EXTREMAL SPECTRAL FUNCTIONS OF A SYMMETRIC OPERATOR

RICHARD C. GILBERT

1. Introduction. Let H_1 be a symmetric operator in a Hilbert space $\mathfrak{H}_1$. If H is a self-adjoint operator in a Hilbert space $\mathfrak{H}$ such that $\mathfrak{H}_1 \subset \mathfrak{H}$ and $H_1 \subset H$, then H is called a *self-adjoint extension* of H_1. If $\mathfrak{H} \ominus \mathfrak{H}_1$ is finite-dimensional, then H is called a *finite-dimensional* self-adjoint extension of H_1. H is called a *minimal* self-adjoint extension if neither $\mathfrak{H} \ominus \mathfrak{H}_1$ nor any of its subspaces different from $\{0\}$ reduces H.

Suppose H is a self-adjoint extension of H_1. If $E(\lambda)$ is the spectral function of H and if P_1 is the operator in $\mathfrak{H}$ of orthogonal projection on $\mathfrak{H}_1$, then the operator function $E_1(\lambda) = P_1E(\lambda)$ restricted to $\mathfrak{H}_1$ is called a *spectral function* of H_1. We shall say that the spectral function $E_1(\lambda)$ is *defined* by the self-adjoint extension H.

The family of spectral functions of H_1 is a *convex set*, i.e., if $E_1'(\lambda)$ and $E_1''(\lambda)$ are spectral functions of H_1 and if a and b are non-negative real numbers such that $a + b = 1$, then $aE_1'(\lambda) + bE_1''(\lambda)$ is also a spectral function of H_1. A spectral function $E_1(\lambda)$ of H_1 is said to be *extremal* if it is impossible to find two different spectral functions $E_1'(\lambda)$, $E_1''(\lambda)$ and positive real numbers a and b, $a + b = 1$, such that $E_1(\lambda) = aE_1'(\lambda) + bE_1''(\lambda)$.

For further information we refer the reader to Achieser and Glasmann [1].

M. A. Naimark [6] has shown that the finite-dimensional extensions of a symmetric operator define extremal spectral functions of the operator. Finite-dimensional extensions exist, however, only for symmetric operators with equal deficiency indices. In § 4 of this paper it is shown that self-adjoint extensions defined by the addition of maximal symmetric operators determine extremal spectral functions for a symmetric operator with unequal deficiency indices. The proof uses the proposition of M. A. Naimark [6] that if $E_1(\lambda)$ is defined by the minimal self-adjoint extension H, then $E_1(\lambda)$ is extremal if and only if every bounded self-adjoint operator A which commutes with H and satisfies the condition $(Af, g) = (f, g)$ for all $f, g \in \mathfrak{H}_1$ is reduced by $\mathfrak{H}_1$. Section 2 is devoted to a description of the self-adjoint extensions of a symmetric operator, and section 3 identifies some extremal spectral functions of a symmetric operator with infinite equal deficiency indices other than the ones defined by finite-dimensional extensions.

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The proof is based on the proposition of M. A. Naimark mentioned above.

2. Self-adjoint extensions of a symmetric operator. The linear operator H in the Hilbert space $\mathfrak{H}$ is said to be *Hermitian* if $(Hf, g) = (f, Hg)$ for all $f, g \in \mathfrak{D}(H)$. H is *symmetric* if it is Hermitian and $\overline{\mathfrak{D}(H)} = \mathfrak{H}$. If H is a closed Hermitian operator and λ is a nonreal number, we define the subspaces $\mathfrak{M}(\lambda)$ and $\mathfrak{L}(\lambda)$ by the equations $\mathfrak{L}(\lambda) = \mathfrak{R}(H - \bar{\lambda}E)$ and $\mathfrak{M}(\lambda) = \mathfrak{H} \ominus \mathfrak{L}(\lambda)$. ($E$ stands for the identity operator.) $\mathfrak{M}(\lambda)$ is called a *deficiency subspace* of H and has the same dimensions for all λ in the same half-plane (upper or lower.) If $m = \dim \mathfrak{M}(\bar{\lambda})$, $n = \dim \mathfrak{M}(\lambda)$, then (m, n) are called the *deficiency indices* of H (with respect to λ). (We add "with respect to λ" because the ordered pair (m, n) depends on the half-plane λ is in.) The operator $U(\lambda) = (H - \bar{\lambda}E)(H - \lambda E)^{-1}$ is an isometry mapping $\mathfrak{L}(\bar{\lambda})$ onto $\mathfrak{L}(\lambda)$. It is called the *Cayley transform* of H. We have that $H = (\lambda U(\lambda) - \bar{\lambda}E)(U(\lambda) - E)^{-1}$. Since λ is a fixed non-real number in the following, we shall write U in place of $U(\lambda)$. For fixed λ the correspondence between a Hermitian operator and its Cayley transform is a one-to-one inclusion-preserving correspondence between the set of closed Hermitian operators H and the set of closed isometric operators U for which $(U - E)^{-1}$ exists. We note, finally, that a subspace $\mathfrak{H}_1$ reduces H if and only if $\mathfrak{H}_1$ reduces U. In this circumstance, if $\mathfrak{H}_2 = \mathfrak{H} \ominus \mathfrak{H}_1$, and if H_i and U_i are H and U respectively restricted to $\mathfrak{H}_i$, then U_i is the Cayley transform of H_i and $H = H_1 \oplus H_2$, $U = U_1 \oplus U_2$.

