

A NEW MEASURE OF A PARTIAL DIFFERENTIAL FIELD EXTENSION

ISRAEL ZUCKERMAN

Let G be a differential field of characteristic zero with the commuting derivations d_1, \dots, d_m . If F is a differential subfield of G , the algebraic and differential degrees of transcendence of G over F , denoted respectively by $d(G/F)$ and $d.d(G/F)$ are numerical invariants of the extension. Unlike the ordinary differential case ($m=1$) $d.d.(G/F)=0$ does not imply that $d(G/F)$ is finite. In this paper an intermediate measure of the extension is constructed, called the limit vector. The first and last components of this vector correspond to $d.d(G/F)$ and $d(G/F)$ respectively, and the limit vector is additive.

Similar concepts have been developed independently by Kolchin in a work not yet published.

Characteristic sets of prime ideals as defined in [6] play a prominent role in the development of the limit vector, as well as in the development of other results of this paper which do not depend on the limit vector. Further, it is shown that an intermediate field of a finitely generated extension is finitely generated. Kolchin has a prior, but different proof of this. Kolchin's analog of Lüroth's Theorem [2, 3] is extended and some results on characteristic sets of length one are obtained.

Raudenbush [5] shows that the dependence axioms of Van der Waerden [7] are satisfied by differential dependence. It is indicated below that these axioms are more readily established by use of the limit vector. A further result is that a proper specialization of $F[a]$ must reduce the limit vector only if a has a characteristic set of length one over F . A short proof of a theorem of Delsarte [1] on partial linear homogeneous differential equations concludes the paper.

2. Ordering the derivatives. The main source of this subject is Ritt [6], especially § 8 and § 2 of Chapter VIII and §§1-16 of Chapter IX. In general, the terminology and notation are as in [6]. Consider the differential ring $F\{y_1, \dots, y_r\}$, where the y_i are differential indeterminates. Then $D = d_1^{i_1} \dots d_m^{i_m}$ will denote a derivative, i.e., the composite of derivations. We associate with D the vector (i_1, \dots, i_m) . The sum of the i_k is called the order of D , or of the associated vector,

Received January 23, 1964. This paper is the author's doctoral dissertation submitted to Rutgers University in March 1963 under the direction of Professor Richard Cohn. The work was partly supported by the National Science Foundation Grant G-19636.

or of Dy_j . y_j is to be considered a derivative of itself of zero order. Let w denote an arbitrary y -derivative. A set of marks will be assigned to the d_i and y_i to achieve a complete ordering of the w . In doing so, the primary objective will be to have w_1 precede w_2 if the order of w_1 is less than the order of w_2 . Associate with each d_i , the marks $u_{i1}, \dots, u_{i,m+2}$; where $u_{i1} = 1$, $u_{i2} = 0$, and for $k > 2$: $u_{ik} = 0$ for $k \neq i+2$, $u_{ik} = 1$ for $k = i+2$. Associate with each y_i the marks $v_{i1}, \dots, v_{i,m+2}$, where $v_{i2} = i$ and $v_{ik} = 0$ for $k \neq 2$. Let $w = d_1^{i_1} \dots d_m^{i_m} y_i$. Then w is assigned $m+2$ marks as follows: The j th mark of w is $v_{ij} + i_1 u_{1j} + \dots + i_m u_{mj}$. Let w_1 and w_2 be two y -derivatives with marks a_i and b_i respectively. Then we shall say that w_1 precedes w_2 or succeeds w_2 according as the first nonzero difference $b_i - a_i$ is positive or is negative. As one can easily verify the system of marks introduced here achieves the desired complete ordering of the y -derivatives. Such ordering will prevail throughout, with the exception of §12.

3. A transcendence basis for a_1, \dots, a_r over F . Let a_1, \dots, a_r be elements in G . If the a_i are differentially dependent over F , let P be the prime ideal in $F\{y_1, \dots, y_r\}$ with (a_1, \dots, a_r) as a generic zero, and let $C = (A_1, \dots, A_k)$ be a characteristic set of P . (C is said to be a characteristic set of the a_i over F .) Denote the leader and separant of A_i by p_i and S_i respectively. If w is a derivative of some p_i , it will be called principal; otherwise, it will be called parametric. We will call an a -derivative principal if it admits a representation $w(a)$ where w is principal; otherwise, it will be called parametric. If the a_i are differentially independent over F , all a -derivatives will be considered to be parametric. The ordering of the y -derivatives carries over to the ordering of the symbols for the a -derivatives.

