ON THE EXTENSIONS OF LATTICE-ORDERED GROUPS

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1. **Introduction.** Throughout this paper $A = 0, a, b, \cdots, \Delta = \theta, \alpha, \beta, \cdots$ and G will be abelian partially ordered groups (p.o. groups). G is a *p.o. extension* of A by Δ if there is an order preserving homomorphism (o-homomorphisn) π of G onto Δ with kernel A such that π induces an o-isomorphism of G/A with Δ, (i.e. $\pi(g) > \theta$ implies $g + A$ contains a positive element). If A and Δ are lattice ordered groups (l-groups) then G is an *l-extension* if G is an l-group, π is an l-homomorphism and π induces an l-isomorphism between G/A and Δ. In this case A is an l-ideal of G.

If G is a p.o. extension of A by Δ then for each $\alpha \in \Delta$ choose $r(\alpha) \in G$ such that $\pi(r(\alpha)) = \alpha$ and $r(\theta) = 0$. Define

$$f(\alpha, \beta) = -r(\alpha + \beta) + r(\alpha) + r(\beta) \quad \text{for all } \alpha, \beta \in \Delta$$

and

$$Q_\alpha = \{a \in A \mid r(\alpha) + a \geqq 0\} \quad \text{for } \alpha \in \Delta^+ = \{\delta \in \Delta \mid \delta \geqq \theta\}\ .$$

Then the following conditions are satisfied for all α, β, γ in Δ.

(i) $f(\alpha, \beta) = f(\beta, \alpha)$

(ii) $f(\alpha, \theta) = f(\theta, \alpha) = 0$

(iii) $f(\alpha, \beta) + f(\alpha + \beta, \gamma) = f(\alpha, \beta + \gamma) + f(\beta, \gamma)$.

Moreover, for $\alpha, \beta \in \Delta^+$ we have

(iv) $Q_\alpha \neq \phi$

(v) $Q_\alpha + Q_\beta + f(\alpha, \beta) \subseteqq Q_{\alpha+\beta}$

(vi) $Q_\theta = A^+$.

Conditions (iv)–(vi) are due to L. Fuchs and can be derived from the results in [5].

Now if $\bar{G} = A \times \Delta$ and we define $(a, \alpha) + (b, \beta) = (a + b + f(\alpha, \beta), \alpha + \beta)$ and (a, α) positive if $\alpha \in \Delta^+$ and $a \in Q_\alpha$, then the mapping $(a, \alpha) \rightarrow r(\alpha) + a$ is an o-isomorphism of $\bar{G}$ onto G. In what follows we usually identify G and $\bar{G}$.

Conversely, if we are given $A, \Delta, f: \Delta \times \Delta \rightarrow A$ and $Q: \Delta^+ \rightarrow$ {subsets of A} such that f and Q satisfy (i)–(vi) then $\bar{G}$ is a p.o. extension of A by Δ and the mapping $(a, \alpha) \rightarrow \alpha$ is the corresponding o-homomorphism.

Two p.o. extensions $G = (A, \Delta, f, Q)$ and $G' = (A, \Delta, f', Q')$ are *o-equivalent* if there is a function $t: \Delta \rightarrow A$ such that

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$$f'(\alpha, \beta) = f(\alpha, \beta) - t(\alpha + \beta) + t(\alpha) + t(\beta)$$

and

$$Q'_\alpha = -t(\alpha) + Q_\alpha .$$

This is equivalent to the fact that there exists an o-isomorphism of G onto G' that induces the identity on A and $G/A = \Delta$.

In Theorem 1 we give necessary and sufficient conditions that a p.o. extension $G = (A, \Delta, f, Q)$ be an l-extension. If G is an l-extension such that for each $\alpha \in \Delta^+$, Q_α is a principal dual ideal, that is, generated by a single element, then Lemma 2.2 shows G is o-equivalent to the cardinal sum $A \boxplus \Delta$. We show in Lemma 2.3, if A is a lexicographic extension of an l-ideal B (notation: $A = \langle B \rangle$) then for each $\alpha \in \Delta^+$, $Q_\alpha = A$ or Q_α is a principal dual ideal. Theorem 2 shows that if G is an l-extension of $A = \langle B \rangle$ then G contains an l-ideal $H \cong A \boxplus J$, $J \subseteqq \Delta$ and G is an l-extension of H by the ordered group (o-group) Δ/J. In addition if Δ is an o-group then $G = \langle A \boxplus J \rangle$.

