PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 1, 1965

WAVE OPERATORS AND UNITARY EQUIVALENCE

TOSIO KATO

This paper is concerned with the wave operators $W_\pm = W_\pm(H_1, H_0)$ associated with a pair H_0, H_1 of selfadjoint operators. Let (M) be the set of all real-valued functions ϕ on reals such that the interval $(-\infty, \infty)$ has a partition into a finite number of open intervals I_k and their end points with the following properties: on each I_k, ϕ is continuously differentiable, $\phi' \neq 0$ and ϕ' is locally of bounded variation. Theorem 1 states that, if $H_1 = H_0 + V$ where V is in the trace class T, then $W'_\pm \pm W_\pm(\phi(H_1), \phi(H_0))$ exist and are complete for any $\phi \in (M)$; moreover, $M'_\pm$ are "piecewise equal" to $W_\pm$ (in the sense to be specified in text). Theorem 2 strengthens Theorem 1 by replacing the above assumption by the condition that $\phi_n(H_1) = \phi_n(H_0) + V_n$, $V_n \in T$, where $\phi_n \in (M)$ and ϕ_n is univalent on $(-n, n)$ for $n = 1, 2, 3, \ldots$. As corollaries we obtain many useful sufficient conditions for the existence and completeness of wave operators.

1. Introduction. The present paper is a continuation of earlier papers of the author on the theory of wave and scattering operators and the related theory of unitary equivalence of selfadjoint operators.

We begin with a brief review of relevant definitions and known results (see Kato [4, 5] and Kuroda [6]), adding some new definitions for convenience. Let $\mathfrak{H}$ be a Hilbert space and let H be a selfadjoint operator in $\mathfrak{H}$ with the spectral representation $H = \int \lambda dE(\lambda)$. A vector $u \in \mathfrak{H}$ is *absolutely continuous* (*singular*) with respect to H if $(E(\lambda)u, u)$ is absolutely continuous (singular) in λ (with respect to the Lebesgue measure). The set of all $u \in \mathfrak{H}$ which are absolutely continuous (singular) with respect to H forms a subspace of $\mathfrak{H}$, which we call the *absolutely continuous* (*singular*) *subspace* with respect to H and denote by $\mathfrak{H}_{ac}(\mathfrak{H}_s)$. These two subspaces are orthogonal complements to each other and reduce H. The part of H in $\mathfrak{H}_{ac}(\mathfrak{H}_s)$ is called the *absolutely continuous* (*singular*) part of H and is denoted by H_{ac} (H_s)

Let H_j, $j = 0, 1$, be two selfadjoint operators in $\mathfrak{H}$ with the spectral representation $H_j = \int \lambda dE_j(\lambda)$, and let P_j be the projection on the absolutely continuous subspace $\mathfrak{H}_{j,ac}$ with respect to H_j. If one or both of the strong limits

$$(1.1) \qquad W_\pm = W_\pm(H_1, H_0) = s - \lim_{t \to \pm\infty} \exp(itH_1) \exp(-itH_0) P_0$$

exist(s), it is (they are) called the (*generalized*) *wave operator*(s).

Received January 8, 1964. This work was sponsored (in part) by Office of Naval Research Contract 222 (62).

W_+ is, whenever it exists, a partial isometry on $\mathfrak{H}$ with initial set $\mathfrak{H}_{0,ac}$ and final set $\mathfrak{M}_+$ contained in $\mathfrak{H}_{1,ac}$. $\mathfrak{M}_+$ reduces H_1, and the part of H_1 in $\mathfrak{M}_+$ is unitarily equivalent to $H_{0,ac}$, with

$$E_1(\lambda) W_+ = W_+ E_0(\lambda) , \qquad -\infty < \lambda < +\infty \tag{1.2}$$

The wave operator W_+ will be said to be *complete* if the final set $\mathfrak{M}_+$ coincides with $\mathfrak{H}_{1,ac}$.

W_+ has the property that, whenever $W_+(H_1, H_0)$ and $W_+(H_2, H_1)$ exist, then $W_+(H_2, H_0)$ exists and is equal to $W_+(H_2, H_1) W_+(H_1, H_0)$. If both $W_+(H_1, H_0)$ and $W_+(H_0, H_1)$ exist, then they are complete and are the adjoints to each other.

Similar results hold for W_+ replaced by W_-.

