ON THE INTEGER SOLUTIONS OF $y(y+1)=x(x+1)(x+2)$

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This paper contains a solution of the following problem proposed to me by Professor Burton Jones to whom it was given by Mr. Edgar Emerson.

Problem. To show that the only integer solutions of

$$y(y+1) = x(x+1)(x+2) \tag{1}$$

are given by

$$x = 0, -1, -2, y = 0, -1; x = 1, y = 2, -3, x = 5, y = 14, -15 . \tag{2}$$

Put

$$2y + 1 = Y, \qquad 2x + 2 = X .$$

Then

$$2Y^2 = X^3 - 4X + 2 . \tag{3}$$

Obviously X in (3) cannot be odd so it must be shown that the only integer solutions of (3) are given by

$$X = 0, \pm 2, 4, 12 . \tag{4}$$

Diophantine equations of the form

$$Ey^2 = Ax^3 + Bx^2 + Cx + D \tag{5}$$

where A, B, C, D, E are integers are well known. I proved[1,2] in 1922 that the equation had only a finite number of integer solutions when the right hand side had no squared factor in x. In fact, this followed immediately from a result[3] I proved in 1913, by quoting Thue's result but which I did not know at that time. Finding these solutions may be a troublesome matter, involving many details, and usually rather difficult or even too difficult, to do.

One method requires a discussion of the field $R(\theta)$ defined by

$$A\theta^3 + B\theta^2 + C\theta + D = 0 . \tag{6}$$

Received November 12, 1962. I should like to thank the referee for his comments on my manuscript. This research was supported by the Air Force Office of Scientific Research.

[1] "Note on the integer solutions of the equation $Ey^2 = Ax^3 + Bx^2 + Cx + D$" Messenger of Math., **51** (1922), 169-171.

[2] "On the integer solutions of the equation $ey^2 = ax^3 + bx^2 + cx + d$" Proc. London Math. Soc., **21** (1923), 415-419.

[3] "Indeterminate Equations of the third and forth degrees" Quarterly Journal of Mathematics **45** (1914).

Less arithmetical knowledge may be needed if the equation is reducible in the rational field R. In general, the field is a cubic field $R(\theta)$ and then we require a knowledge of

(1) The expression for the integers of $R(\theta)$.

(2) The number h of classes of ideals in $R(\theta)$.

(3) The fundamental unit when the equation (6) has only one real solution, and the two fundamental units when all the roots of (6) are real.

Ideal factorization theory may now be applied in (5) on noting that $x - \theta$ is a factor of the right hand side. The results so found require the solution in integers P, Q, R of a finite number of pairs of simultaneous quadratic equations

$$F(P, Q, R) = 0, \ G(P, Q, R) = 1 , \tag{7}$$

where F and G are homogeneous polynomials of the second degree with rational integer coefficients. In general, it is difficult to find all the integer solutions of (7), but in some simple instances, they can be easily found. The difficulty is increased when there are two fundamental units.

In (3), the field $R(\theta)$ is defined by

$$f(\theta) = \theta^3 - 4\theta + 2 = 0 . \tag{8}$$

The arithmetical details are given in the table on page 112, of Delaunay and Faddeev's book (in Russian) on "The theory of the irrationalities of the third degree".

Thus

(1) the integers are given by $a + b\theta + c\theta^2$ where a, b, c are rational integers,

(2) the class number $h = 1$ and so unique factorization exists in $R(\theta)$,

(3) there are two fundamental units

$$\varepsilon = \theta - 1 , \quad \text{and} \quad \eta = 2\theta - 1 .$$

All the units are given by $\pm\varepsilon^l\eta^m$ where $l, m,$ run through all integer values positive, negative and zero.