Theorem 3 gives a method of constructing l-extensions from an abelian extension $G = (A, \Delta, f)$ that depends only on the cardinal summands of A.

In § 4 we use the above to investigate those l-extensions of an l-group A with a finite basis. We show that to an o-equivalence every l-extension of such an l-group A by an l-group Δ is determined by a meet-preserving homomorphism of the semigroup Δ^+ to the semigroup of all cardinal summands of A such that $f(\alpha, \beta) \in H_{\alpha+\beta}$.

2. Extensions of l-groups.

A subset Q of A is a *dual ideal* if $a \in Q$ and $b \geqq a$ implies $b \in Q$.

LEMMA 2.1. *If A is an l-group and $Q \subseteqq A$ is a dual ideal that satisfies*

(*) $\quad Q \cap (b + A^+)$ *has a smallest element for all* $b \in A$,
then Q is a sublattice of A. Thus Q is a lattice dual ideal.

Proof. Let $a, b \in Q$, then $a \vee b \in Q$ since Q is a dual ideal. Also, $a, b \in Q \cap [(a \wedge b) + A^+]$ so by (*) there is an element $x \in Q \cap [(a \wedge b) + A^+]$ such that $x \leqq a$ and $x \leqq b$. Hence, $x \leqq a \wedge b$ so $a \wedge b \in Q$ and Q is a sublattice of A as desired.

If E is a subset of A then the dual ideal generated by E (notation: $DI(E)$) is $\{x \in A \mid x \geqq y$ for some $y \in E\}$. If a dual ideal is generated by a single element we say the dual ideal is *principal*.

THEOREM 1. *Suppose A and Δ are l-groups and $G = (A, \Delta, f, Q)$ is a p.o.-extension of A by Δ. Then G is an l-extension if and only if*

(1) *if* $\alpha \wedge \beta = \theta$ *then* $Q_\alpha \cap [Q_\beta + b + f(\alpha - \beta, \beta)]$ *has a smallest element for all* $b \in A$,

and

(2) $$Q_\alpha + Q_\beta + f(\alpha, \beta) = Q_{\alpha+\beta} \text{ for } \alpha, \beta \in \Delta^+ .$$

Proof. Let G be an l-extension. Suppose $b \in A$ and $\alpha, \beta \in \Delta^+$ are such that $\alpha \wedge \beta = \theta$. Let $\gamma = \alpha - \beta$. For $a \in A$, the mapping of $(a, \alpha) \to \alpha$ is an l-homomorphism so $(b, \gamma) \vee (0, \theta) = (d, \alpha)$ where $d \in A$. Now $(d, \alpha) \geqq (0, \theta)$ implies $d \in Q_\alpha$ and $(d, \alpha) \geqq (b, \gamma)$ implies $(0, \theta) \leqq (d, \alpha) - (b, \gamma) = [d - b - f(\gamma, \beta), \beta]$ so $d - b - f(\gamma, \beta) \in Q_\beta$. Hence, $d \in Q_\alpha \cap [Q_\beta + b + f(\alpha - \beta, \beta)]$. If $c \in Q_\alpha \cap [Q_\beta + b + f(\alpha - \beta, \beta)]$ then a similar argument shows $(c, \alpha) \geqq (b, \gamma)$ and $(c, \alpha) \geqq (0, \theta)$. Hence, $(c, \alpha) \geqq (d, \alpha)$ and $c \geqq d$. Therefore, d is the smallest element in $Q_\alpha \cap [Q_\beta + b + f(\alpha - \beta, \beta)]$ and (1) holds.