If $H_1 - H_0$ is small in the sense that $H_1 = H_0 + V$ with V belonging to the trace class $\boldsymbol{T}$ of operators on $\mathfrak{H}$, then both $W_\pm(H_1, H_0)$ exist and are complete. The main purpose of the present paper is to prove some generalizations of this theorem, which involve what we shall call the *principle of invariance of wave operators.* Roughly speaking, this principle asserts that the wave operators $W_\pm(\phi(H_1), \phi(H_0))$ exist for an "arbitrary" function ϕ and are independent of ϕ for a wide class of functions ϕ. Its precise formulation is given in Theorems 1 and 2 proved below.

The proof of these theorems is rather simple, depending essentially on a single inequality proved in a previous paper (Kato [5]). It will be noted that Theorem 2 contains as special cases most of the sufficient conditions for the existence and completeness of wave operators obtained in recent years (see Kuroda [6, 7], Birman [1, 2], Birman-Krein [3].

2. Principle of invariance of wave operators. Consider the wave operators $W_\pm(\phi(H_1), (\phi(H_0))$ where ϕ is a real-valued, Borel measurable function on $(-\infty, +\infty)$. The principle of invariance asserts that these wave operators do not denoted on ϕ. Of course certain restrictions must be imposed on ϕ and on the relation between H_0 and H_1 To this end it is convenient to introduce a certain class of functions.

DEFINITION. A real-valued function ϕ on $(-\infty, +\infty)$ is said to be of class (M) if the whole interval $(-\infty, +\infty)$ has a partition into a finite number of open intervals I_k, $k = 1, \cdots, r$, and their end points with the following properties: on each I_k, ϕ is strictly monotone and differentiable, with the derivative ϕ' continuous, $\phi' \neq 0$ and (locally) of bounded variation. $\{I_k\}$ will be called a system of intervals associated with ϕ (such a system is not unique).

THEOREM 1. *Let H_0, H_1 be selfadjoint operators such that $H_1 =$*

$H_0 + V$ with $V \in \boldsymbol{T}$. If ϕ is of class (M), $W'_\pm = W_\pm(\phi(H_1), \phi(H_0))$ exist and are complete. Furthermore, $W'_\pm$ are "piecewise equal" either to $W_\pm = W_\pm(H_1, H_0)$ or to $W_\mp$, in the sense that

$$(W'_\pm - W_\pm)E_0(I_k) = 0 \text{ or } (W'_\pm - W_\mp)E_0(I_k) = 0, k = 1, \cdots, r,$$

according as ϕ is increasing or decreasing on I_k. In particular, $W'_\pm = W_\pm (W'_\pm = W_\mp)$ if ϕ is increasing (decreasing) in each I_k, $k= 1, \cdots, r$. (Here $\{I_k\}$ is a system of intervals associated with $\phi \in (M)$ and $E_0(I) = E_0(\beta - 0) - E_0(\alpha)$ if $I = (\alpha, \beta)$.)

Proof. It is known (see Kato [5]) that $W_\pm$ exist under the assumptions of the theorem.

We take a fixed I_k and assume that ϕ is increasing on I_k. We use the inequality (2.9) of the paper cited, which reduces for $s = 0$ to

$$(2.1) \qquad \| (W_+ - 1)x \| \leqq (8\pi m^2 \| V \|_1)^{1/4} \times \left(\int_0^{+\infty} \| |V|^{1/2} \exp(-itH_0)x \|^2 dt\right)^{1/4},$$

where $x \in \mathfrak{H}_{0,ac}$ is subjected to the condition that $d(E_0(\lambda)x, x)/d\lambda \leqq m^2$ almost everywhere. Here $|V|$ is the nonnegative square roof of V^2 and $\| V \|_1$ denotes the trace norm of V, which is finite by assumption.

Now let $u \in \mathfrak{H}_{0,ac}$ be such that $E_0(I_k)u = u$ and $d(E_0(\lambda)u, u)/d\lambda \leqq m^2$. We note that such u with finite m^2 form a dense subset of $E_0(I_k)\mathfrak{H}_{0,ac} = E_0(I_k)P_0\mathfrak{H}$ (see a similar proposition in loc. cit. when I_k is the whole interval). If we set $x=\exp(-is\phi(H_0))u$, we have $(E_0(\lambda)x, x)=(E_0(\lambda)u,u)$ so that the assumptions on x stated above are satisfied. Hence (2.1) gives

$$(2.2) \qquad \| (W_+ - 1) \exp(- is\phi(H_0))u \| \leqq (8\pi m^2 \| V \|_1)^{1/4}\eta(s)^{1/4},$$

$$(2.3) \qquad \eta(s) = \int_0^{+\infty} \| |V|^{1/2} \exp(- itH_0 - is\phi(H_0))u \|^2 dt = \sum_{n=1}^{\infty} |c_n| \int_0^{+\infty} |(\exp(- itH_0 - is\phi(H_0))u, f_n)|^2 dt,$$

where $\{f_n\}$ is a complete orthonormal system of eigenvectors of V and the c_n are the associated eigenvalues.

