SUPERADDITIVITY INEQUALITIES

E. F. BECKENBACH

1. Introduction. In the theory of analytic inequalities, a principal tool is the notion of convex function [6, 1]. A hierarchy of convexity conditions, useful in this theory, can be expressed as follows: Let $K^{\alpha}(a, b)$ denote the class of functions p that are positive and continuous on an interval $a \leqq x \leqq b$ and such that sign (x) $[p(x)]^{\alpha}$ is convex on $[a, b]$ if $\alpha \neq 0$, and $\log p(x)$ is convex on $[a, b]$ if $\alpha = 0$; then for all real α and β with $\beta > \alpha$ we have $K^{\alpha}(a, b) \subset K^{\beta}(a, b)$ [8].

A different sort of hierarchy has been established by Bruckner and Ostrow [3]. In the present paper we are concerned with an illustration and some applications of this latter hierarchy. To describe it, we need a few definitions.

2. Definitions. Let $K(b)$ be the class of real-valued functions f that are continuous and nonnegative on a given closed interval $0 \leqq x \leqq b$ and vanish at the origin, $f(0) = 0$.

The *average function* F of a function $f \in K(b)$ is the function $F \in K(b)$ defined by

$$F(x) \equiv \frac{1}{x}\int_0^x f(t)dt\ , \qquad 0 < x \leqq b\ ,$$
$$F(0) = 0\ .$$

The function $f \in K(b)$, with average function F, will be said to be of *class*

$K_1(b)$ if and only if f is *convex* on $[0, b]$, i.e., if and only if for every x and $y \in [0, b]$, and for every α, $0 \leqq \alpha \leqq 1$, we have

$$f[\alpha x + (1-\alpha)y] \leqq \alpha f(x) + (1-\alpha)f(y)\ ; \tag{1}$$

$K_2(b)$ if and only if $F \in K_1(b)$;

$K_3(b)$ if and only if f is *starshaped* (with respect to the origin) on $[0, b]$, i.e., if and only if for every $x \in [0, b]$, and for every α, $0 \leqq \alpha \leqq 1$, we have

$$f(\alpha x) \leqq \alpha f(x)\ ; \tag{2}$$

$K_4(b)$ if and only if f is *superadditive* on $[0, b]$, i.e., if and only if for every x and $y \in [0, b]$ such that also $(x + y) \in [0, b]$ we have

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$$(3) \qquad f(x+y) \geqq f(x) + f(y) \; ;$$

$K_5(b)$ if and only if $F \in K_3(b)$;
$K_6(b)$ if and only if $F \in K_4(b)$.

If $f \in K_2(b)$, $K_5(b)$, or $K_6(b)$, then f is said [3] to be, respectively, convex, starshaped, or superadditive *on the average* on $[0, b]$.

If $f \in K_i(b_0)$, then clearly $f \in K_i(b)$ for all positive $b < b_0$.

3. The hierarchy. The following class-inclusion implications have been established by Bruckner and Ostrow [3]:

$$K_1(b) \subset K_2(b) \subset K_3(b) \subset K_4(b) \subset K_5(b) \subset K_6(b) \; .$$

They have further given examples to show that none of the reverse implications are valid; i.e., they have given examples showing that

$$(4) \quad K_6(b) \not\subset K_5(b),\ K_5(b) \not\subset K_4(b),\ K_4(b) \not\subset K_3(b),\ K_3(b) \not\subset K_2(b),\ K_2(b) \not\subset K_1(b) \; .$$

Thus they have pointed out that the function f defined on $[0, 1]$ by

$$f(x) \equiv x^2 - x^3$$

is convex on the average on $[0, 4/9]$ but convex only on $[0, 1/3]$, that the function g defined on $[0, \infty)$ by

$$g(x) \equiv \begin{cases} x^2, & 0 \leqq x \leqq 1 \; , \\ x, & x > 1 \; , \end{cases}$$

is starshaped on $[0, b]$ for an arbitrarily large value of b but convex on the average only on $[0, 1]$, that the function h defined on $[0, \infty)$ by

$$h(x) \equiv n + (x-n)^2 \; , \qquad n \leqq x < n+1 \; , \qquad n = 0, 1, 2, \cdots \; ,$$

is superadditive on $[0, b]$ for an arbitrarily larger value of b but starshaped only on $[0, 1]$, etc.

It is our purpose first to use a *single* illustrative function f and its average function F to establish the fact that none of the foregoing reverse implications hold, and secondly to derive some general inequalities for convex, starshaped, and superadditive functions and to apply them to our particular illustrative functions.