M. A. Naimark [5] has proved the following theorem which describes all self-adjoint extensions of a symmetric operator.

THEOREM 1. *Let λ be any fixed nonreal number. Let H_1 be a closed symmetric operator with deficiency indices (m_1, n_1) (with respect to λ). Then every self-adjoint extension H of H_1 is obtained as follows:*

(1) *Let H_2 be a closed Hermitian operator in $\mathfrak{H}_2$ with deficiency indices (m_2, n_2) (with respect to λ) satisfying $m_1 + m_2 = n_1 + n_2$, $m_2 \leqq n_1$.*

(2) *Let $H_0 = H_1 \oplus H_2$ in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. ($H_0$ is therefore a closed Hermitian operator with equal deficiency indices $(m_1 + m_2, n_1 + n_2)$, and if U_i is the Cayley transform of H_i, $i = 0, 1, 2$, then $U_0 = U_1 \oplus U_2$. Further, $\mathfrak{M}_0(\bar{\lambda}) = \mathfrak{M}_1(\bar{\lambda}) \oplus \mathfrak{M}_2(\bar{\lambda})$, $\mathfrak{M}_0(\lambda) = \mathfrak{M}_1(\lambda) \oplus \mathfrak{M}_2(\lambda)$).*

(3) *Let V be an arbitrary isometric operator mapping $\mathfrak{M}_0(\bar{\lambda})$ onto $\mathfrak{M}_0(\lambda)$ satisfying the condition $\varphi \in \mathfrak{M}_2(\bar{\lambda})$, $V\varphi \in \mathfrak{M}_2(\lambda)$ implies $\varphi = 0$.*

(4) *Let $\mathfrak{D}(H)$ be defined as all $g = f + V\varphi - \varphi$, where $f \in \mathfrak{D}(H_0)$, $\varphi \in \mathfrak{M}_0(\bar{\lambda})$.*

(5) *If* $g \in \mathfrak{D}(H)$, *let* $Hg = H_0 f + \lambda V\varphi - \bar{\lambda}\varphi$.

Then, H is self-adjoint extension in $\mathfrak{H}$ of H_1, and every self-adjoint extension of H_1 is obtained in this way. We have that $\mathfrak{D}(H_2) = \mathfrak{D}(H) \cap \mathfrak{H}_2$.

We say that H_2 and V of Theorem 1 *define* the self-adjoint extension H of H_1.

We can put the operator V into correspondence with a matrix (V_{ik}) of operators such that V_{11}: $\mathfrak{M}_1(\bar{\lambda}) \to \mathfrak{M}_1(\lambda)$, V_{12}: $\mathfrak{M}_2(\bar{\lambda}) \to \mathfrak{M}_1(\lambda)$, V_{21}: $\mathfrak{M}_1(\bar{\lambda}) \to \mathfrak{M}_2(\lambda)$, V_{22}: $\mathfrak{M}_2(\bar{\lambda}) \to \mathfrak{M}_2(\lambda)$. Then condition on V in (3) of theorem 1 then becomes $V_{12}\varphi = 0$ implies $\varphi = 0$.

We now give a theorem which gives a more detailed analysis of the structure of V.

THEOREM 2. *Suppose that $\mathfrak{M}_1(\lambda)$, $\mathfrak{M}_1(\bar{\lambda})$, $\mathfrak{M}_2(\lambda)$, $\mathfrak{M}_2(\bar{\lambda})$ are Hilbert spaces and that V is an isometry which maps $\mathfrak{M}_1(\bar{\lambda}) \oplus \mathfrak{M}_2(\bar{\lambda})$ onto $\mathfrak{M}_1(\lambda) \oplus \mathfrak{M}_2(\lambda)$. ($\lambda$ here has nothing to do with the theorem and is retained only as a notational convenience.) If $V = (V_{ik})$ in matrix form, suppose that $V_{12}\varphi = 0$ implies that $\varphi = 0$. Then the following conclusions are true*:

(1) *If $\mathfrak{M}_1^-(\lambda)$ is defined by the equation $\mathfrak{M}_1^-(\lambda) = [V_{12}\mathfrak{M}_2(\bar{\lambda})]^c$ (c indicates closure of a set) and if $\mathfrak{N}_1(\lambda)$ is defined by $\mathfrak{N}_1(\lambda) = \mathfrak{M}_1(\lambda) \ominus \mathfrak{M}_1^-(\lambda)$, then $\mathfrak{N}_1(\lambda)$ is the null space of V_{12}^*. Thus, V_{12}^* is one-to-one on $\mathfrak{M}_1^-(\lambda)$. Further, $\mathfrak{M}_2(\bar{\lambda}) = [V_{12}^*\mathfrak{M}_1^-(\lambda)]^c$.*

(2) *$V^* = V^{-1}$ maps $\mathfrak{N}_1(\lambda)$ onto a subspaces of $\mathfrak{M}_1(\bar{\lambda})$, which we denote by $\mathfrak{N}_1(\bar{\lambda})$. Thus, $\mathfrak{N}_1(\bar{\lambda}) = V^*\mathfrak{N}_1(\lambda)$, $\mathfrak{N}_1(\lambda) = V\mathfrak{N}_1(\bar{\lambda})$.*

(3) *If $\mathfrak{M}_1^-(\bar{\lambda})$ is defined by the equation $\mathfrak{M}_1^-(\bar{\lambda}) = \mathfrak{M}_1(\bar{\lambda}) \ominus \mathfrak{N}_1(\bar{\lambda})$, then V maps $\mathfrak{M}_1^-(\bar{\lambda}) \oplus \mathfrak{M}_2(\bar{\lambda})$ isometrically onto $\mathfrak{M}_1^-(\lambda) \oplus \mathfrak{M}_2(\lambda)$.*

Thus, $V_{11}\mathfrak{M}_1^-(\bar{\lambda}) \subset \mathfrak{M}_1^-(\lambda)$.

(4) *V_{21} is one-to-one on $\mathfrak{M}_1^-(\bar{\lambda})$, and $\mathfrak{N}_1(\bar{\lambda})$ is the null space of V_{21}. $\mathfrak{M}_2(\lambda) = [V_{21}\mathfrak{M}_1^-(\bar{\lambda})]^c$.*

(5) *V_{21}^* is one-to-one on $\mathfrak{M}_2(\lambda)$ and $\mathfrak{M}_1^-(\bar{\lambda}) = [V_{21}^*\mathfrak{M}_2(\lambda)]^c$.*

(6) *If $m_1 = \dim \mathfrak{M}_1(\bar{\lambda})$, $n_1 = \dim \mathfrak{M}_1(\lambda)$, $m_2 = \dim \mathfrak{M}_2(\bar{\lambda})$, $n_2 = \dim \mathfrak{M}_2(\lambda)$, then $m_1 + m_2 = n_1 + n_2$, $m_2 = \dim \mathfrak{M}_2(\bar{\lambda}) = \dim \mathfrak{M}_1^-(\lambda) \leqq n_1$, $n_2 = \dim \mathfrak{M}_2(\lambda) = \dim \mathfrak{M}_1^-(\bar{\lambda}) \leqq m_1$.*

(7) *If $m_2 = n_2$, $m_1 = n_1$.*

Proof. (1) Since $\mathfrak{N}_1(\lambda)$ is the orthogonal complement of the closure of the range of V_{12}, $\mathfrak{N}_1(\lambda)$ is the null space of V_{12}^*, and V_{12}^* is one-to-one on $\mathfrak{M}_1^-(\lambda)$.