The integrals on the right of (2.3) have the form (A1) of Appendix, where $w(\lambda)$ is to be replaced by $d(E_0(\lambda)u, f_n)/d\lambda$ which belongs to $L^2(I_k)$ with L^2-norm not exceeding m (see loc. cit.). Therefore, each term on the right of (2.3) tends to 0 for $s \to +\infty$ (Lemma A3, Appendix). On the other hand, the series on the right of (2.3) is majorized by the convergent series $2\pi m^2 \sum |c_n| = 2\pi m^2 \| V \|_1$. Hence $\eta(s) \to 0$ for $s \to +\infty$ and the left member of (2.2) must tend to 0 for $s \to +\infty$. Since $(W_+ - 1)\exp(-it\phi(H_0))$ is uniformly bounded and the set of u

with the above properties is dense in $E_0(I_k)P_0\mathfrak{H}$ as remarked above, it follows that $(W_+-1)\exp(-is\phi(H_0))P_0E_0(I_k) \to 0$ strongly for $s \to +\infty$. But we have $W_+\exp(-is\phi(H_0)) = \exp(-is\phi(H_1))W_+$ by (1.2). On multiplying the above result from the left with $\exp(is\phi(H_1))$, we thus obtain

$$(2.4) \qquad s-\lim_{s\to+\infty}\exp(is\phi(H_1))\exp(-is\phi(H_0))P_0E_0(I_k) = W_+P_0E_0(I_k) = W_+E_0(I_k) \qquad \text{if } \phi \text{ is increasing on } I_k .$$

Similarly we can show that

$$(2.4') \qquad s-\lim_{s\to+\infty}\exp(is\phi(H_1))\exp(-is\phi(H_0))P_0E_0(I_k) = W_-E_0(I_k) \qquad \text{if } \phi \text{ is decreasing on } I_k.$$

Since $P_0E_0(\lambda)$ is continuous in λ, we have $\sum_k P_0E_0(I_k) = P_0$. Adding (2.4) or (2.4′) for $k = 1, \cdots, r$, we thus arrive at the result

$$(2.5) \qquad s-\lim_{s\to+\infty}\exp(is\phi(H_1))\exp(-is\phi(H_0))P_0 = \sum_{k=1}^{r} W_{(\pm)}E_0(I_k) ,$$

where $W_{(\pm)}$ means that $W_+(W_-)$ should be taken if ϕ is increasing (decreasing) on I_k.

(2.5) shows that the wave operator $W_+(\phi(H_1), \phi(H_0))$ exists and is equal to the right member; it should be noted that the absolutely continuous subspace for $\phi(H_0)$ is identical with $\mathfrak{H}_{0,ac} = P_0\mathfrak{H}$ (Lemma A5, Appendix). Similar results hold for $W_-(\phi(H_1), \phi(H_0))$; we have only to take the opposite choice for $W_{(\pm)}$ in (2.5). These wave operators are complete since they also exist when H_0 and H_1 are exchanged.

3. Generalization. Let us consider a question which is in a sense converse to Theorem 1. Suppose $\psi(H_1) - \psi(H_0)$ belongs to $\boldsymbol{T}$ for some function ψ; then do the wave operators $W_\pm(H_1, H_0)$ exist?

The answer to this question is quite simple if ψ is of class (M) and, in addition, *univalent*. Then the inverse function exists, with domain Δ consisting of a finite number of open intervals and a finite number of points. This inverse function can be extended to a function $\hat{\psi}$ of class (M) by setting, for example, $\hat{\psi}(\lambda) = \lambda$ on the complement of Δ. Therefore, $W_\pm(H_1, H_0) = W_\pm(\hat{\psi}(\psi(H_1)), \hat{\psi}(H_0)))$ exist and are complete by Theorem 1.