4. Example. From

$$F(x) = \frac{1}{x}\int_0^x f(t)dt \; , \qquad 0 < x \leqq b \; ,$$

we obtain

$$F'(x) = \frac{1}{x}f(x) - \frac{1}{x^2}\int_0^x f(t)dt = \frac{1}{x}[f(x) - F(x)] ,$$

whence

$$f(x) = F(x) + xF'(x) . \tag{5}$$

We might call f the *inverse average function* of F.

Let us consider the function F defined on $[0, \infty)$ by

$$\begin{aligned} &F(x) \equiv e^{-1/x} , \qquad 0 < x < \infty , \\ &F(0) = 0 . \end{aligned} \tag{6}$$

Then (5) gives

$$\begin{aligned} &f(x) \equiv \left(1 + \frac{1}{x}\right)e^{-1/x} , \qquad 0 < x < \infty , \\ &f(0) = 0 . \end{aligned} \tag{7}$$

In the following Sections 5–9 we establish the *maximum* values b_i such that the function f defined by (7) is of class $K_i(b_i)$, $i = 1, 2, \cdots, 6$.

5. **Convexity.** A function f of class C'' is convex on an interval if and only if we have $f''(x) \geqq 0$ throughout the interval.

For the function f given by (7), a computation yields

$$f''(x) = \frac{1}{x^5}(1 - 3x)e^{-1/x} .$$

Accordingly, $f \in K_1(b)$ for

$$b = b_1 = \frac{1}{3} , \tag{8}$$

but for no larger value of b. The function is concave on the interval $[1/3, \infty)$.

Similarly, for the function F given by (6), we have

$$F''(x) = \frac{1}{x^4}(1 - 2x)e^{-1/x} .$$

Thus the maximum interval of convexity of F is $[0, 1/2]$, and F is concave on the interval $[1/2, \infty)$. Therefore $f \in K_2(b)$ for

$$b = b_2 = \frac{1}{2} , \tag{9}$$

but for no larger value of b.

A function that is convex on a left-hand portion of its interval of definition, and concave on the complementary right-hand portion, is said to be *convexo-concave* [1]. Thus both the function f given by (7) and the function F given by (6) are convexo-concave on the interval $[0, \infty)$.

6. **Starshapedness.** A function f of class C', $f \in K(b)$, is starshaped on the interval $[0, b]$, i.e., $f \in K_3(b)$, if and only if [3]

$$f'(x) \geqq \frac{f(x)}{x} \text{ for all } x \in (0, b] .$$

For the function f given by (7), we have

$$f'(x) - \frac{f(x)}{x} = \frac{1}{x^3}(1 - x - x^2)e^{-1/x} ,$$

whence it follows that $f \in K_3(b)$ for

$$b = b_3 = \frac{\sqrt{5} - 1}{2} , \tag{10}$$

but for no larger value of b.

Similarly, for the function F given by (6) we obtain

$$F'(x) - \frac{F(x)}{x} = \frac{1}{x^2}(1 - x)e^{-1/x} ,$$

so that $f \in K_5(b)$ for

$$b = b_5 = 1 , \tag{11}$$

but for no larger value of b.

Thus it happens that the maximum interval of starshapedness of the function f forms a golden section [7] of the maximum interval of starshapedness of the function F.

7. **Superadditivity.** Tests for superadditivity appear to be difficult to establish, and more difficult to apply. None are given, for example, in the treatments [5] and [9] of superadditive functions. A few tests, however, have been advanced by Bruckner [2]; see also § 14, below. One of Bruckner's tests, which we shall use in order somewhat to shorten our determination of the maximum interval of superadditivity of the function f given by (7), and of the function F given by (6), can be stated as follows:

BRUCKNER'S TEST. *Let the function $f \in K(b)$ be convexo-concave.*

Then f is superadditive on $[0, b]$, i.e., $f \in K_4(b)$, if and only if

$$f\left(\frac{b}{2} + x\right) + f\left(\frac{b}{2} - x\right) \leqq f(b) \text{ for all } x \in \left[0, \frac{b}{2}\right].$$

In §§ 8 and 9, below, we shall prove the following results:

THEOREM 1. *The function f, defined by*

$$f(x) \equiv \left(1 + \frac{1}{x}\right)e^{-1/x}, \quad 0 < x < \infty,$$
$$f(0) = 0,$$

is superadditive on $[0, b]$ for

$$0 < b \leqq b^*,$$

where b^* is the unique positive solution of the transcendental equation

$$b = \frac{1 - 4e^{-1/b}}{2e^{-1/b} - 1}$$

(approx. $b^* = 0.8955$), but for no larger value of b.

