CUT POINTS IN TOTALLY NON-SEMI-LOCALLY-CONNECTED CONTINUA

EDWARD E. GRACE

1. Introduction. F. Burton Jones has shown [4, Theorem 15] that the set of weak cut points of a compact metric continuum that is not semi-locally-connected at any point of an open set U of M is dense in U (the proofs apply to locally peripherally compact complete metric continua). It is the purpose of this paper to extend Jones' results by establishing stronger cutting properties.

The theory is given here for locally compact metric spaces but applies, with appropriate modifications, to locally peripherally bicompact, regular spaces of the general class mentioned in [2].

2. Definitions and preliminary theorems. A point p of a continuum M is a *weak cut point* of M (or *cuts M weakly*) if there are two points x and y of $M - \{p\}$ such that each subcontinuum of M that contains both x and y contains p also. In this case *p cuts x from y weakly* in M.

A continuum M is *semi-locally-connected* at a point p of M if each open subset U of M containing p contains an open subset V of M containing p, the complement of which relative to M consists of a finite number of components. A continuum M is *totally nonsemi-locally-connected* (on a point set A) if M is not semi-locally-connected at any point (of A). A continuum M is *locally peripherally aposyndetic* at a point p of M if each open subset U of M containing p contains an open subset V of M containing p such that, for some collection $(H_1, \cdots, H_n)$ of subcontinua of M, $(\bigcap_{i=1}^{n} H_i) \cap (\bar{V} - V) = \phi$ and p is in the (nonvoid) interior W of $(\bigcap_{i=1}^{n} H_i) \cap V$, relative to M. In this case W is a *peripheral aposyndesis subset* of U and V is a set *associated with* W and U.

EXAMPLE. The Cartesian product of a cantor set and a simple closed curve, with one of the cantor sets shrunk to a point, is locally peripherally aposyndetic at each point but aposyndetic at only one point.

THEOREM 1. *A locally compact metric continuum M is locally peripherally aposyndetic on a dense G_δ subset of an open subset D*

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of M if it is locally peripherally aposyndetic on a dense subset of D.

Theorem 1 is a corollary of [2, Theorem 1].

THEOREM 2. *If M is a locally compact metric continuum that is not semi-locally-connected at the point x, then x is a limit point of the set of points of M at which M is nonaposyndetic with respect to x.*

Theorem 2 follows from the fact that any open subset D of M which has a compact boundary and contains the point x, contains an open subset, containing x, whose complement with respect to M consists of a finite number of subcontinua of M, if M is aposyndetic at each point of $\bar{D} - D$ with respect to x.

THEOREM 3. *If M is a locally compact metric continuum that is not locally peripherally aposyndetic at the point x then x is a limit point of the set of all points y of M such that M is nonaposyndetic at x with respect to y.*

Theorem 3 follows from the Heine-Borel Theorem by an indirect proof.

3. Cut point theorems.

THEOREM 4. *If M is a locally compact metric continuum which is locally peripherally aposyndetic and nonsemi-locally connected at each point of a G_δ subset I dense in an open subset D, then for each point p of M there is a dense G_δ subset J_p of D such that for each point x of J_p there is a point y at which M is nonaposyndetic with respect to x and x cuts p weakly from each such y.*

Proof. A point x of D has the properties required for membership in J_p if $x \in I - \{p\}$ and each open subset D' of D, containing x, contains an open subset D_1, containing x, and a subset S such that (1) M is aposyndetic at each point of $M - D'$ with respect to each point of D_1 or (2) S separates p from each point y of $M - D'$ at which M is nonaposyndetic with respect to some point of D_1. This observation suggests consideration of distinguished subsets defined as follows (see [2, p.]). An open subset D_1 of an open subset D' of D is a distinguished subset of D' if

(1) M is aposyndetic at each point of $M - D'$ with respect to each point of D_1 or

(2) there is a subset S of D' which separates p from each point

y, in $M - D'$, at which M is nonaposyndetic with respect to some point of D_1.