If ψ is not univalent, we do not know whether the same resalts hold. But we can show that this is true if there is an *approximate univalent sequence* $\{\psi_n\}$ of functions of class (M) such that $\psi_n(H_1) - \psi_n(H_0) \in \boldsymbol{T}$. We call $\{\psi_n\}$ an approximate univalent sequence if ψ_n is univalent on $(-n, n)$, $n = 1, 2, \cdots$

More generally, we can prove

THEOREM 2. *Let H_0, H_1 be selfadjoint and let there exist an approximate univalent sequence $\{\psi_n\}$ of functions of class (M) such that $\psi_n(H_1) = \psi_n(H_0) + V_n$ with $V_n \in \boldsymbol{T}, n = 1, 2, \cdots$ Then, for any $\phi \in (M)$, the wave operators $W'_\pm = W_\pm(\phi(H_1), \phi(H_0))$ exist and are complete. In particular, $W_\pm = W_\pm(H_1, H_0)$ exist and are complete. $W'_\pm$ are piecewise equal either to $W_\pm$ or to $W_\mp$ in the sense stated in Theorem 1.*

Proof. I. The restriction of ψ_n to $(-n, n)$ has inverse function, which can be extended to a $\hat{\psi}_n \in (M)$ in the same way as above.

Set $\phi_n = \phi \circ \hat{\psi}_n \circ \psi_n$; then $\phi_n(\lambda) = \phi(\lambda)$ for $\lambda \in (-n, n)$, and $\phi_n \in (M)$ by Lemma A4 (Appendix). We define the following selfadjoint operators, all functions of $H_j, j = 0, 1$:

$$\begin{aligned} &\psi_n(H_j) = L_{nj} , &&(\hat{\psi}_n \circ \psi_n)(H_j) = H_{nj} , \\ &\phi_n(H_j) = K_{nj} = \int \lambda dF_{nj}(\lambda) , &&\phi(H_j) = K_j = \int \lambda dF_j(\lambda) . \end{aligned} \tag{3.1}$$

Since $K_{nj} = (\phi \circ \hat{\psi}_n)(L_{nj})$ by operational calculus (see Stone [8], Theorem 6.9), where $\phi \circ \hat{\psi}_n \in (M)$ and $L_{n1} = L_{n0} + V_n$, $V_n \in \boldsymbol{T}$, it follows from Theorem 1 that $W'_{n\pm} = W_\pm(K_{n1}, K_{n0})$ exist and are complete.

II. For any function ψ of class (M), $\psi(\pm\infty) = \lim_{\lambda\to\pm\infty} \psi(\lambda)$ exist (the values $\pm\infty$ being permitted for these limits). Thus $\phi_n(\pm\infty)$ and $(\hat{\psi}_n \circ \psi_n)(\pm\infty)$ exist. By replacing $\{\phi_n\}$ by a suitable subsequence (and correspondingly for $\{\psi_n\}$ and $\{\hat{\psi}_n\}$), we may assume that $\alpha_\pm \lim_{n\to\infty}\phi_n(\pm\infty)$ and $\beta_\pm = \lim_{n\to\infty} (\hat{\psi}_n \circ \psi_n)(\pm\infty)$ exist ($\pm\infty$ being permitted for these limits).

Let J be an open interval such that $\alpha_\pm$ and $\phi(\pm\infty)$ are exterior to J, and let $S = \phi^{-1}(J)$, $S_n = \phi_n^{-1}(J)$. S and S_n are unions of a finite number of open intervals and of points. Since $K_j\phi(H_j)$ and $K_{nj} = \phi_n(H_j)$, we have (we denote by $E_j(S)$ the spectral measure determined from $\{E_j(\lambda)\}$)

$$F_j(J) = E_j(S) , \qquad F_{nj}(J) = E_j(S_n) , \qquad j = 0, 1. \tag{3.2}$$

S is bounded since $\phi(\pm\infty)$ are exterior to J. Similarly, S_n is bounded if n is sufficiently large, since $\alpha_\pm$ are exterior to J.