THEOREM 1. *The parametric derivatives of a_1, \dots, a_r are distinct and constitute a transcendence basis for $F(a_1, \dots, a_r)$ over F . Moreover, a principal a -derivative depends algebraically on parametric derivatives of equal or lower order.*

Proof. If the a_i are differentially independent over F , the proof is immediate. Now assume that the a_i are differentially dependent over F , with the associated prime ideal P and the characteristic set C . Since C is a characteristic set of P , every differential polynomial, (d.p.), in P involves a derivative of some p_i . Hence the parametric a -derivatives are distinct and algebraically independent over F . This proves the first statement of the theorem.

Now, the principal a -derivatives are completely ordered with $p_1(a)$ as the first derivative. Since p_1 is the leader of A_1 , $p_1(a)$ depends solely on parametric derivatives. Let $w(a_j)$ be a principal a -derivative suc-

ceeding $p_i(a)$. If w is a proper derivative of some leader, say p_i , then an appropriate derivative D of A_i yields

$0 = DA_i(a_1, \dots, a_r) = S_i(a_1, \dots, a_r)w(a_j) + B(a_1, \dots, a_r)$, where the a -derivatives in $S_i(a)$ and $B(a)$ precede $w(a_j)$. w may also be some p_i for $i > 1$. In either case, $w(a_j)$ depends on principal and parametric derivatives whose symbols precede the symbol for $w(a_j)$. By induction, each of these principal derivatives depends on preceding parametric derivatives; hence on parametric derivatives which precede $w(a_j)$. Thus, $w(a_j)$ depends on preceding parametric derivatives, concluding the proof.

4. Restriction to a -derivatives with orders not exceeding n .

Let A be a subset of G , and n a positive integer. Then the set of derivatives of elements of A of order not exceeding n will be denoted by $(n; A)$. Let F' be an ordinary field contained in G . Then, as is customary, $F'(n; A)$ will denote the ordinary field extension obtained by adjoining $(n; A)$ to F' . $d(F'(n; A)/F')$ will be denoted by $h(F', n; A)$ or simply by $h(n)$ if no ambiguity arises. (This type of abbreviation is repeated throughout.)

With this notation the following corollary is immediate, noting that the ordering of the a -derivatives is such that a principal derivative depends on parametric derivatives of equal or lower order.

COROLLARY 1. $h(F', n; a_1, \dots, a_r)$ is the number of parametric derivatives of the a_i of order not exceeding n .

THEOREM 2. For n sufficiently large,

$$h(F, n; a_1, \dots, a_r) = H(F, n; a_1, \dots, a_r)/m!,$$

where $H(n)$ is a polynomial in n with integral coefficients. $H(n)=0$, or has degree $t \leq m$ and leading coefficient $c_t > 0$.

Proof. Let $p(n)$ denote the number of derivatives of some y of order not exceeding n . Then

$$p(n) = C(n + m, m) = (n + m)(n + m - 1) \cdots (n + 1)/m!$$

Hence, if the a_i are differentially independent over F , $h(n) = rp(n)$ and the theorem is true in such case with $t = m$ and $c_t = r$.

Now assume that the a_i are differentially dependent over F with leaders p_i of a characteristic set for the a_i over F . Partition the p_i into subsets R_j , each consisting of derivatives of the same y_j . Let q_1, \dots, q_s be one such subset. For each nonempty subset T of $\{q_1, \dots, q_s\}$, let $u(T)$ be the m -vector with k th component, $k(T)$, equal to the

maximum of the k th components of the vectors associated with the q_i in T , and let v_T be the sum of the $k(T)$. Let n be a positive integer greater than the maximum of the v over all R_j . Then an m -vector will be called a multiple of $u(T)$ if each of its components is not less than the corresponding component of $u(T)$, and if the sum of its components does not exceed n . Let $S(T)$ denote the set of multiples $u(T)$, and let $N(T)$ denote the number of elements in $S(T)$. Let $B(T)$ denote the number of elements in T .

Then,

$$(1) \quad N(T) = C(n + m - v_T, m).$$

Note that $p(n)$ is of the form (1) with $v = 0$. Therefore we extend the set of subsets T to include the empty set φ and define $v_\varphi = 0$, $N(\varphi) = p(n)$. The number of parametric derivatives of y_j of order not exceeding n is obtained by subtracting from $p(n)$ the number of elements in the union of the $S(T)$. Thus, it is equal to

$$(2) \quad \sum_T (-1)^{B(T)} N(T).$$

(2) is the sum of 2^s expressions, each of the form (1). The term of highest degree in n in each expression is the same, $n^m/m!$. Since there are as many positive as negative expressions in (2), the sum of the terms in n^m is zero. Furthermore, by consideration of large n we see that the effective leading coefficient of (2) is positive.