To show (2) let $\alpha, \beta \in \Delta^+$. If either $\alpha = \theta$ or $\beta = \theta$ then (2) is trivial, so suppose $\alpha > \theta$ and $\beta > \theta$. Since G is a p.o.-extension we have $Q_\alpha + Q_\beta + f(\alpha, \beta) \subseteqq Q_{\alpha+\beta}$. For the reverse containment, let $x \in Q_{\alpha+\beta}$, $y \in Q_\alpha$, $b = x - y - f(\alpha, \beta)$ and $(a, \beta) = (b, \beta) \vee (0, \theta)$. Now $(c, \alpha + \beta) \geqq (0, \theta)$ if and only if $c \in Q_{\alpha+\beta}$; $(c, \alpha + \beta) \geqq (b, \beta)$ if and only if $c \in Q_\alpha + b + f(\alpha, \beta)$. On the other hand, since $(a, \beta) = (b, \beta) \vee (0, \theta)$, $c \in Q_{\alpha+\beta} \cap [Q_\alpha + b + f(\alpha, \beta)]$ if and only if $c \in Q_\alpha + a + f(\alpha, \beta)$. Hence $Q_{\alpha+\beta} \cap [Q_\alpha + b + f(\alpha, \beta)] = Q_\alpha + a + f(\alpha, \beta)$ and by (1) a is the smallest element in $Q_\beta \cap (Q_\theta + b)$. Therefore,

$$\begin{aligned}&[Q_\alpha + b + f(\alpha, \beta)] \cap Q_{\alpha+\beta} \\ &\qquad = Q_\alpha + f(\alpha, \beta) + [Q_\beta \cap (Q_\theta + b)] \subseteqq Q_\alpha + f(\alpha, \beta) + Q_\beta .\end{aligned}$$

By the choice of b, $x \in [Q_\alpha + b + f(\alpha, \beta)] \cap Q_{\alpha+\beta}$ and $Q_\alpha + Q_\beta + f(\alpha, \beta) = Q_{\alpha+\beta}$.

For the sufficiency assume (1) and (2) hold and suppose $(b, \beta) \in G$ and that (b, β) is not comparable with $(0, \theta)$. Let c be the smallest element in $Q_{\beta\vee\theta} \cap [Q_{-(\beta\wedge\theta)} + b + f(\beta, -(\beta \wedge \theta))]$. Then $(c, \beta \vee \theta) \geqq (0, \theta)$ and (b, β). If $(a, \alpha) \geqq (b, \beta), (0, \theta)$ then $a \in Q_\alpha \cap [Q_{\alpha-\beta} + b + f(\alpha - \beta, \beta)]$. Condition (1) implies (*) so $Q_{\alpha-(\beta\vee\theta)}$ is a sublattice of A and from (2) we can derive the equality,

$$\begin{aligned}Q_\alpha \cap [Q_{\alpha-\beta} + b + f(\alpha - \beta, \beta)] &= [Q_{\alpha-(\beta\vee\theta)} + f(\alpha - (\beta \vee \theta), \beta \vee \theta)] \\ &\quad + \{Q_{\beta\vee\theta} \cap [Q_{-(\beta\wedge\theta)} + b + f(\beta, -(\beta \wedge \theta))]\} .\end{aligned}$$

Since c was chosen as the smallest element we have $a \in Q_{\alpha-(\beta\vee\theta)} + f(\alpha - (\beta \vee \theta), \beta \vee \theta) + c$ and therefore $(a, \alpha) \geqq (c, \beta \vee \theta)$. Hence, $(c, \beta \vee \theta) = (b, \beta) \vee (0, \theta)$ and G is an l-extension of A by Δ. It can be shown that conditions (1) and (2) are equivalent to those given by L. Fuchs [5]. The entire proof was given so that this paper will be more self-contained.

An l-group G is a *cardinal sum* of l-ideals $A_1, A_2, \cdots, A_n$ (notation: $G = A_1 \boxplus \cdots \boxplus A_n$) if G is the direct sum (notation: $G = A_1 \oplus A_2 \oplus \cdots \oplus A_n$) of the A_i and if for $a_i \in A_i$, $a_1 + \cdots + a_n \geqq 0$ if and only if $a_i \geqq 0$ for $i = 1, \cdots, n$. It can be shown that a direct sum of l-ideals of an l-group is actually the cardinal sum. G is a *lexico-extension* of an l-group A (notation: $G = \langle A \rangle$) if A is an l-ideal of G, G/A is an o-group, and every positive element in G but not in A exceeds every element in A. In this case we note that if $a + A < b + A$ in G/A then each element of $b + A$ exceeds every element of $a + A$.

