1)$, *and* $Q = \lim (q'\psi^2/q)\int_0^t d\xi/\psi^2$ *exists. Then there exists a unique solution of problem* 2 *for spaces* $\mathscr{F}_b$, $\mathscr{H}_b$ *based on functions* $\omega \notin L^2$ $(\omega \in C^0(0, b])$.

We note now that if $Q \neq 0$ then $q' < 0$ for t small is not possible. Moreover if $\log q/\text{loq} \int_0^t d\xi/\psi^2 \geqq \varepsilon_4 > 0$ then $q \leqq \left(\int_0^t d\xi/\psi^2\right)^{\varepsilon 4}$ and we may assume $\varepsilon_4 < 1$ since if $q \leqq \gamma^{1+\eta}$, $\eta \geqq 0$, $\gamma \to 0$, then $q \leqq \gamma^{\varepsilon 4}$ for any $\varepsilon_4 < 1$ when t is small. In fact $\varepsilon_4 < 1$ is necessary if we are to have $q \geqq c\varphi^{1-\varepsilon_2}$ and thus the case $Q \neq 0$ with $q \geqq \left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2}$ amounts to an estimate of the form $\left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2} \leqq q \leqq \left(\int_0^t d\xi/\psi^2\right)^{\varepsilon 4}$, $0 < \varepsilon_2 < 1$, $\varepsilon_2 + \varepsilon_4 \leqq 1$. Finally we remark that under the hypotheses of Theorem 4 if $\lim q'\psi^2$ exists then by l'Hospital's rule $\lim q'\psi^2 = \lim q\Big/\int_0^t d\xi/\psi^2 = \lim \check{c}\, q/\varphi = \infty$. This implies that $\psi \uparrow \infty$ if q' is bounded but in a case such as $q = t^{1/2}$, $\psi \uparrow \infty$ is not required.

4. Let now $\hat{\mathscr{K}_s}$ be the completion of $\mathscr{K}_s$ for the norm $\|\ \ \|_{\mathscr{K}_s}$. Then we may pose problem 1 for $\hat{\mathscr{K}_s}$ instead of $\mathscr{F}_s$ (call this problem 1′) and repeating the procedures of §§ 2 and 3 there will exist a function $\hat{u} \in \hat{\mathscr{K}_s}$ solving problem 1′ if s is small enough. It may be easily seen that the elements adjoined to $\mathscr{K}_s$ by completion correspond to functions $\hat{k}$ such that $\delta\hat{k} \in L^2(V)$, $\hat{k}'/\varphi\psi \in L^2(H)$, and $\hat{k}(0) = 0$. Moreover the injection $i: \mathscr{K}_s \to \mathscr{F}_s$ may be extended by continuity to a continuous map $\hat{i}: \hat{\mathscr{K}_s} \to \mathscr{F}_s$.

LEMMA 8. $\hat{\mathscr{K}_s} \subset \mathscr{F}_s$ *algebraically and topologically.*

Proof. We need only show, after the above remarks, that $\hat{i}$ is an injection. Let $k_n \to \hat{k}$ in $\hat{\mathscr{K}}_s$, $k_n \in \mathscr{K}_s$, and assume that $i(k_n) = k_n \to 0 = \hat{i}(\hat{k})$. We want to show that $\hat{k} = 0$ in $\hat{\mathscr{K}}_s$. First $k_n = i(k_n) \to 0$ in $\mathscr{F}_s$ means in particular that $\omega k_n \to 0$ in $L^2(V)$. Hence (see [6], p. 133) there is a subsequence $\|\omega k_{n_p}\|^2 \to 0$ almost everywhere. Therefore $\|\delta k_{n_p}\|^2 \to 0$ almost everywhere and by the assumption $k_n \to \hat{k}$ in $\hat{\mathscr{K}}_s$ we know $\delta k_{n_p} \to \delta\hat{k}$ in $L^2(V)$. Theorefore we must have (see [6], p. 133 again) $\delta k_{n_p} \to 0$ in $L^2(V)$, and $\delta\hat{k} = 0$ in $L^2(V)$ (similarly $\hat{k}'/\varphi\psi = 0$ in $L^2(H)$); thus in particular $\hat{k} = 0$ which shows that $\hat{i}(\hat{k}) = 0$ implies $\hat{k} = 0$.