The sum of the number of parametric derivatives of the R_j is then a polynomial with the desired properties. (An empty R_j contributes $p(n)$ elements.)

The following corollary is immediate.

COROLLARY 1. *Let s be an integer with $0 \leq s \leq m$. If a_1, \dots, a_r are differentially dependent over F , then*

$$\lim_{n \rightarrow \infty} n^s h(n)/p(n) = \begin{cases} 0, & \text{if } s < m - t; \\ c_i, & \text{if } s = m - t; \\ \infty, & \text{if } s > m - t. \end{cases}$$

If a_1, \dots, a_r are differentially independent over F , then

$$\lim_{n \rightarrow \infty} n^s h(n)/p(n) = \begin{cases} r, & \text{for } s = 0; \\ \infty, & \text{for } 0 < s < m. \end{cases}$$

5. Introduction of the limit vector. We may now define

$$L_s(F, a_1, \dots, a_r) = \lim_{n \rightarrow \infty} n^s h(n)/p(n) \quad (s = 0, \dots, m),$$

and

$$L(F, a_1, \dots, a_r) = (L_0, \dots, L_m).$$

The latter will be called the limit vector of a_1, \dots, a_r over F .

The following remarks are evident.

$$(1) \quad d.d(F\langle a_1, \dots, a_r \rangle / F) = r \Leftrightarrow L_0 = r.$$

$$(2) \quad \text{For a single element } a, \quad d.d(F\langle a \rangle / F) = 0 \Leftrightarrow L_0 = 0.$$

We will later show that $d.d(F\langle a_1, \dots, a_r \rangle / F) = L_0$, subsuming (1) and (2).

$$(3) \quad d(F\langle a_1, \dots, a_r \rangle / F) = k < \infty \Leftrightarrow L_m = km!$$

In particular, if each a_i is algebraic over F , $L_s = 0$ for all s .

6. Results on L_1 . The following corollaries follow from the proof of Theorem 2. In this section, a will be differentially algebraic over F with characteristic set $C = A_1, \dots, A_k$, $0 < k < \infty$. W_i will denote the vector associated with the leader of A_i .

COROLLARY 2.

$$L_1(F; a) = m \sum_T (-1)^{B(T)+1} v_T.$$

In particular, this shows that $L_1(F; a)$ is divisible by m .

Proof. The coefficient of n^{m-1} in $m! C(n + m - v_T, m)$ is

$$(3) \quad (m - v_T) + \dots + (1 - v_T) = m(m + 1)/2 - mv_T.$$

The first term in (3) is the same for each v_T . Since there are as many positive as negative expressions in (2), these terms cancel in computing C_{m-1} , and the desired result follows.

As a special case of Corollary 2, we have the following.

COROLLARY 3. *If a has a characteristic set of length one over F , then $L_1(F; a) = mg$ where g is the order of such characteristic set.*

The result of Corollary 2 may be carried further so as to depend more directly on the leaders of the characteristic set. We need the following lemma.

LEMMA. *Let $u(1), \dots, u(k)$, $1 \leq k < \infty$, be a sequence S of real numbers with $u(1) = \min(u(i))$. For each subsequence T of S , let $B(T)$ and $M(T)$ denote respectively the number and maximum of its elements. Then,*

$$\sum_T (-1)^{B(T)+1} M(T) = u(1) .$$

Proof. Let T' denote a T which does not have $u(1)$ as its first element; and let T'' be the sequence obtained by adjoining $u(1)$ at the beginning of T' . Then, with the exception of $T = u(1)$, the T can be partitioned into pairs of T' and T'' such that

$$(-1)^{B(T')+1} M(T') + (-1)^{B(T'')+1} M(T'') = 0 .$$

Thus, the desired result is obtained.

COROLLARY 4. *Let w_j denote the minimum of the j th components of the W_i . Then*

$$L_1(F; a) = m \sum_{j=1}^m w_j .$$

Proof. By the lemma, the sum of the j th components of the vectors associated with the T , with the appropriate signs affixed, is w_j .

Hence, if we sum the $(-1)^{B(T)+1} v_T$ component-wise, the result of Corollary 2 yields $L_1(F; a) = m \sum_{j=1}^m w_j$, as desired.

Note that Corollary 4 implies that $L_1(F; a) = 0$ if and only if $w_j = 0$ for all j .

COROLLARY 5. *Let A_i of order v belong to C , and let $k > 1$. Then $L_1(F; a) < mv$.*

Proof. If for $j = 1, \dots, m$, the j th component of W_t is the minimum of the j th components of the W_i , then for $i \neq t$, A_i would not be reduced with respect to A_t .