LEMMA 2.2. *Suppose G is an l-extension of A by Δ.*

(a) *If $Q_\alpha = A$ for all $\theta \neq \alpha \in \Delta^+$ then $G = \langle A \rangle$.*

(b) *If Q_α is a principal dual ideal for each $\alpha \in \Delta^+$ then G is o-equivalent to the cardinal sum, $A \boxplus \Delta$, of A and Δ.*

Proof. Let G be an l-extension of A by Δ.

(a) If $Q_\alpha = A$ for all $\theta \neq \alpha \in \Delta^+$, then every positive element of $G \backslash A$ exceeds every element of A. From (1) it follows that Δ is an o-group and therefore $G = \langle A \rangle$.

(b) If Q_α is a principal dual ideal for each $\alpha \in \Delta^+$, let x_α be the generator of Q_α. By (2) we have $x_\alpha + x_\beta + f(\alpha, \beta) = x_{\alpha+\beta}$. Let $H = A \boxplus \Delta$, then $H = (A, \Delta, f' \equiv 0, Q' \equiv A^+)$ is an l-extension of A by Δ. Define $t' \colon \Delta^+ \to A$ as $t'(\alpha) = x_\alpha$. Then t' induces a function $t \colon \Delta \to A$ and it follows that for $\alpha, \beta \in \Delta$

$$0 = f'(\alpha, \beta) = f(\alpha, \beta) - t(\alpha + \beta) + t(\alpha) + t(\beta)$$

and

$$A^+ = Q'_\alpha = -t(\alpha) + Q_\alpha \quad \text{for } \alpha \in \Delta^+ .$$

Hence G and H are o-equivalent l-extensions.

LEMMA 2.3. *Let $A = \langle B \rangle, A \neq B$ and $G = (A, \Delta, f, Q)$ be an l-extension. Then for $\alpha \in \Delta^+$ either $Q_\alpha = A$ or Q_α is a principal dual ideal.*

Proof. If A is an o-group, $\alpha \in \Delta^+$ and $Q_\alpha \neq A$ then there is $b \in A$ such that $b < a$ for all $a \in Q_\alpha$. Hence, $(b, \alpha) \vee (0, \theta) = (c, \alpha)$ implies c is the smallest element in Q_α and therefore Q_α is a principal dual ideal.

If A is not an o-group then $B \subset A$ and A/B is an o-group. Suppose $\alpha \in \Delta^+$ and $Q_\alpha \neq A$, then there is $0 > b \in A \backslash B$ such that $b + B \neq x + B$ for all $x \in Q_\alpha$. For suppose for each $0 > b \in A \backslash B$ there is an $x \in Q_\alpha$ such that $b + B = x + B$, then $b + h \in Q_\alpha$ for some $h \in B$. Now for any $c \in A$ there is $0 > a \in A\backslash B$ such that $a + B < c + B$ so $c > a + h$ which implies $c \in Q_\alpha$. Thus $Q_\alpha = A$, a contradiction.

Now $Q_\alpha \cap (b + Q_\theta)$ must have a smallest element so it suffices to show $Q_\alpha \subseteqq b + Q_\theta$. To this end let $x \in Q_\alpha$. If $x + B \leqq b + B$ then either $x + B < b + B$ which implies $x < b$ and $b \in Q_\alpha$ or $x + B = b + B$. Both cases lead to contradictions so $x + B > b + B$ which implies $x > b$ and $x \in b + Q_\theta$. The proof is complete.

COROLLARY 2.1. *If* $A = \langle B \rangle$ *then* (1) *may be replaced by*

(1′) *If* $\alpha, \beta \in \Delta^+$ *and* $\alpha \wedge \beta = \theta$ *then either* Q_α *and* Q_β *are principal dual ideals or* Q_α *is principal and* $Q_\beta = A$.

Proof. If G is an l-extension and $\alpha, \beta \in \Delta^+$ such that $\alpha \wedge \beta = \theta$ then (1) implies $Q_\alpha \cap Q_\beta$ must have a smallest element and (1′) follows from Lemma 2.3. Conversely, if x is the smallest element in Q_α, y the smallest in Q_β and $b \in A$ then $x \vee (y + b + f(\alpha - \beta, \beta)$ is the smallest in $Q_\alpha \cap [Q_\beta + b + f(\alpha - \beta, \beta)]$. If $Q_\beta = A$ then x is the smallest and if $Q_\alpha = A$, $y + b + f(\alpha - \beta, \beta)$ is the smallest.