That is, the function $f \in K_4(b)$ for

$$b = b_4 = b^* \doteq 0.8955, \tag{12}$$

but for no larger value of b.

THEOREM 2. *The function F, defined by*

$$F(x) \equiv e^{-1/x}, \quad 0 < x < \infty,$$
$$F(0) = 0,$$

is superadditive on $[0, b]$ for

$$0 < b \leqq \frac{1}{\log 2},$$

but for no larger value of b.

That is, the function $f \in K_6(b)$ for

$$b = b_6 = \frac{1}{\log 2}, \tag{13}$$

but for no larger value of b.

8. **Proof of Theorem 2.** The method of proof we shall use is largely the same for both theorems. Since the formulas are simpler and the details shorter for Theorem 2, we shall treat it first and

then follow substantially the same pattern for Theorem 1.

Relative to the function F given by (6), consider the function G defined for $b \in (0, \infty)$ and $x \in [0, b/2]$ by

$$(14) \qquad \begin{aligned} &G(x; b) \equiv e^{-1/(b/2+x)} + e^{-1/(b/2-x)} - e^{-1/b}, \quad x \neq \frac{b}{2}, \\ &G\left(\frac{b}{2}; b\right) = 0 . \end{aligned}$$

In accordance with Bruckner's test, we shall establish the maximum interval $[0, b]$ of superadditivity of the function F by determining the maximum value b such that

$$G(x; b) \leqq 0 \text{ for all } x \in \left[0, \frac{b}{2}\right].$$

In particular, for F to be superadditive on $[0, b]$, it is necessary that we have

$$(15) \qquad G(0; b) \equiv 2e^{-2/b} - e^{-1/b} \leqq 0 ,$$

or

$$\log 2 - \frac{2}{b} \leqq -\frac{1}{b} ,$$

whence

$$b \leqq b_6 = \frac{1}{\log 2} .$$

Hence the function F is not superadditive on $[0, b]$ for any $b > b_6$. We shall show, however, that F is superadditive on $[0, b_6]$ (and therefore, of course, on $[0, b]$ for every positive $b < b_6$). That is, we show that

$$(16) \qquad G(x; b_6) \leqq 0 \text{ for all } x \in \left[0, \frac{b_6}{2}\right].$$

By (14), we have

$$(17) \qquad G\left(\frac{b_6}{2}; b_6\right) = 0 ,$$

and by the choice of b_6 we have also

$$(18) \qquad G(0; b_6) = 2e^{-2/b_6} - e^{-1/b_6} = 0 .$$

We shall prove somewhat more than is needed for what is claimed in Theorem 2; namely, we shall show that we have not merely (16) but actually the *strict* inequality

$$G(x; b_6) < 0 \text{ for all } x \in \left(0, \frac{b_6}{2}\right). \tag{19}$$

In § 11, below, we shall make essential use of the fact that this is a strict inequality.

If (19) did not hold, then, by (17) and (18), $G(x; b_6)$ would attain a nonnegative maximum value at some interior point x_0 of $(0, b_6/2)$. At x_0 we would have

$$\frac{dG(x: b_6)}{dx} \equiv \frac{1}{(b_6/2 + x)^2} e^{-1/(b_6/2+x)} - \frac{1}{(b_6/2 - x)^2} e^{-1/(b_6/2-x)} = 0 ,$$

and therefore we would have

$$G(x_0; b_6) = \Phi(x_0) \geqq 0 , \tag{20}$$

in which the function Φ is defined by

$$\Phi(x) \equiv \left[1 + \left(\frac{b_6/2 - x}{b_6/2 + x}\right)^2\right] e^{-1/(b_6/2+x)} - e^{-1/b_6} , \qquad x \in \left(\frac{b_6}{2}\right).$$

The function Φ is more tractable than the function G, in that it permits us rigorously to establish the transcendental inequality (19) by investigating only a quadratic function. We shall show that we have

$$\Phi(x) < 0 \text{ for all } x \in \left(0, \frac{b_6}{2}\right), \tag{21}$$

thus contradicting (20) and establishing the theorem.