We are now in a position to prove a converse of Corollary 3.

COROLLARY 6. *If $L_1(F; a) = mg$, $0 \leq g < \infty$, and if a satisfies an irreducible d.p. A over F of order g , then A is a characteristic set for a over F .*

Proof. Since a satisfies A , there exists a d.p. A' in C with order $g' \leq g$. By Corollary 5, if $k > 1$, $L_1(F; a) < mg' \leq mg$. Hence, $C = A'$. By Corollary 3, $g' = g$ and A' has the same order as A . Since A has zero remainder with respect to A' and is irreducible, $A = cA'$ where c is in F ; hence, A is a characteristic set for a over F .

7, Additivity of the limit vector.

LEMMA 1. *Let A be a finite set of elements contained in an ex-*

tension of F . If k , q , and r are nonnegative integers and s an integer such that $0 \leq s \leq m$, then

$$\lim_{n \rightarrow \infty} [(n+k)^s h(n+q, A)/p(n+r)] = \lim_{n \rightarrow \infty} [n^s h(n, A)/p(n)].$$

Proof.

$$\frac{(n+k)^s h(n+q, A)}{p(n+r)} = \frac{(n+q)^s h(n+q, A)}{p(n+q)} \frac{(n+k)^s}{(n+q)^s} \frac{p(n+q)}{p(n+r)}.$$

Clearly,

$$\lim_{n \rightarrow \infty} \frac{p(n+q)}{p(n+r)} = 1, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{(n+k)^s}{(n+q)^s} = 1.$$

Furthermore,

$$\lim_{n \rightarrow \infty} \frac{(n+q)^s h(n+q, A)}{p(n+q)} = \lim_{n \rightarrow \infty} \frac{n^s h(n, A)}{p(n)}.$$

LEMMA 2. Let A be a set and b an element contained in an extension of F with b differentially algebraic over $F\langle A \rangle$. Let t be the maximal order of derivatives of A appearing in a characteristic set C for b over $F\langle A \rangle$. Let A' be a set of A -derivatives containing $(n+t; A)$. Then,

$$h(F\langle A \rangle, n; b) = h(F(A'), n; b).$$

Proof. Let $S(n)$ denote the set of parametric derivatives of b of order not exceeding n with respect to C . Then $S(n)$ is algebraically independent over $F(A')$. Furthermore, $S(n)$ is an algebraic spanning set for $(n; b)$ over $F(A')$ since in the proof of Theorem 1 only derivatives with orders not exceeding n of the $d.p.$ in C are present in the algebraic relations obtained for the principal derivatives.

Note that the result holds more readily if b is differentially transcendental over $F\langle A \rangle$.

THEOREM 3.

$$L(F; a_1, \dots, a_r) = \sum_{i=1}^r L(F\langle a_1, \dots, a_{i-1} \rangle; a_i).$$

Proof. For $i = 1, \dots, r-1$, if a_{i+1} is differentially transcendental over $F\langle a_1, \dots, a_i \rangle$, let $t_i = 0$. Otherwise, t_i be the maximal order of derivatives of a_1, \dots, a_i in a characteristic set for a_{i+1} over $F\langle a_1, \dots, a_i \rangle$. Then,

$$\begin{aligned}
& F(n + t_1 + \cdots + t_{r-1}; a_1, \cdots, a_r) \\
& \supset F(n + t_1 + \cdots + t_{r-1}; a_1)(n + t_2 + \cdots + t_{r-1}; a_2) \\
& \cdots (n + t_{r-1}; a_{r-1})(n; a_r) \supset F(n; a_1, \cdots, a_r).
\end{aligned}$$

Then, by additivity of the transcendence degree and Lemma 2,

$$\begin{aligned}
& h(F, n + t_1 + \cdots + t_{r-1}; a_1, \cdots, a_r) \\
& \geq \sum_{i=1}^r h(F\langle a_1, \cdots, a_{i-1} \rangle, n + t_i + \cdots + t_{r-1}; a_i) \\
& \geq h(F, n; a_1, \cdots, a_r).
\end{aligned}$$

Multiplying by $n^s/p(n)$, then taking the limit of the resulting expressions as n approaches ∞ , by application of Lemma 1, the desired result is obtained.

8. Extension of the limit vector to a measure of an arbitrary differential field extension. Let $f(G)$ denote the set of finite subsets of an extension G of F , including the null set. For S and T in f , suppose that k is the first component for which $L(F; S)$ and $L(F; T)$ differ, and that $L_k(F; S) < L_k(F; T)$. Then we will write $L(F; S) < L(F; T)$ and we may define $L(G/F) = \sup_{s \in f} L(F; S)$.