Suppose $g \in \mathfrak{M}_2(\bar{\lambda})$ and g is perpendicular to $V_{12}^*\mathfrak{M}_1^-(\lambda)$. Then $0 = (g, V_{12}^* f) = (V_{12}g, f)$ for all $f \in \mathfrak{M}_1^-(\lambda)$. Therefore, $V_{12}g = 0$, and, since V_{12} is one-to-one, $g = 0$. Thus, $M_2(\bar{\lambda}) = [V_{12}^*\mathfrak{M}_1^-(\lambda)]^c$.

(2) Since

$$V^* = \begin{pmatrix} V_{11}^* & V_{21}^* \\ V_{12}^* & V_{22}^* \end{pmatrix},$$

$V^*\mathfrak{N}_1(\lambda) = V_{11}^*\mathfrak{N}_1(\lambda) \subset \mathfrak{M}_1(\bar\lambda)$. Thus, $V^* = V^{-1}$ maps $\mathfrak{N}_1(\lambda)$ onto a subspace of $\mathfrak{M}_1(\bar\lambda)$.

(3) Clear, since $\mathfrak{N}_1(\lambda) = V\mathfrak{N}_1(\bar\lambda)$.

(4) We first show that V_{21} is one-to-one on $\mathfrak{M}_1^-(\bar\lambda)$. Suppose $f \in \mathfrak{M}_1^-(\bar\lambda)$, $V_{21}f = 0$. Then, $Vf = V_{11}f + V_{21}f = V_{11}f \in \mathfrak{M}_1^-(\lambda)$. Let $g = V_{11}f = Vf$, so that $f = V^*g = V_{11}^*g + V_{12}^*g$. Since $f \in \mathfrak{M}_1^-(\bar\lambda)$, $V_{11}^*g \in \mathfrak{M}_1^-(\bar\lambda)$, $V_{12}^*g \in \mathfrak{M}_2(\bar\lambda)$, we have that $V_{12}^*g = 0$. By (1) and the fact that $g \in \mathfrak{M}_1^-(\lambda)$, $g = 0$. Thus, $f = V^*g = 0$, and our contention is proved.

Since $\mathfrak{N}_1(\lambda) = V\mathfrak{N}_1(\bar\lambda)$, $V_{21}f = 0$ for all $f \in \mathfrak{N}_1(\bar\lambda)$. On the other hand, we have just shown that V_{21} is one-to-one on $\mathfrak{M}_1^-(\bar\lambda)$. It follows that $\mathfrak{N}_1(\bar\lambda)$ is the null space of V_{21}.

Because $(V_{21}^*)^* = V_{21}$ and the null space of $(V_{21}^*)^*$ is the orthogonal complement of the closure of the range of V_{21}^*, we see that $\mathfrak{M}_1^-(\bar\lambda) = [V_{21}^*\mathfrak{M}_2(\lambda)]^c$.

We claim finally that $\mathfrak{M}_2(\lambda) = [V_{21}M_1^-(\bar\lambda)]^c$. Suppose $g \in \mathfrak{M}_2(\lambda)$ and that g is perpendicular to $V_{21}\mathfrak{M}_1^-(\bar\lambda)$. Therefore, $0 = (V_{21}f, g) = (f, V_{21}^*g)$ for all $f \in \mathfrak{M}_1^-(\bar\lambda)$. Since $V_{21}^*g \in \mathfrak{M}_1^-(\bar\lambda)$, it follows that $V_{21}^*g = 0$. Thus, $V^*g = V_{22}^*g \in \mathfrak{M}_2(\bar\lambda)$. Let $f = V^*g$. Then, $g = Vf = V_{12}f + V_{22}f$, where $g \in \mathfrak{M}_2(\lambda)$, $V_{12}f \in \mathfrak{M}_1^-(\lambda)$, $V_{22}f \in \mathfrak{M}_2(\lambda)$. Hence, $V_{12}f = 0$ and $f = 0$. Whence, $g = Vf = 0$. This proves our claim and completes the proof of (4).

(5) We have already shown in (4) that $\mathfrak{M}_1^-(\bar\lambda) = [V_{21}^*\mathfrak{M}_2(\lambda)]^c$. Since we also showed in (4) that $\mathfrak{M}_2(\lambda) = [V_{21}\mathfrak{M}_1^-(\bar\lambda)]^c$, it follows that the null space of V_{21}^* is empty and therefore V_{21}^* is one-to-one on $\mathfrak{M}_2(\lambda)$.

(6) $m_1 + m_2 = n_1 + n_2$ follows from the fact that V maps $\mathfrak{M}_1(\bar\lambda) \oplus \mathfrak{M}_2(\bar\lambda)$ isometrically onto $\mathfrak{M}_1(\lambda) \oplus \mathfrak{M}_2(\lambda)$.