Let now $\hat{u} \in \hat{\mathscr{K}}_s$ be the solution of problem 1′ above. Then $\hat{u} \in \mathscr{F}_s$ by Lemma 8 and by the uniqueness Theorem 3 we must have $\hat{u} = u$ for s small where u is the solution of problem 1. Hence

THEOREM 5. *Let the hypotheses of Theorem 4 hold. Then there exists a unique solution u of problem 2 which belongs to $\hat{\mathscr{K}}_b$.*

Now consider the proof of the Lions projection theorem given say in [17] (see also [18]). We have $ReE_s(k, k) \geqq \Omega\|k\|^2_{\hat{\mathscr{K}}_s}$ for $k \in \mathscr{K}_s$ and wish to solve $E_s(u, k) = L_s(k)$ for $u \in \hat{\mathscr{K}}_s$ (the equation holding for all $k \in \mathscr{K}_s$). Then we write, following Lions, $L_s(k) = ((\chi, k))_{\hat{\mathscr{K}}_s}$, $\chi \in \hat{\mathscr{K}}_s$, and $E_s(u, k) = ((u, Lk))_{\hat{\mathscr{K}}_s}$, $Lk \in \hat{\mathscr{K}}_s$. Here $L\colon \mathscr{K}_s \to \hat{\mathscr{K}}_s$ is a densely defined linear operator in $\hat{\mathscr{K}}_s$. But $k \in \mathscr{K}_s$

$$\Omega\|k\|^2_{\hat{\mathscr{K}}_s} \leqq |((k, Lk))_{\hat{\mathscr{K}}_s}| \leqq \|k\|_{\hat{\mathscr{K}}_s}\|Lk\|_{\hat{\mathscr{K}}_s} \tag{4.1}$$

which implies L is one-to-one. Moreover if $R_0 = L(\mathscr{K}_s)$ then L^{-1} is a bounded operator on R_0 and may be extended by continuity to $\bar{R}_0$ defining $\hat{L}^{-1}\colon \bar{R}_0 \to \hat{\mathscr{K}}_s$. Let $P\colon \hat{\mathscr{K}}_s \to \bar{R}_0$ be the projection and set $R = \hat{L}^{-1}P$ which is thus everywhere defined and continuous on $\hat{\mathscr{K}}_s$. Then we want to find u such that $((u, Lk)) = ((\chi, L^{-1}Lk)) = ((\chi, RLk)) = ((R^*\chi, Lk))$ for all $k \in \mathscr{K}_s$. Thus a solution is $u = R^*\chi$ and by the subsequent uniqueness result $u = R^*\chi$ is the only solution. Using this sketch of the proof of the projection theorem we can bound u. Indeed $\|u\|_{\hat{\mathscr{K}}_s} \leqq \|R^*\chi\|_{\hat{\mathscr{K}}_s} \leqq c\|\chi\|_{\hat{\mathscr{K}}_s}$ since R^* is bounded. Moreover

$$\begin{aligned} |((\chi, k))| &= \left|\int_0^s\left(\psi f, \frac{h}{\psi}\right)dt\right| \leqq \left(\int_0^s|\psi f|^2\,dt\int_0^s\left|\frac{h}{\psi}\right|^2 dt\right)^{1/2} \\ &\leqq \left(\int_0^s|\psi f|^2\,dt\int_0^s|k'/\varphi\psi|^2\,dt\right)^{1/2} \leqq \left(\int_0^s|\psi f|^2\,dt\right)^{1/2}\|k\|_{\hat{\mathscr{K}}_s} = F\|k\|_{\hat{\mathscr{K}}_s}\,. \end{aligned} \tag{4.2}$$

This means (see [5], p. 111) since $\mathscr{K}_s$ is dense in $\hat{\mathscr{K}}_s$ that $\|\chi\| \leqq F = \left(\int_0^s|\psi f|^2\,dt\right)^{1/2}$. Therefore we have proved

THEOREM 6. *Under the hypotheses of Theorem 4 and for s suf-*

ficiently small the (unique) solution of problem 1 *satisfies the estimate* $||u||_{\hat{\mathscr{K}}_s} \leqq c\left(\int_0^s |\psi f|^2\, dt\right)^{1/2}$.