COROLLARY 1. *If G is finitely generated over F , i.e., $G = F\langle a_1, \cdots, a_r \rangle$, then $L(G/F) = L(F; a_1, \cdots, a_r)$.*

Proof. Let $A = \{a_1, \cdots, a_r\}$ and B belong to f . Then $G = F\langle A, B \rangle$. By Theorem 3, $L(F; B) \leq L(F; A, B) = L(F; A) + L(F\langle A \rangle, B)$. Since $B \subset F\langle A \rangle$, $L(F\langle A \rangle; B) = 0$. Hence, $L(F; B) \leq L(F; A)$ for all B in f .

Immediate consequences are:

COROLLARY 2. *Given $G = F\langle A \rangle = F\langle B \rangle$, where both A and B belong to f . Then $L(F; A) = L(F; B)$.*

COROLLARY 3. *If H is finitely generated over G , and G is finitely generated over F , then*

$$L(H/F) = L(H/G) + L(G/F).$$

9. A general additivity theorem.

THEOREM 4. *Given $F \subset G \subset H$. Then, $L(H/F) = L(G/F) + L(H/G)$.*

Proof. (Part I): To prove $L(H/F) \geq L(G/F) + L(H/G)$. Let $g(H)$ denote the set of finite subsets of H , including the null set, which contain no elements of G . Then every V in $f(H)$ is the unique union

of an S in $f(G)$ and a T in $g(H)$, and conversely. For a particular $V = S \cup T$,

$$L(F; V) = L(F; S) + L(F\langle S \rangle; T) \geq L(F; S) + L(G; T).$$

Hence,

$$\begin{aligned} \sup_{V \in f(H)} L(F; V) &\geq \sup_V (L(F; S) + L(G; T)) \\ &= \sup_{S \in F(G)} L(F; S) + \sup_{T \in g(H)} L(G; T) = L(G; F) + L(H; G). \end{aligned}$$

Proof. (Part II): To prove $L(H/F) \leq L(G/F) + L(H/G)$. It suffices to show that $L(F\langle h_1, \dots, h_r \rangle/F) \leq L(H/G) + L(G/F)$ for any finite subset h_1, \dots, h_r of H . Let $H_1 = G\langle h_1, \dots, h_r \rangle$, and let P be the prime ideal in $G\{y_1, \dots, y_r\}$ with h_1, \dots, h_r as a generic zero. If $P = 0$, let $G_1 = F$. Otherwise, let $G_1 = F\langle A \rangle$, where A is the set of coefficients of a characteristic set C of P . Let P_1 be the prime ideal in $G_1\{y_1, \dots, y_r\}$ with h_1, \dots, h_r as a generic zero. Then C is also a characteristic set of P_1 . Hence, $L(H_1/G) = L(G_1\langle h_1, \dots, h_r \rangle/G_1)$.

By additivity for the finitely generated case,

$$\begin{aligned} L(F\langle h_1, \dots, h_r \rangle/F) &\leq L(G_1\langle h_1, \dots, h_r \rangle/F) \\ &= L(G_1\langle h_1, \dots, h_r \rangle/G_1) + L(G_1/F) \\ &= L(H_1/G) + L(G_1/F) \\ &\leq L(H/G) + L(G/F). \end{aligned}$$

10. Remarks on L_0 . (1) Almost all of the previous results on L_0 could be obtained more readily: If a is differentially algebraic over F , it is sufficient to consider a single d.p. $B(y)$ in $F\{y\}$ which has a as a nonsingular solution. (i.e., a does not annul the separant of $B(y)$.)

(2) In order to establish the theory of differential dependence for arbitrary m , Raudenbush showed in [3] that differential dependence satisfied the dependence axioms of Van der Waerden [5]. However this theory follows immediately from the results on L_0 . Furthermore, since it suffices to consider differentially independent subsets of G in determining $L_0(G/F)$, it is clear that $\text{d.d.}(G/F) = L_0(G/F)$. Hence the additivity of d.d. follows from the additivity of L_0 .

11. A result on finitely generated extensions.

THEOREM 5. *Given $F \subset G \subset H = F\langle a_1, \dots, a_r \rangle$. Then G is finitely generated over F .*

Proof. If the a_i are differentially independent over G , then it follows readily that $F = G$. Hence the theorem is true in such case.