We claim now that $\dim \mathfrak{M}_2(\bar\lambda) = \dim \mathfrak{M}_1^-(\lambda)$. Let $\{\varphi_\alpha\}$ be a complete orthonormal system in $\mathfrak{M}_2(\bar\lambda)$. Then $\{V_{12}\varphi_\alpha\}$ is a fundamental set in $\mathfrak{M}_1^-(\lambda)$. (See Nagy [4] for definitions.) Therefore $\dim \mathfrak{M}_2(\bar\lambda) = P\{\varphi_\alpha\} = P\{V_{12}\varphi_\alpha\} \geqq \dim \mathfrak{M}_1^-(\lambda)$, where P stands for cardinality. Using V_{12}^* and an analogous argument, we obtain that $\dim \mathfrak{M}_1^-(\lambda) \geqq \dim \mathfrak{M}_2(\bar\lambda)$. Thus, $\dim \mathfrak{M}_2(\bar\lambda) = \dim \mathfrak{M}_1^-(\lambda)$, and $m_2 = \dim \mathfrak{M}_2(\bar\lambda) = \dim \mathfrak{M}_1^-(\lambda) \leqq n_1$. Similarly, $n_2 = \dim \mathfrak{M}_2(\lambda) = \dim \mathfrak{M}_1^-(\bar\lambda) \leqq m_1$.

(7) The proof is clear from the inequalities in (6).

Theorem 2 is therefore completely proved.

THEOREM 3. (M. A. Naimark [5]). *For each self-adjoint extension H in $\mathfrak{H}$ of a symmetric operator H_1 in $\mathfrak{H}_1$ there exists a minimal self-adjoint extension H_0 in $\mathfrak{H}_0$ such that*

(1) $\mathfrak{H}_1 \subset \mathfrak{H}_0 \subset \mathfrak{H}$;

(2) $H_1 \subset H_0 \subset H$;

(3) *H_0 and H define the same spectral function of H_1.*

THEOREM 4. *Suppose that H_1 is a closed symmetric operator and that H_2 and V define a self-adjoint extension H of H_1. Let H_0 be a self-adjoint extension of H_1 having the properties that $\mathfrak{H}_1 \subset \mathfrak{H}_0 \subset \mathfrak{H}$ and $H_1 \subset H_0 \subset H$. Then the following statements are true*:

(1) *If we write $\mathfrak{H}_0 = \mathfrak{H}_1 \oplus \mathfrak{H}_3$, $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_4 = \mathfrak{H}_1 \oplus \mathfrak{H}_3 \oplus \mathfrak{H}_4$, $\mathfrak{H}_2 = \mathfrak{H}_3 \oplus \mathfrak{H}_4$, then H is reduced by $\mathfrak{H}_4$ and $H = H_0 \oplus H_4$, where H_4 is a self-adjoint operator in $\mathfrak{H}_4$.*

(2) $\mathfrak{H}_4 \subset \mathfrak{L}_2(\bar{\lambda}) \cap \mathfrak{L}_2(\lambda)$, $\mathfrak{M}_2(\bar{\lambda}) \subset \mathfrak{H}_3$, $\mathfrak{M}_2(\lambda) \subset \mathfrak{H}_3$.

(3) *H_2 is reduced by $\mathfrak{H}_4$ and $H_2 = H_3 \oplus H_4$, where H_3 is a closed Hermitian operator in $\mathfrak{H}_3$ with the same deficiency subspaces $M_2(\bar{\lambda})$, $\mathfrak{M}_2(\lambda)$ as H_2.*

(4) *H_0 is defined by H_3 and V.*

(5) *H and H_0 define the same spectral function of H_1.*

Proof. (1) Since $H_1 \subset H_0 \subset H$, we have that $U_1 \subset U_0 \subset U$. Because U_0 maps $\mathfrak{H}_0$ isometrically onto $\mathfrak{H}_0$ and U maps $\mathfrak{H}$ isometrically onto $\mathfrak{H}$, we have that U maps $\mathfrak{H}_4$ isometrically onto $\mathfrak{H}_4$. Thus, $\mathfrak{H}_4$ reduces U, and hence $U = U_0 \oplus U_4$, $H = H_0 \oplus H_4$, where H_4 is a self-adjoint operator in $\mathfrak{H}_4$ with Cayley transform U_4. This proves (1).

(2) We claim first that $\mathfrak{H}_4 \subset \mathfrak{L}_2(\bar{\lambda})$. Let $f \in \mathfrak{H}_4$. Since $H_4 \subset \mathfrak{H}_2 = \mathfrak{M}_2(\bar{\lambda}) \oplus \mathfrak{L}_2(\bar{\lambda})$, $f = f' + f''$, where $f' \in \mathfrak{M}_2(\bar{\lambda})$, $f'' \in \mathfrak{L}_2(\bar{\lambda})$. Hence, $Uf = Uf' + Uf'' = Vf' + U_2 f'' = V_{12} f' + V_{22} f' + U_2 f''$, where $Uf \in \mathfrak{H}_4 \subset \mathfrak{H}_2$, $V_{12} f' \in \mathfrak{M}_1(\lambda) \subset \mathfrak{H}_1$, $V_{22} f' \in \mathfrak{M}_2(\lambda) \subset \mathfrak{H}_2$, $U_2 f'' \in \mathfrak{L}_2(\lambda) \subset \mathfrak{H}_2$. Thus, $V_{12} f' = 0$, and therefore $f' = 0$. It follows that $f = f'' \in \mathfrak{L}_2(\bar{\lambda})$ and that $\mathfrak{H}_4 \subset \mathfrak{L}_2(\bar{\lambda})$.

Since $\mathfrak{H}_4 \subset \mathfrak{H}_2(\bar{\lambda})$, and since U maps $\mathfrak{H}_4$ isometrically onto $\mathfrak{H}_4$ and $\mathfrak{L}_2(\bar{\lambda})$ isometrically onto $\mathfrak{L}_2(\lambda)$, we conclude that $\mathfrak{H}_4 \subset \mathfrak{L}_2(\lambda)$. Hence, $\mathfrak{H}_4 \subset \mathfrak{L}_2(\bar{\lambda}) \cap \mathfrak{L}_2(\lambda)$. It follows immediately that $\mathfrak{M}_2(\bar{\lambda}) \subset \mathfrak{H}_3$, $\mathfrak{M}_2(\lambda) \subset \mathfrak{H}_3$. (2) is therefore completely proved.