From the above it follows that if $A = \langle B \rangle$ and Δ is an o-group then (1) may be replaced by

(1″) For each $\alpha \in \Delta^+$, $Q_\alpha = A$ or Q_α is a principal dual ideal.

From (2) of Theorem 1 we have: The only l-extensions of $A = \langle B \rangle$ by an Archimedean o-group Δ are o-isomorphic to the cardinal extension or the lexico-extension.

THEOREM 2. *Let* $A = \langle B \rangle$ *and* Δ *be l-groups and* $G = (A, \Delta, f, Q)$ *be an l-extension. Then* G *contains an l-ideal* H *which is o-isomorphic to* $A \boxplus J$, $J \subseteqq \Delta$, *and* G *is an l-extension of* H *by the o-group* Δ/J.

Proof. By Lemma 2.3 either $Q_\alpha = A$ or Q_α is principal for all $\alpha \in \Delta^+$. Let $J^+ = \{\alpha \in \Delta^+ \mid Q_\alpha \neq A\}$. Then by (2) of Theorem 1, J^+ is a convex subsemigroup of Δ^+. Let J be the l-ideal of Δ generated by J^+ and let $H = (A, J, f', Q')$ where $f' = f \mid (J \times J)$ and $Q'_\alpha = Q_\alpha$, $\alpha \in J^+$. Then H is an l-ideal of G and Q'_α is a principal dual ideal for all $\alpha \in J^+$. Therefore by Lemma 2.2, we have H o-isomorphic to $A \boxplus J$.

By way of contradiction, if Δ/J is not an o-group then there are $X, Y \in (\Delta/J)^+$ such that $X \wedge Y = J$. Let $X = \alpha + J$, $Y = \beta + J$ then $X \wedge Y = (\alpha + J) \wedge (\beta + J) = (\alpha \wedge \beta) + J = J$ so $\alpha \wedge \beta \in J$. Now $\alpha = (\alpha \wedge \beta) + \gamma$, $\beta = (\alpha \wedge \beta) + \delta$ where $\gamma \wedge \delta = \theta$ and $\gamma, \delta \notin J$, hence $Q_\gamma = A = Q_\delta$. This contradicts Corollary 2.1. Thus Δ/J is an o-group.

Finally, the natural mappings induce an o-isomorphism of G/H onto Δ/J. Hence, G is an l-extension of H by the o-group Δ/J.

We note that if $\alpha \in \Delta^+ \backslash J^+$ then $Q_\alpha = A$ so if $0 < g \in G \backslash H$ then $g > a$ for all $a \in A$.

COROLLARY 2.2. *If Δ is an o-group and $G = (A, \Delta, f, Q)$ is an l-extension then $G = \langle A \boxplus J \rangle$.*

Proof. If Δ is an o-group then $\Delta = \langle J \rangle$. The corollary follows from the results of Conrad [3, p 235] since $A \boxplus J$ contains all the nonunits of G.

We note that if G is an l-group with two disjoint elements but not three then G is an l-extension of an o-group by an o-group and hence we have the structure theorem of Conrad and Clifford [4] for the abelian case.

3. l-extensions with each Q_α generated by a coset of an l-ideal. Throughout this section we will consider those l-extensions $G = (A, \Delta, f, Q)$ where, for each $\alpha \in \Delta^+$, $Q_\alpha = DI(x_\alpha + H_\alpha)$, H_α an l-ideal of A.

LEMMA 3.1. *Suppose $G = (A, \Delta, f, Q)$ is an l-extension of the above type. Then there is an l-extension $G' = (A, \Delta, f', Q')$ o-equivalent to G with $Q'_\alpha = DI(H_\alpha)$ for each $\alpha \in \Delta^+$.*

Proof. If G is an l-extension and $Q_\alpha = DI(x_\alpha + H_\alpha)$ for each $\alpha \in \Delta^+$, then there is a mapping $t: \Delta^+ \to A$ defined as $t'(\alpha) = x_\alpha$. Since each $\alpha \in \Delta$ has a unique representation $\alpha = \alpha^+ - \alpha^-$ where $\alpha^+ = \alpha \vee \theta$, $\alpha^- = -(\alpha \wedge \theta)$, we can extend t' to a mapping $t: \Delta \to A$ by defining $t(\alpha) = t'(\alpha^+) - t'(\alpha^-)$.