A computation yields

$$\frac{d\Phi}{dx} = \frac{2e^{-1/(b_6/2+x)}}{(b_6/2 + x)^4} Q(x) , \tag{22}$$

where Q is the quadratic polynomial function defined by

$$Q(x) \equiv (1 + \log 2)x^2 - (1 - \log 2)\left(\frac{b_6}{2}\right)^2. \tag{23}$$

Since

$$Q(0) = -(1 - \log 2)\left(\frac{b_6}{2}\right) < 0 ,$$

$$Q\left(\frac{b_6}{2}\right) = 2 \log 2\left(\frac{b_6}{2}\right) > 0 ,$$

and the coefficient of x^2 in (23) is positive, it follows that $Q(x)$ has precisely one zero on $(0, b_6/2)$, actually at

$$x_0 = \frac{b_6}{2}\sqrt{\frac{1-\log 2}{1+\log 2}}\,,$$

being negative on $(0, x_0)$ and positive on $(x_0, b_6/2)$. Accordingly, by (22), $\Phi(x)$ is strictly decreasing on $(0, x_0)$ and strictly increasing on $(x_0, b_6/2)$, whence the desired inequality (21) follows from

$$\Phi(0) = \Phi\left(\frac{b_6}{2}\right) = 0\,.$$

9. Proof of Theorem 1. In place of the function G of § 8, relative to the function F given by (6), we now consider, relative to the function f given by (7), the function g defined for $b \in (0, \infty)$ and $x \in [0, b/2]$ by

$$\begin{aligned} g(x; b) &\equiv \left(1 + \frac{1}{b/2 + x}\right)e^{-(b/2+x)} \\ &\quad + \left(1 + \frac{1}{b/2 - x}\right)e^{-1/(b/2-x)} - \left(1 + \frac{1}{b}\right)e^{-1/b}\,, \quad x \neq \frac{b}{2}\,, \\ g\left(\frac{b}{2}; b\right) &= 0\,. \end{aligned} \tag{24}$$

To prove the theorem, we shall show that the maximum value b such that

$$g(x; b) \leqq 0 \text{ for all } x \in \left[0, \frac{b}{2}\right]$$

is given by (12).

In particular, for f to be superadditive on $[0, b]$, it is necessary that we have

$$\begin{aligned} g(0; b) &\equiv 2\left(1 + \frac{2}{b}\right)e^{-2/b} - \left(1 + \frac{1}{b}\right)e^{-1/b} \\ &\equiv \frac{e^{-1/b}}{b}[b(2e^{-1/b} - 1) - (1 - 4e^{-1/b})] \leqq 0\,. \end{aligned} \tag{25}$$

Now, as we see through differentiation, on $[0, \infty)$ the function α, defined by

$$\begin{aligned} \alpha(b) &\equiv b(2e^{-1/b} - 1)\,, \qquad 0 < b < \infty\,, \\ \alpha(0) &= 0\,, \end{aligned} \tag{26}$$

is convex; $\alpha(b)$ is strictly decreasing from the value 0 at $b = 0$ to a negative value at the root b_0 (approx. $b_0 = 0.60$) of the transcendental equation

$$b = \frac{2}{e^{1/b} - 2} ,$$

which expresses the relation $d\alpha/db = 0$, and then $\alpha(b)$ is strictly increasing on $[b_0, \infty)$.

On the other hand, the function β, defined on $[0, \infty)$ by

$$\beta(b) \equiv 1 - 4e^{-1/b} , \qquad 0 < b < \infty , \tag{27}$$
$$\beta(0) = 1 ,$$

is strictly decreasing on its entire interval of definition.

Since

$$\alpha(b_0) < 0 \text{ and } \beta(b_0) = \frac{1 - b_0}{1 + b_0} > 0 ,$$

it therefore follows from (25), (26), and (27) that the equation

$$g(0; b) = 0$$

has a single root $b \in (0, \infty)$, namely, at the solution

$$b = b_4 = b^* \doteq 0.8955$$

of the transcendental equation

$$\alpha(b) = \beta(b) ,$$

and that further $g(0; b)$ satisfies the inequalities

$$g(0; b) < 0 , \qquad 0 < b < b_4 , \tag{28}$$
$$g(0; b) > 0 , \qquad b > b_4 .$$

By (28), the function f is not superadditive on $[0, b]$ for any $b > b_4$; it remains for us to show that f is superadditive on $[0, b_4]$.