If the a_i are differentially dependent over G let Q be the set of

coefficients of a characteristic set A_1, \dots, A_k of the a_i over G . Then we assert that $G = F\langle Q \rangle$. For let p_i be the leader of A_i and R the set of parametric derivatives of the a_i . Construct

$$G' = F\langle Q \rangle, \quad G'^* = G'(R), \quad G^* = G(R).$$

Then

$$H = G'^*(p_1, \dots, p_k) = G^*(p_1, \dots, p_k),$$

and

$$[H : G'^*] = [H : G^*] = \prod_{i=1}^k (\deg A_i \text{ in } p_i) < \infty.$$

Thus

$$G'^* = G^*.$$

Moreover, since the parametric derivatives of the a_i are transcendental over both G and G' , $[G : G'] = [G^* : G'^*] = 1$. Hence, $G = G'$ and the theorem is proved.

12. Characteristic sets of length one and Lüroth's theorem. Kolchin [1] and [2] proves the differential analog of Lüroth's theorem for ordinary differential fields of characteristic zero; explicitly, if $F \subset G \subset F\langle y \rangle$, then $G = F\langle a \rangle$, for some a in G . With minor changes Kolchin's proof goes through for the partial case, provided that y has a characteristic set of length one over G .¹

Using the ideas in [1] and [2], we establish the following converse.

THEOREM 6. *If $F \subset F\langle a \rangle \subset F\langle y \rangle$, where $a \notin F$, then y has a characteristic set of length one over $F\langle a \rangle$.*

Proof. Since $a \in F\langle y \rangle$, $a = P(y)/Q(y)$, where P and Q may be taken to be relatively prime d.p. over F . We assert that $A(z) = aQ(z) - P(z)$ is a characteristic set of y over $F\langle a \rangle$. To begin with, A is clearly irreducible over $F\langle a \rangle$.

Let Y be the prime ideal over $F\langle a \rangle$ with y as a generic zero. Let X be the prime ideal over $F\langle a \rangle$ with A as a characteristic set, and let x be a generic zero of X . x is differentially transcendental over F ; for otherwise a , and hence y would be differentially algebraic over F . Therefore, mapping y onto x determines an F -isomorphism

1. In Kolchin's proof, [2] on the 9th and 10th lines from the top on page 400, the result follows by considering the derivatives ordered so that every z derivative is higher than every y derivative. On the 16th line from the top on the same page, the y derivatives need to be considered higher. These orderings differ from those used in this paper.

of $F\langle y \rangle$ onto $F\langle x \rangle$. Under this isomorphism, a remains fixed, for $a = P(y)/Q(y) = P(x)/Q(x)$. Hence the isomorphism leaves $F\langle a \rangle$ fixed. Thus y and x satisfy the same d.p.'s over $F\langle a \rangle$, and y is a generic zero of Q . Hence, $X = Y$ and A is a characteristic set of X .

13. The length of a characteristic set is not a property of the extension. If $F\langle a \rangle = F\langle b \rangle$, and if a has a characteristic set of length one over F , must this also be true of b ? The answer is in the negative as the following example will indicate.

EXAMPLE. Let u denote a differential indeterminate and let $m = 2$ with the derivations denoted by subscripts x and y as in u_x and u_y .

Let P be the set of d.p. with zero remainder with respect to u_x . Then P is a prime differential ideal. Since the initial and separant of u_x are both one, P is also generated by u_x over F . Let a be a generic zero of P . Define

$$(1) \quad b = a + pa_y,$$

where p is in F and is differentially transcendental over the field of rationals contained in F . Then

$$(2) \quad b_x = p_x a_y,$$

$$(3) \quad b_y = a_y + p_y a_y + pa_{yy} = (1 + p_y)a_y + pa_{yy},$$

$$(4) \quad b_{xx} = p_{xx} a_y.$$

Then (1) and (2) imply

$$(5) \quad a = b - pa_y = b - p/p_x(b_x),$$

showing that a is in $F\langle b \rangle$. (2) and (4) imply

$$(6) \quad b_x - p_x b_{xx}/p_{xx} = 0,$$

showing that b satisfies a second order d.p. over F .

Equations (1), (2), and (3) may be solved for a , a_y , a_{yy} in terms of b , b_x and b_y . Hence, since a , a_y , a_{yy} are algebraically independent over F , b cannot satisfy a first order d.p. over F . Equation (6) yields the irreducible d.p. satisfied by b ,

$$(6') \quad A = u_x - p_x u_{xx}/p_{xx},$$

which may be chosen as a d.p. in a characteristic set for b .

By Corollary 3 to Theorem 2, $L_1(F; a) = 2$. Since $F\langle a \rangle = F\langle b \rangle$, $L_1(F; b) = 2$. By the same corollary, A cannot be the sole d.p. in a characteristic set for b .