Let $f'(\alpha, \beta) = f(\alpha, \beta) - t(\alpha + \beta) + t(\alpha) + t(\beta)$ and $Q'_\alpha = -t(\alpha) + Q_\alpha$. It is easily verified that f' and Q' satisfy conditions (i)–(vi) so $G' = (A, \Delta, f', Q')$ is a p.o. extension of A by Δ. From Theorem 1 it follows that G' is an l-extension. Clearly, G' is o-equivalent to G and $Q' = DI(H_\alpha)$.

For those l-extensions G of A by Δ with Q_α as above the question of o-equivalence leads to an investigation of the l-ideals of A. To show this we need the following.

LEMMA 3.2. *If A is an l-group, H and K l-ideals of A and $DI(y + H) = DI(z + K)$ then $y + H = z + K$ and $H = K$.*

Proof. Suppose $DI(y + H) = DI(z + K)$ where H and K are l-ideals of A. If $x = z - y$ then $DI(H) = DI(x + K)$. Since $H \subseteqq DI(x + K)$, $0 \in DI(x + K)$. If $0 \notin x + K$ then $0 > x + k, k \in K$ so $x + K$ contains a negative element. Since $DI(H)$ is a semigroup, $2(x + k) \in DI(x + K)$

so $2x + 2k \geqq x + 1$, $1 \in K$. Hence, $x + (2k - 1) \geqq 0$. This is a contradiction since $x + K$ can contain no positive elements. Thus $0 \in x + K$ and $x \in K$. Moreover, we have $DI(H) = DI(K)$ which implies $H = K$. For if $H \neq K$ then, without loss of generality, there is $0 > h \in H \backslash K$. But $h \in DI(K)$ so $h > k \in K$. Hence, $0 > h > k$, and by convexity $h \in K$, a contradiction. Thus, $H = x + K = z - y + K$ and $y + H = z + K$.

Now if $G = (A, \Delta, f, Q)$ and $G' = (A, \Delta, f', Q')$ are two l-extensions with Q_α and Q'_α generated by l-ideals H_α and H'_α of A, then G and G' are o-equivalent if and only if there is a function $t: \Delta \rightarrow A$ such that

$$f'(\alpha, \beta) = f(\alpha, \beta) - t(\alpha + \beta) + t(\alpha) + t(\beta)$$
$$H'_\alpha = H_\alpha \text{ and } t(\alpha) \in H'_\alpha \, .$$

The question at this point is which l-extensions will have Q_α generated by a coset of an l-ideal. We give a partial answer to this question in the next section.

We complete this section by giving a method for the construction of l-extensions of l-groups.

THEOREM 3. *Suppose A and Δ are l-groups and $G = (A, \Delta, f)$ is an abelian extension of A by Δ. For each $\alpha \in \Delta^+$, let H_α be a cardinal summand of A such that*

(1*) *if $\alpha \wedge \beta = \theta$ then $H_\alpha \cap H_\beta = 0$*

(2*) *$H_\alpha + H_\beta = H_{\alpha+\beta}$ and $f(\alpha, \beta) \in H_{\alpha+\beta}$.*

If $Q_\alpha = DI(H_\alpha)$ then $G = (A, \Delta, f, Q)$ is an l-extension of A by Δ.

Proof. Clearly (iv) is satisfied and for any $\alpha \in \Delta^+$, (2*) implies $H_\theta \subseteqq H_\alpha$. From (1*) it follows that $H_\theta = 0$. Thus $Q_\theta = A^+$ and (vi) is satisfied. Moreover, from (2*) we have $DI(H_\alpha + H_\beta + f(\alpha, \beta)) = DI(H_{\alpha+\beta})$ so $DI(H_\alpha) + DI(H_\beta) + f(\alpha, \beta) = DI(H_{\alpha+\beta})$ and (2) of Theorem 1 holds.