For this, it is sufficient that we establish the inequality

$$g(x; b_4) \leqq 0 \text{ for all } x \in \left[0, \frac{b_4}{2}\right] . \tag{29}$$

By (24), we have

$$g\left(\frac{b_4}{2}; b_4\right) = 0 , \tag{30}$$

and by the choice of b_4 we have also

$$g(0; b_4) = \frac{e^{-1/b_4}}{b_4}[b_4(2e^{-1/b_4} - 1) - (1 - 4e^{-1/b_4})] = 0 . \tag{31}$$

We shall prove that

$$g(x; b_4) < 0 \text{ for all } x \in \left(0, \frac{b_4}{2}\right), \tag{32}$$

thus establishing (29) and with it the validity of the theorem.

If (32) did not hold, then, by (30) and (31), $g(x; b_4)$ would attain a nonnegative maximum value at some interior point x_0 of $(0, b_4/2)$. At x_0 we would have

$$\frac{dg(x; b_4)}{dx} \equiv \frac{1}{(b_4/2 + x)^3} e^{-1/(b_4/2+x)} - \frac{1}{(b_4/2 - x)^3} e^{-1/(b_4/2-x)} = 0,$$

and therefore

$$g(x_0; b_4) = \varphi(x_0) \geqq 0, \tag{33}$$

in which the function φ is defined by

$$\varphi(x) \equiv \left[1 + \frac{1}{b_4/2 + x} + \left(1 + \frac{1}{b_4/2 - x}\right)\left(\frac{b_4/2 - x}{b_4/2 + x}\right)^3\right] e^{-1/(b_4/2+x)}$$
$$- \left(1 + \frac{1}{b_4}\right) e^{-1/b_4}, \qquad x \in \left(0, \frac{b_4}{2}\right).$$

We shall show that we have

$$\varphi(x) < 0 \text{ for all } x \in \left(0, \frac{b_4}{2}\right), \tag{34}$$

thus contradicting (33) and establishing the theorem.

A computation yields

$$\frac{d\varphi}{dx} = \frac{2e^{-1/(b_4/2+x)}}{(b_4/2 + x)^5} q(x), \tag{35}$$

where q is the cubic polynomial function defined by

$$\begin{aligned} q(x) \equiv {} & \left[-3\left(\frac{b_4}{2}\right) - 1\right] x^3 + \left[3\left(\frac{b_4}{2}\right)^2 + 4\left(\frac{b_4}{2}\right) + 1\right] x^2 \\ & + \left[3\left(\frac{b_4}{2}\right)^3 - \left(\frac{b_4}{2}\right)^2\right] x + \left[-3\left(\frac{b_4}{2}\right)^4 - 2\left(\frac{b_4}{2}\right)^3 + \left(\frac{b_4}{2}\right)^2\right]. \end{aligned} \tag{36}$$

Since

$$q(0) = \left[-3\left(\frac{b_4}{2}\right)^2 - 2\left(\frac{b_4}{2}\right) + 1\right]\left(\frac{b_4}{2}\right)^2 \doteq -0.497\left(\frac{b_4}{2}\right)^2 < 0,$$
$$q\left(\frac{b_4}{2}\right) = 2\left(\frac{b_4}{2}\right)^2 > 0,$$

and the coefficient of x^3 in (36) is negative, it follows that $q(x)$ has precisely one zero on $(0, b_4/2)$, say at $x = x_0$. Then $q(x)$ is negative on $(0, x_0)$ and positive on $(x_0, b_4/2)$. Accordingly, by (35), $\varphi(x)$ is strictly decreasing on $(0, x_0)$ and strictly increasing on $(x_0, b_4/2)$, whence the desired inequality (34) follows from

$$\varphi(0) = \varphi\left(\frac{b_4}{2}\right) = 0 .$$

10. The reverse implications. The numbers b_i, $i = 1, 2, \cdots, 6$, as given by (8)-(13), satisfy

$$b_{i-1} < b_i , \qquad i = 2, 3, \cdots, 6,$$

in accordance with the following table of approximations:

i	b_i
1	0.3333
2	0.5000
3	0.6180
4	0.8955
5	1.0000
6	1.4428

Accordingly, since b_i is the maximum of all numbers b such that the function f given by (7) is of class $K_i(b)$, it follows that $f \in K_i(b_i)$ but $f \notin K_{i-1}(b_i)$, whence

$$K_i(b_i) \not\subset K_{i-1}(b_i), \qquad i = 2, 3, \cdots, 6.$$

This establishes (4).

11. The sign of equality. In determining maximum intervals of superadditivity, we have established the following results, except for the specification of the conditions under which the sign of equality holds.