Take an n so large that S_n is bounded and $S \subset (-n, n)$. Since $\phi_n(\lambda) = \phi(\lambda)$ for $\lambda \in (-n, n)$, we have $S = (-n, n) \cap S_n$. Further take an $m > n$ such that $S_n \subset (-m, m)$. We have $S = (-m, m) \cap S_m$ as above, so that $S_m \cap S_n = S_m \cap (-m, m) \cap S_n = S \cap S_n = S$. Hence

$$\begin{aligned} F_{nj}(J)F_{mj}(J) &= F_j(S_n)E_j(S_m) \\ &= E_j(S_n \cap S_m) = E_j(S) = F_j(J) . \end{aligned} \tag{3.3}$$

III. Now we have, for any $u \in \mathfrak{H}_{0,ac} = P_0\mathfrak{H}$,

$$\begin{aligned}(3.4)\qquad &\exp(itK_{n1})(1-F_{n1}(J))\exp(-itK_{n0})P_0F_0(J)\\ &=(1-F_{n1}(J))\exp(itK_{n1})\exp(-itK_{n0})P_0F_0(J)\\ &\to(1-F_{n1}(J))W'_{n+}F_0(J)\qquad \text{strongly for } t\to+\infty\ .\end{aligned}$$

Since $(1-F_{n1}(J))W'_{n+}=W'_{n+}(1-F_{n0}(J))$ by (1.2) applied to W'_{n+}, and since $F_0(J)\leqq F_{n0}(J)$ by (3.3), the last member of (3.4) vanishes. On the other hand $\exp(-itK_{n0})F_0(J)=\exp(-itK_0)F_0(J)$ since $\phi_n(\lambda)=\phi(\lambda)$ for $\lambda\in(-n,n)$ and $F_0(J)=E_0(S)\leqq E_0((-n,n))$. On multiplying (3.4) from the left by $\exp(-itK_{n1})$, we thus obtain

$$(3.5)\qquad s-\lim_{t\to+\infty}(1-F_{n1}(J))\exp(-itK_0)P_0F_0(J)=0\ .$$

The same is true when n is replaced by the $m>n$ considered above. Now multiply the latter from the left by $F_{n1}(J)$ and add to (3.5). In view of (3.3), we then obtain

$$(3.6)\qquad s-\lim_{t\to+\infty}(1-F_1(J))\exp(-itK_0)P_0F_0(J)=0\ .$$

Multiply again (3.6) from the left by $\exp(itK_1)$; then

$$\begin{aligned}(3.7)\qquad &s-\lim_{t\to+\infty}\exp(itK_1)\exp(-itK_0)P_0F_0(J)\\ &=s-\lim_{t\to+\infty}F_1(J)\exp(itK_{n1})\exp(-itK_{n0})P_0F_0(J)\\ &=F_1(J)W'_{n+}F_0(J)\ ,\end{aligned}$$

where we have again used the relation

$$\exp(-itK_0)F_0(J)=\exp(-itK_{n0})F_0(J)$$

and similarly $\exp(itK_1)F_1(J)=\exp(itK_{n1})F_1(J)=F_1(J)\exp(itK_{n1})$.

(3.7) shows that $\lim_{t\to+\infty}\exp(itK_1)\exp(-itK_0)u$ exists and is equal to $F_1(J)W'_{n+}u$ whenever u belongs to $P_0F_0(J)\mathfrak{H}$, where J is any interval with the four points $\alpha_\pm$ and $\phi(\pm\infty)$ in its exterior. Since such u forms a dense set in $P_0\mathfrak{H}$, the existence of $W'_+=W_+(K_1,K_0)$ has been proved. The existence of W'_- can be proved in the same way. Since K_0 and K_1 can be exchanged, all these wave operators are complete.

Incidentally, it follows from (3.7) that $W'_+u=F_1(J)W'_{n+}u$ for $u\in P_0F_0(J)\mathfrak{H}$. But $\|W'_+u\|=\|u\|=\|W'_{n+}u\|$ since W'_+ and W'_{+n} are isometric on $P_0\mathfrak{H}$. Since $F_1(J)$ is a projection, we must have $W'_+u=W'_{n+}u$. Similar result holds for W'_-. Thus

$$(3.8)\qquad (W'_\pm-W'_{n\pm})F_0(J)=0\ .$$

Note that this is true for sufficiently large n (depending on J).

IV. To prove the piecewise equality of $W'_\pm$ and $W_\pm$ or $W_\mp$, let I_k be one of the intervals associated with $\phi\in(M)$. We may assume that $\phi' > 0$ on I_k; we have to show that $(W'_\pm - W_\pm)E_0(I_k) = 0$. For this it suffices to show that $(W'_\pm - W_\pm)E_0(I) = 0$ for any finite subinterval I of I_k; we may further assume that $\beta_\pm$ are exterior to I and $\alpha_\pm$, $\phi(\pm\infty)$ are exterior to the interval $\phi(I)$.