If $\alpha \wedge \beta = \theta$ then $H_\alpha \cap H_\beta = 0$ so $H_{\alpha+\beta} = H_\alpha \oplus H_\beta$ and since H_α and H_β are l-ideals we have $H_{\alpha+\beta} = H_\alpha \boxplus H_\beta$. Since $H_{\alpha+\beta}$ is a cardinal summand we conclude $A = H_{\alpha+\beta} \boxplus D = H_\alpha \boxplus H_\beta \boxplus D$ where D is an l-ideal of A. Suppose $b \in A$ and $b + f(\alpha - \beta, \beta) = (a_1, a_2, a_3)$ where $a_1 \in H_\alpha$, $a_2 \in H_\beta$ and $a_3 \in D$. We show $(a_1, 0, a_3 \vee 0)$ is the smallest element in

$$Q_\alpha \cap (b + f(\alpha - \beta, \beta) + Q_\beta) = DI(H_\alpha) \cap DI(b + f(\alpha - \beta, \beta) + H_\beta) \, .$$

Now $(a_1, 0, a_3 \vee 0) \geqq (a_1, 0, 0)$ so $(a_1, 0, a_3 \vee 0) \in DI(H_\alpha)$. Also $(a_1, 0, a_3) = (a_1, 0, a_3) = (a_1, a_2, a_3) - (0, a_2, 0)$ so $(a_1, 0, a_3) \in b + f(\alpha - \beta, \beta) + H_\beta$ and $(a_1, 0, a_3 \vee 0) \in DI(b + f(\alpha - \beta, \beta) + H_\beta)$. If

$$(u, v, w) \in DI(H_\alpha) \cap DI(b + f(\alpha - \beta, \beta) + H_\beta)$$

then $u \geqq h_\alpha \in H_\alpha$, $v \geqq 0$ and $w \geqq 0$. Also $u \geqq a_1$, $v \geqq a_2 + h_\beta$ where

$h_\beta \in H_\beta$ and $w \geqq a_3$. Hence, $(u, v, w) \geqq (a_1, 0, a_3 \vee 0)$ and $(a_1, 0, a_3 \vee 0)$ is the smallest element in $Q_\alpha \cap (b + f(\alpha - \beta, \beta) + Q_\beta)$. Thus G is an l-extension of A by Δ.

We note that, since any two representations of an l-group as a cardinal sum have a common refinement, the cardinal summands of an l-group form an additive semigroup closed with respect to intersection. That is, if $H = A \boxplus A'$ and $H = B \boxplus B'$ then $A = (A \cap B) \boxplus (A \cap B')$, $A' = (A' \cap B) \boxplus (A' \cap B')$ and $B = (A \cap B) \boxplus (A' \cap B)$. Thus $H = A \boxplus A' = (A + B) \boxplus (A' \cap B')$. Hence, $A + B$ is a cardinal summand of G.

4. Extensions of l-groups with a finite basis.

An element g of an l-group G is *basic* if $0 < g$ and $\{x \in G \mid 0 < x \leqq g\}$ is ordered. A subset S of G is a *basis* for G if S is a maximum set of disjoint elements and each $g \in S$ is basic. Conrad [2] has shown that an l-group A with a finite basis of n elements is a lexico-sum of n ordered subgroups. In particular, A is the cardinal sum of two l-groups each with a basis of fewer than n elements, or A is a lexico-extension of such an l-group. In this section we are concerned with l-extensions of l-groups with finite bases.

LEMMA 4.1. *Suppose A has a finite basis and $G = (A, \Delta, f, Q)$ is an l-extension of A. Then for $\alpha \in \Delta^+$, $Q_\alpha = DI(x_\alpha + H_\alpha)$ where H_α is an l-ideal of A.*

Proof. Let A have a basis of n elements. The proof is by induction on n.

It follows from Lemma 2.3 that we need only consider $A = B \boxplus C$ and if $n = 1$ then $H_\alpha = A$ or $H_\alpha = 0$.

So suppose the theorem is true for all l-groups with a basis of fewer than n elements. Let $\varphi: A \rightarrow B$ and $\psi: A \rightarrow C$ be the projections. Now B has a basis of fewer than n elements and $G' = (B, \Delta, \varphi f, \varphi Q)$ is an l-extension of B so by induction $\varphi Q_\alpha = DI(x + M)$ where $x \in B$ and M is an l-ideal of B. Similarly, $\psi Q_\alpha = DI(y + N)$ where $y \in C$ and N is an l-ideal of C. Since Q_α is a sublattice of A, a straight forward argument shows $Q_\alpha = DI((x + y) + (M + N))$ and $M + N$ is an l-ideal of A. The proof is complete.