THEOREM 3. *With the notation of Theorem* 1, *we have*

$$f(x + y) \geqq f(x) + f(y) \tag{37}$$

for all nonnegative x and y satisfying

$$x + y \leqq b^* .$$

The sign of equality holds in (37) *if and only if either at most one of x and y is different from* 0 *or else*

$$x = y = \frac{b^*}{2} .$$

THEOREM 4. *With the notation of Theorem* 2, *we have*

$$F(x+y) \geqq F(x) + F(y) \tag{38}$$

for all nonnegative x *and* y *satisfying*

$$x + y \leqq \frac{1}{\log 2} .$$

The sign of equality holds in (38) *if and only if either at most one of* x *and* y *is different from* 0 *or else*

$$x = y = \frac{1}{2\log 2} .$$

Proof. We have only to discuss the conditions under which the sign of equality holds in (37) and (38).

To establish the validity of Bruckner's test, which we have used in the proof of Theorems 1 and 2, we observe that if $f \in K(b)$ is convexo-concave, then the difference

$$[f(x) + f(y)] - f(x+y) \tag{39}$$

is either nonincreasing, or nondecreasing, or first nonincreasing and then nondecreasing, in each of its variables, in the triangular region

$$x \geqq 0 , \qquad y \geqq 0 , \qquad x + y \leqq b ,$$

and hence attains its maximum value either on the line $x + y = b$ or at the origin. For the functions with which we are dealing, however, the above difference is either strictly decreasing, or strictly increasing, or first strictly decreasing and then strictly increasing, in each of its variables, *except* when the other is 0.

Hence, in applying Bruckner's test, the only points we have bypassed at which the sign of equality might hold lie along the axes, and thus the difference attains it maximum value *only* on the triangular boundary.

The boundary consists of the segments $0 \leqq x \leqq b_i$, $0 \leqq y \leqq b_i$, and the portion of the line $x + y = b_i$ in the first quadrant, where $b_i = b_4 = b^*$ and $b_i = b_6 = 1/(\log 2)$ for Theorems 3 and 4, respectively. The difference (39), for the functions of Theorems 3 and 4, vanishes identically on the axes, whereas on the interior of the remaining side, by (18) and (19), and by (31) and (32), it vanishes at the midpoint and otherwise is negative, as specified in the statement of the two theorem.

We note, in passing, that to establish Theorems 2 and 4 without recourse to Bruckner's test, we might adjust the foregoing proofs as follows. For any b', $0 < b' \leqq b_6$, by (14) we have

$$G\left(\frac{b'}{2}; b'\right) = 0 ; \tag{40}$$

further, by (15), we have

$$G(0; b') \leqq 0 , \tag{41}$$

with equality if and only if $b' = b_6$. The proof of (19) can now be extended to give

$$G(x; b') < 0 \text{ for all } x \in \left(0, \frac{b'}{2}\right) . \tag{42}$$

The conclusion of Theorems 2 and 4, including the condition for equality, follows from (40), (41), and (42). Analogous remarks hold for Theorems 1 and 3.

12. Superadditivity inequalities. Let $f \in K_4(b)$. An immediate induction on (3) yields

$$\sum_{i=1}^{n} f(x_i) \leqq f\left(\sum_{i=1}^{n} x_i\right) , \qquad 0 \leqq x_i \leqq b , \qquad \sum_{i=1}^{n} x_i \leqq b . \tag{43}$$

Since, by definition, any function $f \in K_4(b)$ is nonnegative, it follows from (3) that f is nondecreasing. Therefore, by (43), we have

$$\sum_{i=1}^{n} f(x_i) \leqq f(b) , \qquad 0 \leqq x_i \leqq b , \qquad \sum_{i=1}^{n} x_i \leqq b . \tag{44}$$

Thus, for example, for positive numbers x_i, $i = 1, 2, \cdots, n$, $n \geqq 1$, such that

$$\sum_{i=1}^{n} x_i = x_0 \leqq b^* , \tag{45}$$

by Theorem 3 we have

$$\sum_{i=1}^{n}\left(1 + \frac{1}{x_i}\right)e^{-1/x_i} \leqq \left(1 + \frac{1}{x_0}\right)e^{-1/x_0} ,$$

with equality if and only if either (a) $n = 1$, or (b) $n = 2$ and $x_1 = x_2 = b^*/2$.