We set $J = \phi(I)$ and apply the preceding results to J. Since $S = \phi^{-1}(J) \supset I$, we have $E_j(I) \leqq E_j(S) = F_j(J)$ and hence by (3.8)

$$(W'_\pm - W'_{n\pm})E_0(I) = 0 \tag{3.9}$$

for sufficiently larg n.

We have similar results when $\phi(\lambda)$ is replaced by the identity function λ (since $\beta_\pm$ and $\pm\infty$ are exterior to I). Then $W'_\pm$, $W'_{n\pm}$ are to be replaced respectively by $W_\pm = W_\pm(H_1, H_0)$ and $W_{n\pm} = W_\pm(H_{n1}, H_{n0})$. Thus

$$(W_\pm - W_{n\pm})E_0(I) = 0 \tag{3.10}$$

for sufficiently large n.

We may assume that n is so large that $I \subset (-n, n)$. I can be expressed as the union of a finite number of subintervals $\varDelta_p$ (and a finite number of points) in each of which ψ_n is monotonic. Then $\hat{\psi}_n$ is monotonic on $\varDelta'_p = \psi_n(\varDelta_p)$ since ψ_n is univalent on $(-n, n)$. $\phi \circ \hat{\psi}_n$ is also monotonic on $\varDelta'_p$ since $\phi' > 0$ on $\hat{\psi}_n(\varDelta'_p) = \varDelta_p$; it is increasing or decreasing with $\hat{\psi}_n$. Since $K_{nj} = (\phi \circ \hat{\psi}_n)(L_{nj})$, $H_{nj} = \hat{\psi}_n(L_{nj})$ and $L_{n1} = L_{n0} + V_n$, $V_n \in \boldsymbol{T}$, it follows from Theorem 1 that $(W'_{n\pm} - W_{n\pm})E_0(\varDelta_p) = 0$; note that $E_0(\varDelta_p) \leqq E_0(\psi_n^{-1}(\varDelta'_p)) = G_0(\varDelta'_p)$ where $\{G_0(\lambda)\}$ is the resolution of the identity for $L_{n0} = \psi_n(H_0)$. Adding the results obtained for $p = 1, 2, \cdots$, we have

$$(W'_{n\pm} - W_{n\pm})E_0(I) = 0\ . \tag{3.11}$$

The desired result $(W'_\pm - W_\pm)E_0(I) = 0$ follows from (3.9), (3.10) and (3.11).

4. Applications.

A number of sufficient conditions for the existence and completeness of wave operators can be deduced from Theorem 1 or 2. We shall mention only a few.

(a) Let neither H_0 nor H_1 have the eigenvalue 0. If $H_1^{-p} = H_0^{-p} + V$ with $V \in \boldsymbol{T}$ for some *odd* integer p, then $W_\pm(\phi(H_1), \phi(H_0))$ exist and are complete for any $\phi \in (M)$.

The proof follows by applying Theorem 2 with $\psi_n = \psi$ (independent of n) where $\psi(\lambda) = \lambda^{-p}$ for $\lambda \neq 0$ and $\psi(0) = 0$.

(b) In (a) we may allow *even* integers p if we assume in addition that H_0 and H_1 are nonnegative.

In this case we need only to replace the above ψ by $\psi(\lambda) = (\operatorname{sign}\lambda)\,|\,\lambda\,|^{-p}$ for $\lambda \neq 0$.

(c) Let $(H_1 - \zeta)^{-1} - (H_0 - \zeta)^{-1} \in \boldsymbol{T}$ for some nonreal complex number ζ. Then $W_\pm(\phi(H_1), \phi(H_0))$ exist and are complete for any $\phi \in (M)$.

For the proof we first note that, if the assumption is true for some $\zeta = \zeta_0$, then it is true also for all nonreal ζ. This can be seen first for $|\,\zeta - \zeta_0\,| < |\operatorname{Im}\zeta_0\,|$ by considering the Neumann series for the resolvents. The result can then be extended to all ζ of the half-plane $(\operatorname{Im}\zeta)(\operatorname{Im}\zeta_0) > 0$ by a standard procedure. The other half-plane can be taken care of by considering the adjoints.