We require the factorization of 2. Since $f(x) \equiv x^3 \pmod 2$, the general theory shows that 2 is a perfect cube except for unit factors, and since

$$2 = \theta(2 + \theta)(2 - \theta) , \tag{9}$$

we would expect $2 \pm \theta$ to be associated with θ, namely that $\varphi = 2/\theta \pm 1$ is an integral unit. In fact,

$$\frac{8}{(\varphi \mp 1)^3} - \frac{8}{\varphi \mp 1} + 2 = 0 ,$$

or

$$(\varphi \mp 1)^3 - 4(\varphi \mp 1)^2 + 4 = 0 ;$$

and this proves the result about φ. Hence $2 = \zeta\theta^3$ where ζ is a unit, and it is easily seen from this that θ is a prime in $R(\theta)$.

Write (3) in the form

$$(X - \theta)(X^2 + \theta X + \theta^2 - 4) = \zeta\theta^3 Y^2 . \tag{10}$$

We now show that θ is a common divisor of the two left hand factors of (10). Since $X \equiv 0 \pmod 2$, and so $X \equiv 0 \pmod{\theta^3}$, θ divides $X - \theta$ and $X^2 + X\theta + \theta^2 - 4$ but θ^2 does not divide $X - \theta$. We show that $4\theta - 3$ is a possible common divisor and note that $4\theta - 3$ is a prime since its norm is given by $N(4\theta - 3) = -37$. Any common divisor must also divide

$$\theta^2 + \theta^2 + \theta^2 - 4 = 3\theta^2 - 4 = (3\theta^3 - 4\theta)/\theta = (8\theta - 6)/\theta = 2(4\theta - 3)/\theta ,$$

and so $4\theta - 3$ may occur.

We have two cases according as $X - \theta$ is not or is divisible by $4\theta - 3$.

The second case reduces very simply to the first case. Let $(4\theta - 3)^n$ be the highest power of $4\theta - 3$ dividing $X - \theta$. Then since unique factorization exists in the cubic field $R(\theta)$,

$$X - \theta = \pm\theta(4\theta - 3)^n \varepsilon^l \eta^m (a + b\theta + c\theta^2)^2 ,$$

where l, m are any integers positive, negative or zero. Take the norm of both sides. Then

$$X^3 - 4X + 2 = \pm 2(37)^n z^2 ,$$

say. Put $X = 2X$, and then

$$4X_1^3 - 4X_1 + 1 = \pm(37)^n z^2 .$$

Since the left hand side is $\equiv 1 \pmod 8$, we have an impossible congruence (mod 8) if n is odd. If n is even, we can absorb $(4\theta - 3)^n$ in $(a + b\theta + c\theta^2)^2$. We are then led to an equation of the same form as in the first case, that is, $X - \theta$ is prime to $4\theta - 3$, and is divisible by θ but not by θ^2.

From (10), we have in the first case

$$X - \theta = \pm\theta\varepsilon^l \eta^m (a + b\theta + c\theta^2)^2 . \tag{11}$$

Clearly it suffices to consider only the four cases

$$(l, m) = (0, 0), (1, 0), (0, 1), (1, 1)$$

on absorbing even powers of ε, η in the square term. On using (8), we can write

$$(12)\qquad (a + b\theta + c\theta^2)^2 = a^2 - 4bc + \theta(2ab + 8bc - 2c^2) \\ + \theta^2(b^2 + 2ac + 4c^2) .$$

Take first $(l, m) = (0, 0)$. Then using (8) and equating terms in θ, θ^2 on both sides of (11), we have

$$(13)\qquad \begin{aligned} 0 &= 2ab + 8bc - 2c^2 , \\ \mp 1 &= a^2 - 4bc + 4(b^2 + 2ac + 4c^2) . \end{aligned}$$

A congruence mod 4, shows that the plus sign should be taken. Write (13) as

$$b(a + 4c) = c^2, (a + 4c)^2 + 4b^2 - 4bc = 1 .$$

When $b = 0$, this gives $c = 0, a = \pm 1$.
If $b \neq 0$,

$$c^4/b^2 + 4b^2 - 4bc = 1 .$$

The minimum value of the left hand side is b^2 given when $c = b$, and so $b = \pm 1, c = \pm 1, a = \mp 3$.