Also, for positive numbers x_i satisfying (45), we have the weaker inequality

$$\sum_{i=1}^{n}\left(1 + \frac{1}{x_i}\right)e^{-1/x_i} \leqq \left(1 + \frac{1}{b^*}\right)e^{-1/b^*} ,$$

with equality if and only if either (a) $n = 1$ and $x_1 = b^*$, or (b) $n = 2$ and $x_1 = x_2 = b^*/2$.

Similarly, for positive numbers x_i, $i = 1, 2, \cdots, n$, such that

$$\sum_{i=1}^{n} x_i = x_0 \leqq \frac{1}{\log 2},$$

we have

$$\sum_{i=1}^{n} e^{-1/x_i} \leqq e^{-1/x_0} \leqq e^{-\log 2},$$

with analogous conditions for the sign of equality to hold.

13. Whittaker's inequality. If, for any number $a > 1$, in the foregoing discussions we substitute $x/\log a$ for x, then we obtain the following results:

The function f_a, defined by

$$f_a(x) \equiv a^{-1/x}\left(1 + \frac{1}{x}\log a\right), \qquad 0 < x < \infty,$$
$$f_a(0) = 0,$$

is convex on the interval $[0, (1/3)\log a]$, starshaped on the interval $[0, (1/2)(\sqrt{5} - 1)\log a]$, and superadditive on the interval $[0, b^* \log a]$.

The function F_a, defined by

$$F_a(x) \equiv a^{-1/x}, \qquad 0 < x < \infty,$$
$$F_a(0) = 0,$$

is convex on the interval $[0, (1/2)\log a]$, starshaped on the interval $[0, \log a]$, and superadditive on the interval $[0, \log a/\log 2]$.

In particular, the function F_2 is superadditive on the interval $[0, 1]$. Therefore, for positive numbers x_i, $i = 1, 2, \cdots, n$, $n \geqq 1$, such that

$$\sum_{i=1}^{n} x_i = x_0 \leqq 1, \tag{46}$$

we have the inequality

$$\sum_{i=1}^{n} 2^{-1/x_i} \leqq 2^{-1/x_0}$$

and the weaker inequality

$$\sum_{i=1}^{n} 2^{-1/x_i} \leqq 2^{-1}. \tag{47}$$

Substituting $1/(y_i + 1)$ for x_i in (46) and (47), we obtain the following result:

If the nonnegative numbers y_i, $i = 1, 2, \cdots, n$, $n \geqq 1$, are such that

$$\sum_{i=1}^{n} \frac{1}{1+y_i} \leqq 1 ,$$

then we have

$$\sum_{i=1}^{n} 2^{-y_i} \leqq 1 , \tag{48}$$

with equality if and only if either (a) $n = 1$ *and* $y_1 = 0$, *or* (b) $n = 2$ *and* $y_1 = y_2 = 1$.

The relation (48) is Whittaker's inequality [10, 4].

14. The method of Boas. The following sufficient condition for superadditivity was suggested to the author by R. P. Boas in personal correspondence:

BOAS'S TEST. *If the function* $f \in K(b)$ *is of class* C', *and there are numbers* $a \leqq b/2$ *and* $c \leqq a$ *such that*

(i) f *is starshaped on* $[0, 2a]$,

(ii) f *is concave and satisfies* $f(x/2) \leqq (1/2)f(x)$ *on* $[c, b]$,

(iii) $f'(0) < f'(b)$,

(iv) $f'(x) - f'(b - x)$ *has at most one zero in* $(0, a)$,

then f *is superadditive on* $[0, b]$.

The validity of the test can be established by considering separately the following three cases:

(i) $0 \leqq x \leqq a, \quad 0 \leqq y \leqq a$,

(ii) $x \geqq a, \quad y \geqq a, \quad x + y \leqq b$,

(iii) $x < a < y < b, \quad x + y \leqq b$.

Boas has observed that his test applies to such convexo-concave functions, or functions having ogive-shaped graphs, as $e^{-1/x\alpha}$ for $0 < \alpha \leqq 1$, $\log(1 + x^\lambda)$, and arc tan x^λ, yielding intervals of superadditivity and consequent inequalities typified by the inequality of Whittaker given in § 13, above.

A systematic tabulation of maximum intervals of superadditivity of such functions, of their average functions, and of their inverse average functions, might well be desirable.