(3) Because $U_2 = U$ on $\mathfrak{L}_2(\bar{\lambda})$, we see that U_2 maps $\mathfrak{H}_4$ isometrically onto $\mathfrak{H}_4$. We know, however, that U_2 maps $\mathfrak{L}_2(\bar{\lambda})$ isometrically onto $\mathfrak{L}_2(\lambda)$. It follows that $\mathfrak{H}_4$ reduces U_2. Thus, $U_2 = U_3 \oplus U_4$, where U_3 maps $\mathfrak{L}_2(\bar{\lambda}) \ominus \mathfrak{H}_4$ isometrically onto $\mathfrak{L}_2(\lambda) \ominus \mathfrak{H}_4$, and $H_2 = H_3 \oplus H_4$, where H_3 is a closed Hermitian operator in $\mathfrak{H}_3$ with Cayley transform U_3. Noting that $\mathfrak{H}_3 = \mathfrak{M}_2(\lambda) \oplus [\mathfrak{L}_2(\bar{\lambda}) \ominus \mathfrak{H}_4] = \mathfrak{M}_2(\lambda) \oplus [\mathfrak{L}_2(\lambda) \ominus \mathfrak{H}_4]$, we see that H_3 has deficiency subspaces $\mathfrak{M}_2(\bar{\lambda})$, $\mathfrak{M}_2(\lambda)$. This proves (3).

(4) By Theorem 1, H_3 and V define a self-adjoint extension H_0' of H_1 in $\mathfrak{H}_0 = \mathfrak{H}_1 \oplus \mathfrak{H}_3$. If U_0' is the Cayley transform of H_0', then $U_0' = U_1 = U$ on $\mathfrak{L}_1(\bar{\lambda})$, $U_0' = V = U$ on $\mathfrak{M}_1(\bar{\lambda}) \oplus \mathfrak{M}_2(\bar{\lambda})$, $U_0' = U_3 = U$ on $\mathfrak{L}_2(\bar{\lambda}) \ominus \mathfrak{H}_4$. It follows that $U_0' = U$ on $\mathfrak{H}_1 \oplus \mathfrak{H}_3 = \mathfrak{H}_0$. But since $U_0 \subset U$, $U_0 = U$ on $\mathfrak{H}_0$, hence, $U_0 = U_0'$, and therefore $H_0 = H_0'$. This

proves (4).

(5) As we have shown, $H = H_0 \oplus H_4$. Thus, $E(\lambda) = E_0(\lambda) \oplus E_4(\lambda)$, and therefore $E(\lambda)f = E_0(\lambda)f$ for all $f \in \mathfrak{H}_1$. If P is the operator of orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_1$ and if P_0 is the operator of orthogonal projection of $\mathfrak{H}_0$ onto $\mathfrak{H}_1$, $PE(\lambda)f = PE_0(\lambda)f = P_0E_0(\lambda)f$ for all $f \in \mathfrak{H}_1$, so that H and H_0 define the same spectral function of H_1. This proves (5), and the proof of theorem 4 is completed.

3. Extremal spectral functions of a symmetric operator with equal deficiency indices.

THEOREM 5. *Let H be a self-adjoint extension of the closed symmetric operator H_1. Suppose that H is defined by H_2 and V. Then the following statements are equivalent*:

(1) $\mathfrak{D}(H_2) = \{0\}$.

(2) $\mathfrak{M}_2(\bar{\lambda}) = \mathfrak{M}_2(\lambda) = \mathfrak{H}_2$.

(3) $\mathfrak{D}(H) \cap \mathfrak{H}_2 = \{0\}$.

Proof. That (1) implies (2) is clear from the definition of $\mathfrak{M}_2(\bar{\lambda})$ and $\mathfrak{M}_2(\lambda)$. Suppose, on the other hand, that $\mathfrak{M}_2(\bar{\lambda}) = \mathfrak{M}_2(\lambda) = \mathfrak{H}_2$. Then, $\mathfrak{R}(H_2 - \lambda E) = \mathfrak{R}(H_2 - \bar{\lambda} E) = \{0\}$. If $f \in \mathfrak{D}(H_2)$, $H_2 f - \lambda f = 0$ and $H_2 f - \bar{\lambda} f = 0$. Subtracting the first equation from the second, $(\lambda - \bar{\lambda})f = 0$, and therefore $f = 0$. Thus, $\mathfrak{D}(H_2) = \{0\}$, and we have proved that (2) implies (1).

By Theorem 1, $\mathfrak{D}(H_2) = \mathfrak{D}(H) \cap \mathfrak{H}_2$, so that (1) and (3) are clearly equivalent.

THEOREM 6. *Let H_1 be a closed symmetric operator. Suppose that H is a self-adjoint extension of H_1 defined by H_2 and V. If $\mathfrak{D}(H_2) = \{0\}$, the following statements are true*:

(1) $m_1 = n_1$, i.e., *the deficiency indices of H_1 are equal.*

(2) *H is minimal.*

(3) *The spectral function $E_1(\lambda)$ of H_1 defined by H is extremal.*

Proof. (1) By Theorem 5, $\mathfrak{D}(H_2) = \{0\}$ implies that $m_2 = n_2$. By theorem 2, (7), $m_1 = n_1$.

(2) By Theorem 5, $\mathfrak{D}(H_2) = \{0\}$ implies that $\mathfrak{M}_2(\bar{\lambda}) = \mathfrak{M}_2(\lambda) = \mathfrak{H}_2$. Hence, $\mathfrak{L}_2(\bar{\lambda}) = \mathfrak{L}_2(\lambda) = \{0\}$. It follows from Theorem 3 and Theorem 4, (2), that H is minimal.

(3) Let A be any bounded operator in $\mathfrak{H}$ having a matrix representation,

$$A \sim \begin{pmatrix} E & B \\ B^* & C \end{pmatrix},$$

where E is the identity in $\mathfrak{H}_1$, B maps $\mathfrak{H}_2$ into $\mathfrak{H}_1$, C maps $\mathfrak{H}_2$ into $\mathfrak{H}_2$, and C is self-adjoint. Suppose that A commutes with H. We shall show that this implies that $B \equiv 0$. By the proposition of M. A. Naimark [6] mentioned in the introduction, then, it follows that the spectral function $E_1(\lambda)$ defined by H is extremal.