Set now $\psi_n(\lambda) = -i[(n - i\lambda)^{-1} - (n + i\lambda)^{-1}] = 2\lambda(n^2 + \lambda^2)^{-1}$. It follows from the above remark that $\psi_n(H_1) - \psi_n(H_0) \in \boldsymbol{T}$. But it is easy to see that $\{\psi_n\}$ is an approximate univalent sequence of functions of class (M). Hence the proposition follows by Theorem 2.

(b) It should be remarked that the existence of $W_\pm(\phi(H_1), \phi(H_0))$ implies the existence of

$$s - \lim_{n\to\pm\infty} U_1^n U_0^{-n} = W_\pm(H_1, H_0)\ , \tag{4.1}$$

where $U_j = (H_j - i)(H_j + i)^{-1}$ is the Cayley transform of H_j. In fact, $U_j = \exp(i\phi(H_j))$ where $\phi(\lambda) = -2\operatorname{arccot}\lambda$, and ϕ belongs to (M), being strictly increasing on $(-\infty, +\infty)$.

Appendix. We prove here some lemmas which are used in the text.

LEMMA A1. *Let f, g be complex-valued, continuous functions on a closed interval $[a, b]$. Let f be of bounded variation with total variation V_f. Let $G(\lambda) = \int_a^\lambda g(\lambda)d\lambda$ and let $M_G = \max|\,G(\lambda)\,|$, $M_f = \max|\,f(\lambda)\,|$. Then $\left|\int_a^b f(\lambda)g(\lambda)d\lambda\right| \leqq (M_f + V_f)M_G$.*

The proof is simple and will be omitted.

LEMMA A2. *Let ϕ be a real-valued differentiable function on $[a, b]$ such that the derivative ϕ' is continuous, positive and of bounded variation. We have for any $t, s > 0$*

$$\left|\int_a^b \exp(it\lambda - is\phi(\lambda))d\lambda\,\right| \leqq \frac{2(c + V_{\phi'})}{c(t + cs)}\ ,$$

where $c = \min\phi'(\lambda) > 0$ and $V_{\phi'}$ is the total variation of ϕ'.

Proof. The integral in question is equal to

$$\int_a^b i(t + s\phi'(\lambda))^{-1}(d/d\lambda)\exp(-it\lambda - is\phi(\lambda))d\lambda .$$

We apply Lemma A1 to estimate this integral, setting $f(\lambda) = i(t + s\phi'(\lambda))^{-1}$ and $g(\lambda) = (d/d\lambda)\exp(-it\lambda - is\phi(\lambda))$. Then $M_f = (t+cs)^{-1}$, $M_G \leqq 2$ and it is easily seen that $V_f \leqq sV_{\phi'}/(t + cs)^2 \leqq V_{\phi'}/c(t + cs)$. This proves the desired inequality.

LEMMA A3. *Let ϕ be of class (M) with an associated system of of intervals $\{I_k\}$ (see definition in text). For a fixed k, let $w \in L^2(I_k)$. If ϕ is increasing on I_k, we have*

$$\text{(A1)} \qquad \int_0^{+\infty} dt \left| \int_{-\infty}^{+\infty} \exp(-it\lambda - is\phi(\lambda))w(\lambda)d\lambda \right|^2 \longrightarrow 0 , \qquad s \rightarrow +\infty .$$

If ϕ is decreasing on $I_{k'}$ (A1) is true if $\int_0^{+\infty} dt$ is relpaced by $\int_{-\infty}^0 dt$.

Proof. We may assume that $w \in L^2(-\infty, +\infty)$, on setting $w(\lambda) = 0$ for λ outside I_k. Let H be the selfadjoint operator $Hu(\lambda) = \lambda u(\lambda)$ acting in $L^2(-\infty, +\infty)$, and let U be the unitary operator defined by the Fourier transformation. The inner integral of (A1) represents the function $(U\exp(-is\phi(H))w(t)$, and the left member of (A1) is equal to $\| EU\exp(-is\phi(H))w \|^2$, where E is the projection of $L^2(-\infty, +\infty)$ onto the subspace consisting of all functions that vanish on $(-\infty, 0)$. Thus (Al) is equivalent to that $EU\exp(-is\phi(H))w \rightarrow 0$, $s \rightarrow +\infty$. Since $EU\exp(-is\phi(H))$ is uniformly bounded with norm $\leqq 1$, it suffices to prove (A1) for all w belonging to a fundamental subset of $L^2(I_k)$. Thus we may restrict ourselves to considering only characteristic functions w of closed finite subintervals $[a, b]$ of I_k.