14. A simply generated extension with no generator having a characteristic set of length one. *Example:* Let P be the prime differential ideal generated by u_x and u_y over F . Then u_x and u_y also constitute a characteristic set of P . Let a be a generic zero of P , and let $G = F\langle a \rangle$.

Then $h(F, n; a) = 1$, for all positive integral n . Hence $L_1(G/F) = 0$. By Corollary 3 to Theorem 2, if G has a generator b with a characteristic set of length one, then $L_1(G/F) = mg$ where g is the order of such characteristic set. Hence, $g = 0$ and b is algebraic over F . This implies that a is algebraic over F which is a contradiction of the fact that a is a generic zero of P .

15. Specializations. a' is called a specialization of a if there exists a differential homomorphism of $F\{a\}$ onto $F\{a'\}$, taking a into a' , and leaving F elementwise fixed. Since

$$h(F, n; a') \leq h(F, n; a), \quad L(F; a') \leq L(F; a).$$

We investigate when equality holds.

PROPOSITION 1. If a has a characteristic set of length one over F and if a' is a proper specialization of a , (i.e., $F\{a\}$ and $F\{a'\}$ are not isomorphic), then $L(F; a') < L(F; a)$.

Proof. Assume $L(F; a') = L(F; a)$, and let A be a characteristic set for a over F . Then by Corollaries 3 and 6 of Theorem 2, A is also a characteristic set for a' over F . Hence, $F\{a\}$ and $F\{a'\}$ are isomorphic. Thus the proposition is proved.

PROPOSITION 2. If a has a characteristic set of length exceeding one, a proper specialization need not reduce the limit vector.

Proof. The following example will prove the point. As in the example of § 13, we consider $F\{u\}$ with u a differential indeterminate and two derivations denoted by subscripts x and y . Let t be a generic zero of the prime differential ideal P in $F\{u\}$ with characteristic set and generator u_x . Also, let a be a generic zero of the prime differential ideal Q in $F\{u\}$ generated by and with a characteristic set, u_{xx} and u_{xy} . Since Q is properly contained in P , t is a proper specialization of a . But, by direct computation, or by Corollary 4 of Theorem 2, $L(F; a) = L(F; t) = (0, 2, \infty)$, proving the proposition.

16. Order of a prime ideal and systems of linear homogeneous d.p. Let P be a prime differential ideal in $F\{y\}$ with generic zero a .

The algebraic degree of transcendence of $F\langle a \rangle$ over F is called the order of P , (ord P). By Theorem 1, this is the number of parametric derivatives of a . We will show that this use of order agrees with the "order" of a system of linear homogeneous d.p. as used in the study of differential equations.

The following lemmas will lead to this result. The first two are nondifferential and are stated without proof. Lemma 3 is Kolchin's Lemma 1 in [4], where it is proved. Lemmas 4 and 5 are the differential analogues of Lemmas 1 and 2, respectively.

LEMMA 1. *Let S be a system of linear homogeneous polynomials in $K[x_1, \dots, x_n]$ which is a vector space over K , where K is a non-differential field. Then*

(a) *(S) is prime and contains no linear homogeneous polynomials which are not in S .*

(b) $S \cap K[x_1, \dots, x_r] = 0 = (S) \cap K[x_1, \dots, x_r] = 0$.

LEMMA 2. *Let a set T of linear homogeneous polynomials with coefficients in a field K generate a vector space over K and over an extension L of K . Then elements of T linearly independent over K remain linearly independent over L , and the number of such elements in a maximal set is the dimension of both vector spaces.*

LEMMA 3. *Let C be the field of constants of the partial differential field F , and let a_1, \dots, a_n belong to F . If a_1, \dots, a_n are linearly dependent over C , then $W_{D_1, \dots, D_n}(a_1, \dots, a_n) = 0$ for every choice of derivatives D_1, \dots, D_n where $W_{D_1, \dots, D_n}(a_1, \dots, a_n) = \det(D_i a_j)$. Conversely, if $W_{D_1, \dots, D_n}(a_1, \dots, a_n) = 0$ for every choice of D_1, \dots, D_n of orders $\leq n-1$, then a_1, \dots, a_n are linearly dependent over C .*

LEMMA 4. *Let S be a system of homogeneous linear partial d.p. in $F\{y\}$ which is a vector space over F and is closed under the derivations d_1, \dots, d_m . Let T denote a set of y -derivatives. Then $[S]$ is prime and*

$$S \cap F[T] = 0 \Rightarrow [S] \cap F[T] = 0.$$

Proof. Let V denote the set of finite sets of y -derivatives. Then by Part (a) of Lemma 1, if $U \in V$, $S \cap F[U]$ generates a prime ideal S_U in $F[U]$. Since the union of the S_U over all U in V is $[S]$, $[S]$ is prime.