The estimate can clearly be extended to $[0, b]$ which given

COROLLARY. *Under the hypotheses of Theorem* 6 *the unique solution of problem* 2 *satisfies the estimate* $||u||_{\hat{\mathscr{K}}_b} \leqq c\left(\int_0^b (\psi f|^2\, dt\right)^{1/2}$.

REFERENCES

1. R. Barancev, *Expansion theorems connected with boundary value problems for the equation* $u_{xx} - K(x)u_{tt} = 0$ *in the strip* $0 \leqq x \leqq 1$ *with a degeneracy or a singularity on the boundary,* Doklady Akad. Nauk, SSSR, T. 121, (1958), 9-12.
2. I. Berezin, *On the Cauchy problem for linear equations of the second order with initial data on the parabolic line,* Mat. Sbornik, **24** (1949), 301-320.
3. L. Bers, *On the continuation of a potential gas flow across the sonic line,* NACA Tech. Note 2058, 1950.
4. ———, *Mathematical aspects of subsonic and transonic gas dynamics,* Wiley, New York, 1958
5. N. Bourbaki, *Espaces vectoriels topologiques,* Chap. 3-5, Paris, 1955.
6. ———, *Int´gration,* Chap. 1-4, Paris, 1952.
7. ———, *Intégration vectorielle,* Chap. 6, Paris, 1959.
8. R. Carroll, *Quelques problèmes de Cauchy dégénérés avec des coefficients opérateurs,* Comptes Rendus, Paris, T. 253, (1961), 1193-1195.
9. R. Conti, *Sul problema di Cauchy per l'equazione* $y^{2\alpha}K^2(x, y)t_{xx} - t_{yy} = f(x, y, t, t_x, t_y)$ *con i dati sulla linea parabolica,* Annali di Matematica, **31** (1950), 303-326.
10. F. Frankl', *On Cauchy's problem for equations of mixed elliptic-hyperbolic type with initial data on the transition line,* Izvestia Akad. Nauk, SSSR, **8** (1944), 195-224.
11. P. Germain and P. Bader, *Solutions élémentaires de certaines équations aux deriveés partielles du type mixte, Bull.* de la Soc. Math. de France, T. 81 (1953), 145-174.
12. G. Hellwig, *Anfangs-und Randwertprobleme bei partiellen Differential gleichungen von wechselndem Typus auf dem Rändern,* Math. Zeitschrift, Bd. 58, (1953), 337-357.
13. ———, *Anfangswertprobleme bei partiellen Differential-gleichungen mit Singularitäten,* Journal of Rational Mech. and Analysis, **5** (1956), 395-418.
14. Hu-Hsien Sun, *On the uniqueness of the solution of degenerate equations and the rigidity of surfaces,* Doklady Akad. Nauk, SSSR. T. 122. (1958), 770-773.
15. M. Krasnov, *Mixed boundary value problems and the Cauchy problem for degenerate hyperbolic equations,* Doklady Akad. Nauk, SSSR. T. 107, (1956), 789-792.
16. ———, *Mixed boundary value problems for degenerate linear hyperbolic differential equations of the second order,* Mat. Sbornik, **49** (31), (1959), 29-84.
17. J. L. Lions, *Problemi misti nel senso di Hadamard classici e generalizzati,* Rend. del Sem. Mat. e Fis. di Milano, **28** (1959), 1-47.
18. ———, *Equations différentielles opérationelles et problèmes aux limites,* Grund. d. Math. Wiss., Bd. 111, Berlin, 1961.
19. E. McShane, *Integration,* Princeton, 1949.
20. M. Protter, *The Cauchy problem for a hyperbolic second order equation with data on the parabolic line,* Canadian Journal of Math., **6** (1954), 542-553.
21. L. Schwartz, *Theorie des distributions,* Vols 1, 2, Paris, 1950-51.
22. ———, *Espaces de fonctions différentiables à valeurs vectorielles,* Journal d'analyse Math., **4** (1954-55), 88-148.

23. ———, *Théorie des distributions à valeurs vectorielles*, Annales de l'institut Fourier, pp. 1-141, 1957 et pp. 1-209, 1958.
24. S. Tersenov, *On an equation of hyperbolic type degenerating on the boundary*, Doklady Akad. Nauk. SSSR, T. **129** (1959), 276-279.
25. A. Weinstein, *The singular solutions and the Cauchy problem for generalized Tricomi equations*, Comm. Pure and Appl. Math., **7** (1954), 105-116.