Since from (11), we have

$$-X = -2(b^2 + 2ac + 4c^2) ,$$

we have

$$X = 0, -2 .$$

Take next $l = 0, m = 1$. On multiplying (11) by θ, absorbing θ^2 in the squared term, and with a slight change of notation, we have

$$\pm(\theta X - \theta^2) = (1 - 2\theta)\big(a^2 - 4bc + \theta(2ab + 8bc - 2c^2) \\ + \theta^2(2ac + b^2 + 4c^2)\big) ,$$

and so

$$(14)\qquad \begin{aligned} 0 &= a^2 - 4bc + 4(2ac + b^2 + 4c^2) , \\ \mp 1 &= 2ac + b^2 + 4c^2 - 2(2ab + 8bc - 2c^2) , \\ \mp X &= 2ab + 8bc - 2c^2 - 2(a^2 - 4bc) - 8(2ac + b^2 + 4c^2) . \end{aligned}$$

Clearly a is even and so ∓ 1 must be $+1$. Then

$$\begin{aligned} 1 &= a(2c - 4b) + b^2 - 16bc + 8c^2 , \\ 0 &= (a + 4c)^2 + 4b(b - c) . \end{aligned}$$

Suppose first that $c = 2b$ and so $b^2 = 1$. Since solutions (a, b, c), $(-a, -b, -c)$ give the same value for X, we need only take $b = 1$, $c = 2$, and then $(a + 8)^2 - 4 = 0$ and so $a = -6, -10$. Then $b = 1$,

$c = 2, a = -6$ gives $X = 4$.

Also $b = 1, c = 2, a = -10$ gives from (14), $X_1 = 12$. Suppose next that $c \neq 2b$. Then

$$a = \frac{b^2 - 16bc + 8c^2 - 1}{4b - 2c}, \; a + 4c = \frac{b^2 - 1}{4b - 2c};$$

and

$$\left(\frac{b^2 - 1}{4b - 2c}\right)^2 + 4b(b - c) = 0 .$$

Since $b \mid (b^2 - 1)$, $b = \pm 1, c = \pm 1, a = \mp 4$. Then from (14), $X = 2$. All the solutions (4) have now been obtained.

We take next $l = 1, m = 0$ and so

$$\pm(X - \theta) = (\theta - \theta^2)\big(a^2 - 4bc + \theta(2ab + 8bc - 2c^2) + \theta^2(2ac + b^2 + 4c^2)\big) .$$

Then

$$0 = 2ab + 8bc - 2c^2 - a^2 + 4bc - 4(2ac + b^2 + 4c^2) ,$$
$$\mp 1 = a^2 - 4bc + 6(2ac + b^2 + 4c^2) - 4(2ab + 8bc - 2c^2) .$$

The first equation shows that a is even and the second that a is odd; and so no solutions arise.

Suppose finally that $l = m = 1$, so that (11) can be written as say,

$$\pm(X - \theta) = \theta(1 - \theta)(1 - 2\theta)F^2 .$$

On multiplying by θ and absorbing θ^2 in F^2, we can write this as, say

$$\pm(X\theta - \theta^2) = (1 - 3\theta + 2\theta^2)\big(a^2 - 4bc + \theta(2ab + 8bc - 2c^2) + \theta^2(2ac + b^2 + 4c^2)\big) .$$

Hence

$$0 = a^2 - 4bc + 6(2ac + b^2 + 4c^2) - 4(2ab + 8bc - 2c^2) .$$
$$\mp 1 = 9(2ac + b^2 + 4c^2) - 3(2ab + 8bc - 2c^2) + 2(a^2 - 4bc) .$$

The first equation shows that a is even and then that b is even since $6b^2 \equiv 0 \pmod 4$. The second equation shows that b is odd. Hence no solutions arise.

This finishes the proof.

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