Let T' be a finite subset of T . By Part (b) of Lemma 1,

$$S \cap F[T'] = 0 \Rightarrow S_{T'} \cap F[T'] = 0 \Rightarrow [S] \cap F[T] = 0 .$$

LEMMA 5. Let S denote a set of homogeneous linear partial d.p. in $F\{y\}$, and let G be an extension of F . Let T denote a set of y -derivatives. If P and Q denote the prime differential ideals generated by S in

$$F\{y\} \quad \text{and} \quad G\{y\} \quad \text{respectively,}$$

then

$$P \cap F[T] = 0 \Leftrightarrow Q \cap G[T] = 0 .$$

This implies that P and Q have the same orders.

Proof. Let S^* denote the system consisting of the d.p. in S and of their derivatives. Let B be a maximal linearly independent subset of S^* over F , therefore, by Lemma 2, over G . By Lemma 4, P contains a nonzero polynomial in members of T if and only if it contains such a polynomial which is linear homogeneous; that is, if and only if there is a linear dependence among members of T and B over F . In the same way Q contains a nonzero polynomial in the members of T if and only if the members of T and B are linearly dependent over G . But Lemma 2 shows that these conditions are equivalent. Thus the lemma is proved.

THEOREM 7. Let S be a system of homogeneous linear partial d.p. in $F\{y\}$. If the set of solutions of S is of linear dimension k over constants, then $\text{ord}[S] = k$.

Proof. (a) $\text{ord}[S] \geq k$:

Let u_1, \dots, u_k be a linear basis for the solutions over constants. Let c_1, \dots, c_k be new constants algebraically independent over $F\langle u_1, \dots, u_k \rangle$. Let $v = \sum c_i u_i$. Since the u_i are linearly independent over constants, by Lemma 3, for some set of derivatives D_1, \dots, D_k of orders $\leq k-1$, $W_{D_1}, \dots, W_{D_k}(u_1, \dots, u_k) \neq 0$. Hence, $D_1 v, \dots, D_k v$ are linearly independent over F . Since v belongs to the manifold of $[S]$, $[S]$ contains no linear homogeneous polynomial, and therefore by Lemma 4, contains no polynomial in $D_1 y, \dots, D_k y$. Hence, $\text{ord}[S] \geq k$.

(b) $k \geq \text{ord}[S]$:

Let $\text{ord}[S] = k$. Let v_1 be a generic zero of $[S]$. Then k derivatives, D_1, \dots, D_k , of v_1 are algebraically independent over F . We define inductively a sequence of elements v_j which are solutions of S by the requirement that v_j be a generic zero of the differential ideal generated

by S over $F\langle v_1, \dots, v_{j-1} \rangle\{y\}$.

Then, by Lemma 5, D_1v_j, \dots, D_kv_j are algebraically independent over $F\langle v_1, \dots, v_{j-1} \rangle$. In particular, the $D_iv_j, i, j = 1, \dots, k$, are algebraically independent over F . Hence, $W_{D_1, \dots, D_k}(v_1, \dots, v_k) \neq 0$. Then Lemma 3 implies that the v_j are linearly independent over constants. Thus $k \geq \text{ord}[S]$.

THEOREM 8. *Let P be a prime differential ideal in $F\{y\}$. If P contains a set of nonzero d.p. $A_i, i = 1, \dots, m$, such that A_i involves only y -derivatives of the form $d_i^t y$ where t is a nonnegative integer, then $\text{ord } P$ is finite.*

Proof. If P contains a d.p. free of proper y -derivatives, then $\text{ord } P = 0$, and the theorem holds. Hence, we may assume that the A_i involve proper y -derivatives. Then A_i involves a derivative of the leader of some member of a characteristic set C of P . Hence, there exist leaders of members of C of the form $d_i^{t_i} y, t_i > 0, i = 1, \dots, m$. Thus, by Theorem 1, $\text{ord } P \leq t_1 \cdots t_m$, proving the theorem.

If we specify in Theorem 8 that $P = [S]$, where S is a system of linear partial homogeneous d.p. in $F\{y\}$, then we have the hypothesis of a theorem of Delsarte [1], Proposition A, page 37. Then by Theorems 7 and 8, the linear dimension of the solution space of S is finite, which is Delsarte's conclusion.

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