15. Combination inequalities. If the function f is convex for $x \in [a, b]$, then, by Jensen's inequality [6, 1], for any numbers

$$x_i \in [a, b] , \qquad i = 1, 2, \cdots, n ,$$

and any weights

$$\alpha_i, \alpha_i > 0, \sum_{i=1}^{n} \alpha_i = 1 ,$$

we have

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leqq \sum_{i=1}^{n} \alpha_i f(x_i) . \tag{49}$$

This inequality is an extension of the defining inequality (1).

Analogues of the inequality (49), for functions of the sort treated in this paper, are given in the two theorems that follow.

THEOREM 5. *If the function $f \in K(b)$ is convex for $x \in [0, a]$ and starshaped for $x \in [0, b]$, $b > a$, then for any numbers*

$$x_i \in [0, b], \qquad i - 1, 2, \cdots, n ,$$

and any weights

$$\alpha_i,\ \alpha_i > 0,\ \sum_{i=1}^{n} \alpha_i = 1 ,$$

we have

$$f\left(\frac{a}{b} \sum_{i=1}^{n} \alpha_i x_i\right) \leqq \frac{a}{b} \sum_{i=1}^{n} \alpha_i f(x_i) . \tag{50}$$

Proof. Since the numbers x_i satisfy $0 \leqq x_i \leqq b$, we have $0 \leqq ax_i/b \leqq a$, so that Jensen's inequality (49) can be applied for the numbers ax_i/b, yielding

$$f\left(\sum_{i=1}^{n} \alpha_i \frac{a}{b} x_i\right) \leqq \sum_{i=1}^{n} \alpha_i f\left(\frac{a}{b} x_i\right) . \tag{51}$$

Now since $0 \leqq x_i \leqq b$, and $a/b < 1$, the defining inequality (2) for starshapedness gives

$$f\left(\frac{a}{b} x_i\right) \leqq \frac{a}{b} f(x_i) , \qquad i = 1, 2, \cdots, n , \tag{52}$$

and (50) follows from (51) and (52).

For example, for the function f defined by (7) we have $a = 1/2$, $b = 1$, so that for positive numbers $x_i \leqq 1$ and weights α_i we have

$$\exp \frac{-2}{\sum_{i=1}^{n} \alpha_i x_i} \leqq \frac{1}{2} \sum_{i=1}^{n} \alpha_i e^{-1/x_i} .$$

THEOREM 6. *If the function $f \in K(c)$ is convex for $x \in [0, a]$, starshaped for $x \in [0, b]$, and superadditive for $x \in [0, c]$, $c > b > a$, then for any numbers*

$$x_i \in [0, b] , \qquad i = 1, 2, \cdots, n ,$$

satisfying

$$\sum_{i=1}^{n} x_i = c_0 \leqq c \,, \tag{53}$$

we have

$$f\left(\frac{ac_0}{bn}\right) = f\left(\frac{a}{bn}\sum_{i=1}^{n} x_i\right) \leqq \frac{a}{bn}\sum_{i=1}^{n} f(x_i) \leqq \frac{a}{bn} f\left(\sum_{i=1}^{n} x_i\right) \tag{54}$$
$$= \frac{a}{bn} f(c_0) \leqq \frac{a}{bn} f(c) \,.$$

Proof. By (50), we have

$$f\left(\frac{a}{bn}\sum_{i=1}^{n} x_i\right) \leqq \frac{a}{bn}\sum_{i=1}^{n} f(x_i) \,, \tag{55}$$

and from (43) and (53) we obtain

$$\sum_{i=1}^{n} f(x_i) \leqq f\left(\sum_{i=1}^{n} x_i\right) \,, \tag{56}$$

whence (54) follows from (55), (56), and the fact that f is a non-decreasing function.

By way of illustration, for positive numbers $x_i \leqq 1$ satisfying

$$\sum_{i=1}^{n} x_i = c_0 \leqq \frac{1}{\log 2} \,,$$

we have both lower and upper bounds for

$$\sum_{i=1}^{n} e^{-1/x_i} \,,$$

given by

$$e^{-2n/c_0} = \exp \frac{-2n}{\sum_{i=1}^{n} x_i} \leqq \frac{1}{2n}\sum_{i=1}^{n} e^{-1/x_i} \leqq \frac{1}{2n} \exp \frac{-1}{\sum_{i=1}^{n} x_i}$$
$$= \frac{1}{2n} e^{-1/c_0} \leqq \frac{1}{2n} e^{-\log 2} = \frac{1}{4n} \,.$$

For a function having a relatively longer interval of superadditivity, a more useful inequality would result.

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UNIVERSITY OF CALIFORNIA, LOS ANGELES