The following theorem shows that for an l-group A with a finite basis every l-extension G of A by an l-group Δ is o-equivalent to an l-extension constructed by the method described in Theorem 3. That is, to an o-equivalence, every such l-extension is determined by a meet-preserving homomorphism from the semigroup Δ^+ to the semigroup of all cardinal summands of A such that $f(\alpha, \beta) \in H_{\alpha+\beta}$.

In what follows we may, by Lemmas 3.1 and 4.1, assume for each $\alpha \in \Delta^+$ that $Q_\alpha = DI(H_\alpha)$.

THEOREM 4. *If A has a finite basis and $G = (A, \Delta, f, Q)$ is an l-extension of A by an l-group Δ then, for $\alpha, \beta \in \Delta^+$*

(a) *if $\alpha \wedge \beta = \theta$ then $H_\alpha \cap H_\beta = 0$*

(b) *$H_\alpha + H_\beta = H_{\alpha+\beta}$ and $f(\alpha, \beta) \in H_{\alpha+\beta}$*

(c) *H_α is a cardinal summand of A.*

Proof. Let A have a finite basis of n elements and G be an l-extension. By (1) if $\alpha \wedge \beta = \theta$ then $Q_\alpha \cap Q_\beta$ must have a smallest element w. Since $0 \in Q_\alpha \cap Q_\beta$, $w \leqq 0$ and therefore $w \in H_\alpha \cap H_\beta$. If $H_\alpha \cap H_\beta \neq 0$ then there is $h \in H_\alpha \cap H_\beta$ such that $h < w$ and $h \in Q_\alpha \cap Q_\beta$, a contradiction. Thus (a) holds.

From (2) we have

$$DI(H_\alpha) + DI(H_\beta) + f(\alpha, \beta) = DI(H_{\alpha+\beta})$$

so

$$DI(H_\alpha + H_\beta + f(\alpha, \beta)) = DI(H_{\alpha+\beta}) .$$

Thus by Lemma 2.3, $H_\alpha + H_\beta = H_{\alpha+\beta}$ and $f(\alpha, \beta) \in H_{\alpha+\beta}$ and (b) holds.

Now if $A = \langle B \rangle$ then for each $\alpha \in \Delta^+$, $H_\alpha = 0$ or $H_\alpha = A$ and (c) follows in a trivial way. So suppose $A = B \boxplus C$ and (c) is true for all l-groups with a basis of fewer then n elements. If $\varphi: A \to B$ and $\psi: A \to C$ are the projections then $G' = (B, \Delta, \varphi f, \varphi Q)$ and $G'' = (C, \Delta, \psi f, \psi Q)$ are l-extensions where $\varphi Q_\alpha = DI(\varphi H_\alpha)$ and $\psi Q_\alpha = DI(\psi H_\alpha)$. Hence, by induction, φH_α is a cardinal summand of B and ψH_α is a cardinal summand of C and we have $A = B \boxplus C = \varphi H_\alpha \boxplus M \boxplus \psi H_\alpha \boxplus N = \varphi H_\alpha \boxplus \psi H_\alpha \boxplus M \boxplus N = H_\alpha \boxplus M \boxplus N$ where M is an l-ideal of B and N is an l-ideal of C.

Using the results of Conrad [3, p. 223] we conclude that the minimal cardinal summands of an l-group A with a finite basis are those l-ideals of A that are lexico-extensions and are not bounded in A.

Added in Proof. The results of this paper have been extended by the author to include central extensions G of an abelian l-group A by an arbitrary l-group Δ. For central extensions, Theorem 1 (1) reads: if $\alpha \wedge \beta = \theta$ then $Q_\alpha \cap [Q_\beta + b + f(\beta, \alpha - \beta)]$ has a smallest element for all $b \in A$. In Theorem 2, G/H is still o-isomorphic to the o-group Δ/J but G need not be a central extension of H by Δ/J. The remaining results are unchanged for central extensions.

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