Since A commutes with H, it commutes with the Cayley transform U of H. If we represent U as a matrix, $U \sim (U_{jk})$, where U_{jk} maps $\mathfrak{H}_k$ into $\mathfrak{H}_j$, then the fact that A commutes with U implies that $BU_{21} = U_{12}B^*$. Taking adjoints, we also have that $U_{21}^*B^* = BU_{12}^*$. We observe, further, that $U = V$ on $\mathfrak{M}_1(\bar{\lambda}) \oplus \mathfrak{M}_2(\bar{\lambda})$ and that $U^* = U^{-1} = V^{-1} = V^*$ on $\mathfrak{M}_1(\lambda) \oplus \mathfrak{M}_2(\lambda)$.

Using the equation $BU_{12}^* = U_{21}^*B^*$, the fact that $\mathfrak{M}_2(\lambda) = \mathfrak{H}_2$, and Theorem 2, we obtain that $BV_{12}^*\mathfrak{M}_1^-(\lambda) = BU_{12}^*\mathfrak{M}_1^-(\lambda) = U_{21}^*B^*\mathfrak{M}_1^-(\lambda) \subset U_{21}^*\mathfrak{H}_2 = U_{21}^*\mathfrak{M}_2(\lambda) = V_{21}^*\mathfrak{M}_2(\lambda) \subset \mathfrak{M}_1(\bar{\lambda})$. Since by Theorem 2 $V_{12}^*\mathfrak{M}_1^-(\lambda)$ is dense in $M_2(\bar{\lambda}) = \mathfrak{H}_2$ and since B is bounded, it follows that $B\mathfrak{H}_2 \subset \mathfrak{M}_1(\bar{\lambda})$.

Similarly, using the equation $BU_{21} = U_{12}B^*$, we obtain that $BV_{21}\mathfrak{M}_1^-(\bar{\lambda}) = BU_{21}\mathfrak{M}_1^-(\bar{\lambda}) = U_{12}B^*\mathfrak{M}_1^-(\bar{\lambda}) \subset U_{12}\mathfrak{H}_2 = U_{12}\mathfrak{M}_2(\bar{\lambda}) = V_{12}\mathfrak{M}_2(\bar{\lambda}) \subset \mathfrak{M}_1(\lambda)$, and therefore $B\mathfrak{H}_2 \subset \mathfrak{M}_1(\lambda)$.

Thus, $B\mathfrak{H}_2 \subset \mathfrak{M}_1(\bar{\lambda}) \cap \mathfrak{M}_1(\lambda)$. But $\mathfrak{M}_1(\bar{\lambda}) \cap \mathfrak{M}_1(\lambda) = \{0\}$, because $\mathfrak{M}_1(\bar{\lambda})$ and $\mathfrak{M}_1(\lambda)$ are the deficiency subspaces of a symmetric operator. Hence, $B \equiv 0$. This complete the proof of Theorem 6.

By use of a somewhat less general form of Theorem 6, M. A. Naimark [6] has shown that every finite-dimensional extension H of a closed symmetric operator H_1 defines an extremal spectral function of H_1.

THEOREM 7. *If H is a finite-dimensional extension of a closed symmetric operator H_1, then H_1 must have equal deficiency indices.*

Proof. Suppose that H is defined by H_2 and V. Then H_2 is a Hermitian operator in the finite-dimensional space $\mathfrak{H}_2$. Since U_2 maps $\mathfrak{L}_2(\bar{\lambda})$ isometrically onto $\mathfrak{L}_2(\lambda)$, it follows that $\dim \mathfrak{L}_2(\bar{\lambda}) = \dim \mathfrak{L}_2(\lambda)$. Hence $\dim \mathfrak{M}_2(\bar{\lambda}) = \dim \mathfrak{M}_2(\lambda)$, i.e., $m_2 = n_2$. By Theorem 2, (7), $m_1 = n_1$. This proves Theorem 7.

4. Extremal spectral functions of a symmetric operator with unequal deficiency indices. We first introduce the notion of a partial isometry and some of the properties thereof. (See Murray and von Neumann [3].) A bounded linear operator W in a Hilbert space $\mathfrak{H}$ is called a *partial isometry* if it maps a subspace $\mathfrak{E}$ isometrically onto another subspace $\mathfrak{F}$, while it maps $\mathfrak{H} \ominus \mathfrak{E}$ onto $\{0\}$. $\mathfrak{E}$ is called the *initial set* of W, and $\mathfrak{F}$ is called the *final set* of W. If W is a partial isometry, then the following statements hold:

(1) If $P(\mathfrak{E})$ is the operator of orthogonal projection on $\mathfrak{E}$ and if $P(\mathfrak{F})$ is the operator of orthogonal projection on $\mathfrak{F}$, then $P(\mathfrak{E}) = W^*W$;

$P(\mathfrak{F}) = WW^*$.

(2) U^* is a partial isometry with initial set $\mathfrak{F}$ and final set $\mathfrak{E}$.

(3) As a mapping of $\mathfrak{F}$ onto $\mathfrak{E}$, U^* is the inverse of U as a mapping of $\mathfrak{E}$ onto $\mathfrak{F}$.

THEOREM 8. *Suppose that W is a partial isometry with initial set $\mathfrak{M}$ and final set $\mathfrak{H}$. Let $\mathfrak{N} = \mathfrak{H} \ominus \mathfrak{M}$. Then, $\mathfrak{M} = \mathfrak{M}' \oplus \mathfrak{M}''$, where*

(1) W maps $\mathfrak{M}''$ isometrically onto $\mathfrak{M}''$;

(2) if $f \in \mathfrak{N} \oplus \mathfrak{M}'$, $\lim_{p\to\infty} W^p f = 0$.

Proof. Let $\mathfrak{M}_i = (W^*)^i \mathfrak{N}$, $i = 0, 1, 2, \cdots$. Then each $\mathfrak{M}_i$ is a subspace (i.e., a closed linear manifold), and the following statements are true:

(a) $\mathfrak{M}_i \subset \mathfrak{M}$ for $i = 1, 2, \cdots$. This is clear because W^* is a partial isometry with initial set $\mathfrak{H}$ and final set $\mathfrak{M}$.