Assume that ϕ is increasing on I_k. If we denote by $v_s(t)$ the inner integral of (A1) for the characteristic function w of $[a, b] \subset I_k$, we have by Lemma A2

$$|v_s(t)| \leqq \frac{2(c + V_{\phi'})}{c(t + cs)} \text{ so that } \int_0^{+\infty} |v_s(t)|^2 dt \leqq \frac{4(c + V_{\phi'})^2}{c^3 s} \longrightarrow 0$$

for $s \rightarrow +\infty$, where c is the minimum of $\phi'(\lambda)$ on $[a, b]$ and $V_{\phi'}$ is the total variation of ϕ' on $[a, b]$. A similar proof applies to the case $\phi' < 0$ on I_k, with $\int_0^{+\infty} dt$ replaced by $\int_{-\infty}^0 dt$.

LEMMA A4. *Let ϕ, ψ be of class (M). Then the composed function $\phi \circ \psi$ also belongs to (M), and there exists a system of intervals associated with $\phi \circ \psi$ such that, in each interval of the system, both ψ and $\phi \circ \psi$ are monotonic.*

Proof. Let $\{I_k\}$ and $\{J_h\}$ be systems of intervals associated with ϕ and ψ, respectively. For each h, ψ maps J_h one-to-one onto an open interval J'_h. Let J_{kh} be the inverse image under this map of $J'_h \cap h_k$. Obviously all J_{kh} are open and mutually disjoint, and cover the whole interval $(-\infty, +\infty)$ except for a finite number of points. It is easy to see that $\phi \circ \psi$ is monotonic and continuously differentiable on each J_{kh}, with $(\phi \circ \psi)'(\lambda) = \phi'(\psi(\lambda))\psi'(\lambda)$. Furthermore, $(\phi \circ \psi)'$ is locally of bounded variation on J_{kh}, for the same is true with ϕ' and ψ' by assumption. The intervals J_{kh} form a system stated in the lemma.

LEMMA A5. *Let ϕ be of class (M). For any selfadjoint operator H, the absolutely continuous subspace for $\phi(H)$ is identical with the absolutely continuous subspace for H.*

Proof. Let $H = \int \lambda dE(\lambda)$, $\phi(H) = \int \lambda dF(\lambda)$ be the spectral representations of the operators considered. We denote by $E(S)$, $F(S)$ the spectral measures constructed from $\{E(\lambda)\}$, $\{F(\lambda)\}$, respectively. For any Borel subsets S of the real line, we have $F(S) = E(\phi^{-1}(S))$. If $|S| = 0$ (we denote by $|S|$ the Lebesgue measure of S), then $|\phi^{-1}(S)| = 0$ by the properties of $\phi \in (M)$, so that $F(S)u = 0$ if u is absolutely continuous with respect to H. On the other hand, $F(\phi(S)) = E(\phi^{-1}(\phi(S)) \geqq E(S)$. If $|S| = 0$, we have $|\phi(S)| = 0$ so that $\|E(S)u\| \leqq \|F(\phi(S))u\| = 0$ if u is absolutely continuous with respect to $\phi(H)$. This proves the lemma.

BIBLIOGRAPHY

1. M. Sh. Birman, *On the conditions for the existence of wave operators,* Doklady Akad. Nauk USSR **143** (1962), 506-509.
2. ———, *A test for the existence of wave operators,* Doklady Akad. Nauk USSR **147** (1962), 1008-1009.
3. M. Sh. Birman, and M.G. Krein, *On the theory of wave and scattering operators,* Doklady Akad. Nauk USSR **144** (1962), 475-478.
4. T. Kato, *On finite-dimensional perturbations of self-adjoint operators,* J. Math. Soc. Japan **9** (1957), 239-249.
5. ———, *Perturbation of continuous spectra by trace class operators,* Proc. Japan Acad. **33** (1957), 260-264.
6. S. T. Kuroda, *On the existence and the unitary property of the scattering operators,* Nuovo Cimento **12** (1959), 431-454.
7. ———, *Perturbation of continuous spectra by unbounded operators, I and II,* J. Math. Soc. Japan **11** (1959), 247-262; **12** (1960), 243-257.
8. M. H. Stone, *Linear transformations in Hilbert space and their applications to analysis,* AMS Colloq. Publ., 1932.