If each open set in D contains a distinguished open subset then [2, Theorem 1] the set of distinguished points of D is a dense G_δ subset of D the intersection of which with I is the desired set J_p. To prove that each open set in D contains a distinguished open subset, let

(1) D' be any open subset of D,

(2) S be a peripheral aposyndesis subset of D' that does not contain p (such a set exists since M is locally peripherally aposyndetic on a dense subset of D) and

(3) D'' be a set associated with S and D'. If $M - S$ is connected then $D_1 = S$ is a distinguished subset of D', since M is aposyndetic at each point of $M - D'$ with respect to each point of D_1. If $M - S = A \cup B$, where A and B are mutually separated and $p \in A$, then $B \cap (D'' - S) \neq \phi$. Let H be a continuum containing S but not containing $B \cap (D'' - S)$. Let $D_1 = (M - H) \cap B \cap (D'' - S)$. Then M is aposyndetic at each point of A with respect to each point of D_1, since $H \cup A$ is a continuum containing A in its interior. But S separates p from each point of B and $A \cup B \supset M - D'$. It follows that D_1 is a distinguished subset of D'.

That there is a space satisfying the hypothesis of Theorem 4 is seen from the example [3, Example 2] of a bounded plane continuum which is both connected im kleinen and nonsemi-locally-connected at each point of a dense G_δ subset.

Local peripheral aposyndesis is used instead of aposyndesis in Theorem 4 in order that the complementary case (covered in Theorem 5) will be such that each x in J_p will be a limit point of A_x.

COROLLARY 4.1. *If a compact metric continuum M is aposyndetic, but not semi-locally-connected, at each point of a dense G_δ subset of M and P is a countable, dense subset of M, then there is a dense G_δ subset J of M each point x of which cuts each point p of P from each point y at which M is nonaposyndetic with respect to x.*

Proof. First, if M is aposyndetic at x then M is locally peripherally aposyndetic at x. Second, if $p(1), p(2), \cdots$ is a counting of P let $J = \bigcap_{n=1}^{\infty} J_{p(n)}$, where $J_{p(n)}$ is as given in Theorem 4.

Theorem 5 is a corollary of [1, Theorem 2].

THEOREM 5. *If M is a locally compact metric continuum and D is an open subset of M such that M is nonlocally peripherally aposyndetic at each point of a dense G_δ subset of D, then for each* *point p of M there is a dense G_δ subset J_p of D such that each point x of J_p*

(1) *is cut from p weakly by each point of the set A_x, of all points $y \neq p$ such that M is nonaposyndetic at x with respect to y, and*

(2) *is a limit point of A_x.*

The following theorem is a consequence of Theorems 4 and 5.

THEOREM 6. *If M is a locally compact metric continuum which is totally nonsemi-locally-connected on some dense G_δ subset of an subset D of M then D is contained in the union of the closures of two* (possibly void) *open subsets D_1 and D_2 such that M is locally peripherally aposyndetic at each point of a G_δ set dense in D_1 but not at any point of D_2 and for each point p of M there is a dense G_δ subset J_p of D such that*

(1) *each point x of $J_p \cap D_1$ cuts p weakly from each point y such that M is nonaposyndetic at y with respect to x, and*

(2) *each x in $J_p \cap D_2$ is cut from p weakly by each point of $A_x = \{y \mid y \neq p$ and M is nonaposyndetic at x with respect to $y\}$ and is a limit point of A_x.*

COROLLARY 6.1. *If M is a compact metric continuum which is totally nonsemi-locally-connected on some dense G_δ set and P is a countable dense subset of M then M contains a dense set A each point of which cuts some point of M weakly from each point of P.*

Proof. Let $p(1), p(2), \cdots$ be a counting of P and let D_1 and D_2 be as given in Theorem 6. For each natural number n, let $J_{p(n)}$ be as given in Theorem 6. Let $J = \bigcap_{n=1}^{\infty} J_{p(n)}$. Let $B = J \cap D_1$ and C be the set of all $x \in M$ such that M is nonaposyndetic at some point y of $J \cap D_2$ with respect to x. Then $\bar{C} \supset J \cap D_2$ and each point of C cuts some point of $J \cap D_2$ weakly from each point of P. The union of B and C is the desired dense set of points.

Question. Does each totally nonsemi-locally-connected, compact metric continuum contain a dense G_δ set of weak cut points ?

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ARIZONA STATE UNIVERSITY