(b) If $f \in \mathfrak{M}_n$, where $n \geqq 0$, then $W^p f \in \mathfrak{M}_{n-p}$ for $1 \leqq p \leqq n$, and $W^p f = 0$ for $p > n$. Proof: If $f \in \mathfrak{M}_n$, then $f = (W^*)^n g$ for some $g \in \mathfrak{N}$. Since $WW^* = E$, $W^p f = (W^*)^{n-p} g \in \mathfrak{M}_{n-p}$, $1 \leqq p \leqq n$. If $p > n$, $W^p f = W^{p-n} g = 0$.

(c) If $f \in \mathfrak{M}_i$, $i = 0, 1, 2, \cdots$, and if n is a positive integer, then $(W^*)^n f \in \mathfrak{M}_{i+n}$. Proof: If $f \in \mathfrak{M}_i$, $f = (W^*)^i g$, where $g \in \mathfrak{N}$. Therefore, $(W^*)^n f = (W^*)^{i+n} g \in \mathfrak{M}_{i+n}$.

(d) $\mathfrak{M}_i$ is perpendicular to $\mathfrak{M}_j$ if $i \neq j$. Proof: Suppose $i < j$, and let $f \in \mathfrak{M}_i$, $g \in \mathfrak{M}_j$. Then there exists $f_1 \in \mathfrak{N}$ and $g_1 \in \mathfrak{N}$ such that $f = (W^*)^i f_1$, $g = (W^*)^j g_1$. Hence, $(f, g) = ((W^*)^i f_1, (W^*)^j g_1) = (f_1, (W^*)^{j-i} g_1) = 0$, since $f_1 \in \mathfrak{N}$, $(W^*)^{j-i} g_1 \in \mathfrak{M}_{j-i} \subset \mathfrak{M}$.

Now let $\mathfrak{M}' = \sum_{i=1}^{\infty} \mathfrak{M}_i$. Then $\mathfrak{M}'$ is a subspace of $\mathfrak{M}$. Let $\mathfrak{M}'' = \mathfrak{M} \ominus \mathfrak{M}'$. We shall show that $\mathfrak{M}'$ and $\mathfrak{M}''$ satisfy (1) and (2).

Since $\mathfrak{M} = \mathfrak{M}' \oplus \mathfrak{M}''$ and $\mathfrak{H} = \mathfrak{N} \oplus \mathfrak{M}' \oplus \mathfrak{M}''$, and since W maps $\mathfrak{M}$ isometrically onto $\mathfrak{H}$, in order to prove (1) it is sufficient to show that W maps $\mathfrak{M}'$ onto $\mathfrak{N} \oplus \mathfrak{M}'$. Suppose $f \in \mathfrak{M}'$. Then, $f = \sum_{i=1}^{\infty} f_i$, where $f_i \in \mathfrak{M}_i$, and $Wf = \sum_{i=1}^{\infty} Wf_i$. Because by (b) $Wf_i \in \mathfrak{M}_{i-1}$, we see that $Wf \in \mathfrak{N} \oplus \mathfrak{M}'$. Thus, W maps $\mathfrak{M}'$ into $\mathfrak{N} \oplus \mathfrak{M}'$. To show that the map is onto, let $g \in \mathfrak{N} \oplus \mathfrak{M}'$. Then, $g = \sum_{i=0}^{\infty} f_i$, where $f_i \in \mathfrak{M}_i$. If $f = W^* f = \sum_{i=0}^{\infty} W^* f_i \in \mathfrak{M}'$, by (c). Further, $Wf = WW^* g = g$. Hence, W maps $\mathfrak{M}'$ onto $\mathfrak{N} \oplus \mathfrak{M}'$.

We now prove (2). Let $f \in \mathfrak{N} \oplus \mathfrak{M}'$. Then, $f = \sum_{i=0}^{\infty} f_i$, where $f_i \in \mathfrak{M}_i$. By (b), $W^p f = \sum_{i=0}^{\infty} W^p f_i = \sum_{i=p}^{\infty} W^p f_i$. Hence, $\| W^p f \|^2 = \sum_{i=p}^{\infty} \| W^p f_i \|^2 = \sum_{i=p}^{\infty} \| f_i \|^2$. Thus, $\lim_{p\to\infty} \| W^p f \|^2 = 0$. This proves (2) and completes the proof of the theorem.

THEOREM 9. *Let λ be a fixed nonreal number. Suppose that H_1 is a closed symmetric operator in $\mathfrak{H}_1$ with deficiency indices (m, n)*

(with respect to λ*), and suppose that* $m \neq n$*. Let* H *be a self-adjoint extension of* H_1 *defined by* H_2 *and* V*, where* H_2 *is a closed Hermitian operator with deficiency indices* $(0, s)$*,* $n + s = m$*, if* $m > n$ *and* $(s, 0)$*,* $m + s = n$*, if* $m < n$*. Then the spectral function defined by* H *is extremal.*

Proof. Assume that $m > n$. The case $m < n$ then follows by interchanging the roles of $\bar{\lambda}$ and λ in Theorem 1 and defining H by H_2 and V^*.

By Theorem 3 there exists a minimal self-adjoint extension H_0 of H_1 such that $\mathfrak{H}_1 \subset \mathfrak{H}_0 \subset \mathfrak{H}$, $H_1 \subset H_0 \subset H$, and H_0 and H define the same spectral function of H_1. By Theorem 4, H_0 is defined by V and a Hermitian operator H_3 with the same deficiency subspaces as H_2. Since we can always consider H_0 instead of H, it follows that without loss of generality we can consider H to be a minimal self-adjoint extension.

Since $\mathfrak{M}_2(\bar{\lambda}) = \{0\}$ and $\mathfrak{L}_2(\bar{\lambda}) = \mathfrak{H}_2$, we have that if $f \in \mathfrak{H}_2$, $Uf \in \mathfrak{L}_2(\lambda) \subset \mathfrak{H}_2$. If we represent U as a matrix, $U \sim (U_{jk})$, where U_{jk} maps $\mathfrak{H}_k$ into $\mathfrak{H}_j$, then it follows that $U_{12} \equiv 0$ on $\mathfrak{H}_2$. Further, $Uf = U_{22}f$ for all $f \in \mathfrak{H}_2$, so that U_{22} maps $\mathfrak{H}_2$ isometrically onto $\mathfrak{L}_2(\lambda)$. U_{22} is thus a partial isometry in $\mathfrak{H}_2$ with initial set $\mathfrak{H}_2$ and final set $\mathfrak{L}_2(\lambda)$, while U_{22}^* is a partial isometry with initial set $\mathfrak{L}_2(\lambda)$ and final set $\mathfrak{H}_2$. We have that $E = P(\mathfrak{H}_2) = U_{22}^* U_{22}$, while $P(\mathfrak{L}_2(\lambda)) = U_{22}U_{22}^*$.

Now let A be any bounded operator in $\mathfrak{H}$ with matrix representation

$$A \sim \begin{pmatrix} E & B \\ B^* & C \end{pmatrix},$$

where E is the identity in $\mathfrak{H}_1$, B maps $\mathfrak{H}_2$ into $\mathfrak{H}_1$, C maps $\mathfrak{H}_2$ into $\mathfrak{H}_2$, and C is self-adjoint. Suppose that A commutes with H. We shall show that this implies $B \equiv 0$. Then by the proposition of M. A. Naimark [6] mentioned in the introduction, it follows that the spectral function $E_1(\lambda)$ defined by H is extremal.

Since A commutes with H, it commutes with the Cayley transform U of H. This implies that $BU_{21} = U_{12}B^*$ and $U_{12} + BU_{22} = U_{11}B + U_{12}C$. Since $U_{12} \equiv 0$, these equations become $BU_{21} \equiv 0$ and $BU_{22} = U_{11}B$. On $\mathfrak{M}_1(\bar{\lambda})$, $U_{21} = V_{21}$ and therefore $BV_{21}\mathfrak{M}_1(\bar{\lambda}) = BU_{21}\mathfrak{M}_1(\bar{\lambda}) = \{0\}$. Becaese by Theorem 2, $V_{21}\mathfrak{M}_1(\bar{\lambda})$ is dense in $\mathfrak{M}_2(\lambda)$, $B\mathfrak{M}_2(\lambda) = \{0\}$, i.e., $BP(\mathfrak{M}_2(\lambda)) = 0$. From the equation $BU_{22} = U_{11}B$ we have that $BP(\mathfrak{L}_2(\lambda)) = BU_{22}U_{22}^* = U_{11}BU_{22}^*$. Adding $BP(\mathfrak{L}_2(\lambda)) = U_{11}BU_{22}^*$ with $BP(\mathfrak{M}_2(\lambda)) = 0$, we obtain that $B = U_{11}BU_{22}^*$. By iterating this equation we see that $B = U_{11}^pB(U_{22}^*)^p$ for every positive integer p. Since $\| U_{11} \| \leqq 1$, $\| Bf \| \leqq \| B \| \ \| (U_{22}^*)^pf \|$ for each $f \in \mathfrak{H}_2$ and each positive integer p.

By Theorem 8, $\mathfrak{L}_2(\lambda) = \mathfrak{M}' \oplus \mathfrak{M}''$, where U_{22}^* maps $\mathfrak{M}''$ isometrically onto $\mathfrak{M}''$, and if $f \in \mathfrak{M}_2(\lambda) \oplus \mathfrak{M}'$, then $\lim_{p\to\infty} \| (U_{22}^*)^p f \| = 0$. But if U_{22}^* maps $\mathfrak{M}''$ isometrically onto $\mathfrak{M}''$, then U_{22} and therefore U maps $\mathfrak{M}''$ isometrically onto $\mathfrak{M}''$. This means that U and therefore H is reduced by $\mathfrak{M}''$, a subspace of $\mathfrak{H}_2$. Since H is a minimal self-adjoint extension of H_1, $\mathfrak{M}'' = \{0\}$. Hence, $\mathfrak{H}_2 = \mathfrak{M}_2(\lambda) \oplus \mathfrak{M}'$, and thus if $f \in \mathfrak{H}_2$, $\lim_{p\to\infty} \| (U_{22}^*)^p f \| = 0$. Since $\| Bf \| \leqq \| B \| \; \| (U_{22}^*)^p f \|$ for each $f \in \mathfrak{H}_2$ and for every positive integer p, it follows that $B \equiv 0$ on $\mathfrak{H}_2$. This completes the proof of Theorem 9.

Since the operator H_2 in Theorem 9 is a *Hermitian* operator with deficiency indices $(0, s)$ or $(s, 0)$, it may seem that we are dealing with a wider class of operators than the maximal symmetric operators. That this is not so is shown by Theorem 10 below.

THEOREM 10. *If H is a Hermitian operator with deficiency indices* $(0, s)$ *or* $(s, 0)$, *then H is a maximal symmetric operator. If H is a Hermitian operator with deficiency indices* $(0, 0)$, *then H is a self-adjoint operator.*

Proof. If H is a Hermitian operator and $\mathfrak{B} = \mathfrak{H} \ominus [\mathfrak{D}(H)]^c$, then $\mathfrak{B} \cap \mathfrak{L}(\bar{\lambda}) = \{0\}$. (If $h \in \mathfrak{B} \cap \mathfrak{L}(\bar{\lambda})$, then $h = (H - \lambda E)g$, Hence, $0 = (h, g) = (Hg, g) - \lambda(g, g)$. Since (Hg, g) is real while λ is not, $g = 0$. This simple argument is due to M. A. Krasnosel'skii [2, Lemma 2].) If H has deficiency indices $(0, s)$, $\mathfrak{M}(\bar{\lambda}) = \{0\}$ so that $\mathfrak{B} \subset \mathfrak{L}(\bar{\lambda})$. Thus, $\mathfrak{B} = \{0\}$ and H is symmetric. Similarly, H is symmetric if its deficiency indices are $(s, 0)$. It follows immediately that if H has deficiency indices $(0, 0)$, H is self-adjoint. Theorem 10 is proved.

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MATHEMATICS RESEARCH CENTER, U.S. ARMY
UNIVERSITY OF WISCONSIN, AND
UNIVERSITY OF CALIFORNIA, RIVERSIDE