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EUCLIDEAN FIBERINGS OF SOLVMANIFOLDS

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This paper is concerned with the problem of finding conditions on a solvable Lie group G and a closed subgroup H which are sufficient for G/H to have topological structure of a fiber bundle with compact base space and euclidean fiber (if this is the case, we say that G/H has a euclidean fibering). The main results are the following two theorems.

THEOREM 5.3. **Let G be a connected solvable linear Lie group, and H a closed subgroup which splits in G. Then G/H has a euclidean fibering.**

THEOREM 5.4. **Let G be a connected solvable matrix group, and assume that G is of finite index in its algebraic group hull. Then for any closed subgroup H of G, G/H has a euclidean fibering.**

To the best of the author's knowledge, these are the first results on existence of such fiberings which do not require that the isotropy subgroup H have a finite number of connected components.

A solvmanifold is a Hausdorff space X on which a solvable Lie group G acts transitively. It is well-known that, if $x \in X$ and H is the subgroup of G leaving x fixed, then X is homeomorphic to the coset space G/H. The problem investigated in this paper is that of finding conditions on G and H which are sufficient for G/H to have the topological structure of a fiber bundle with compact base space and euclidean fiber, and such that the action of the structure group on the fiber is equivalent to the action of a linear group. Such a fibering will be called a euclidean fibering. We are led to suspect the existence of euclidean fiberings by Mostow's result of [4], in which he shows that G/H is covered a finite number of times by the Cartesian product of a compact solvmanifold and a euclidean space.

Our main results on euclidean fiberings are the two theorems quoted above.

In addition to these results on euclidean fiberings of solvmanifolds, we obtain in § 3 generalizations of some results of Togo on splittable matrix groups (cf. [9], [10], [11]). Togo [10] has proved Theorem 3.1 for the case that H is a Zariski-connected C^∞-group over an algebraically closed perfect field, and has given an example (p. 319

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in [11]) to show that the theorem is not generally true if $H \cap N$ is not connected.

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2. Notation and preliminaries. If G is a Lie group, we shall always mean by $\mathfrak{G}$ the Lie algebra of G, and Ad will denote the adjoint representation of G by automorphisms of $\mathfrak{G}$; if H is a subgroup of G, AdH denotes the image of H in AdG under Ad.

If G is a connected Lie group (we shall also say, equivalently, that G is an analytic group), and H is a subgroup, H is said to be full in G if no proper analytic subgroup of G contains H. H is said to be uniform in G if $G/\bar{H}$ is compact, $\bar{H}$ being the topological closure of H. It is shown in [3] that if G is nilpotent, and simply connected, then there is a unique subgroup $\tilde{H}$ in which the subgroup H is full, and $\tilde{H}$ may also be characterized as the unique analytic subgroup of G in which H is uniform.

We shall often use without explicit reference the following fact: Let G be a Lie group, H a closed subgroup, and F a closed subgroup invariant under inner automorphism by elements of H, and such that $F \cap H$ is uniform in F. Then $HF = \{h \cdot f \mid h \in H, f \in F\}$ is a closed subgroup of G. Clearly HF is a subgroup, and to see that it is closed we need only to observe that HF/H, being a continuous image of the compact space $F/F \cap H$, is a compact, hence closed, subset of G/H, and consequently the inverse image HF of HF/H in G is closed.

If ρ is a representation of the Lie group G, we say that a subgroup H of G is ρ-reductive if $\rho(H)$ is a reductive (i.e., fully reducible) subgroup of $\rho(G)$. It is known that a connected solvable matrix group is reductive if and only if it is abelian and consists only of semisimple matrices (a matrix is called semisimple if its minimal polynomial has no repeated roots).

All linear transformations considered will be on vector spaces over the real or complex numbers. A linear transformation x is said to be unipotent if, for some integer n, $(x - I)^n = 0$, I being the identity. Any nonsingular linear transformation x may be written uniquely in the form $x = su = us$, where s is semisimple and u is unipotent; s and u are called respectively the semisimple and unipotent parts of x, or the Jordan components of x.

By an algebraic matrix group we shall mean a subgroup of $GL(n, R)$ (resp. $GL(n, C)$) which is the intersection of some algebraic variety in E^{n^2} with $GL(n, R)$ (or in E^{2n^2} with $GL(n, C)$), $GL(n)$ being embedded in euclidean space in the usual way. If H is a matrix group, the intersection of all algebraic groups containing H is called the algebraic group hull of H. If x and y are in $GL(n)$, y is called an adherent of x if y is in every algebraic group containing x. In particular, the Jordan components of x are adherents of x.

We shall say that the space X has a euclidean fibering if X is the total space of a fiber bundle having a compact base space, euclidean fiber, and a group whose action on the fiber is equivalent to the action of a linear group.

The term Γ-invariant exp-set will be used as defined in [5], namely: Let Γ be a group of automorphisms of a Lie group G, S be a subset of G, and let $d\Gamma$ denote the group of automorphisms of $\mathfrak{G}$ induced by Γ. S is called a Γ-invariant exp-set if there exist linearly independent subspaces $\mathfrak{S}_1, \cdots, \mathfrak{S}_n$ of $\mathfrak{G}$, each invariant under $d\Gamma$, such that the mapping $s_1 + s_2 + \cdots + s_n \to \exp s_1 \cdot \exp s_2 \cdots \exp s_n$ is a homeomorphism of $\mathfrak{S}_1 + \cdots + \mathfrak{S}_n = \mathfrak{S}$ onto S (s_i in $\mathfrak{S}_i$). Thus the operation of Γ on S is equivalent to the operation of the linear group $d\Gamma$ on $\mathfrak{S}$. If H is a subgroup of G and Γ_H is the group of inner automorphisms of G by elements in H, we shall also call a Γ_H-invariant exp-set an H-invariant exp-set.

Topological terms applied to matrix groups will always refer to the euclidean topology induced on $GL(n)$, not the Zariski topology, unless otherwise stated.

3. Splittability of solvable groups. In this section, we consider different definitions of splittability for solvable groups. We first define splittability for linear groups.

DEFINITION. Let G be a solvable linear group, and H a subgroup.

(1) G is said to be Jordan-splittable if it contains the Jordan components of each of its elements;

(2) G is said to be splittable if $G = A \cdot N$ (semi-direct), where A is a maximal abelian subgroup of semisimple elements of G, and N is the subgroup of unipotent elements in G;

(3) H is said to split in G if there is a semi-direct decomposition $G = A \cdot N$ as in (2), with $H = (H \cap A) \cdot (H \cap N)$ (semi-direct).

It has been shown by Togo in [9] that for connected linear groups, (1) and (2) above are equivalent. We generalize this result in the following theorem.

THEOREM 3.1. *Let G be a connected splittable solvable linear group, and H a subgroup of G with $H \cap N$ connected, where N is the subgroup of unipotent matrices in G. Then* (1), (2), *and* (3) *above are equivalent conditions on H. Furthermore, if any of these are* *satisfied, then maximal reductive subgroups of* H *are conjugate by inner automorphism from* $H \cap N$.

Proof. It is obvious that (3) implies (2). Conversely, if H is splittable, so that $H = A_H(H \cap N)$ (semi-direct), then A_H is contained in a maximal reductive subgroup A of G; since G is connected we have, by the result in [9], $G = A \cdot N$ (semi-direct), and $A_H = H \cap A$, so H satisfies (3).

Proof that (1) implies (2). Let $\tilde{H}$ be the algebraic group hull of H, and x a unipotent element of $\tilde{H}$, so x is an adherent of some element $y \in H$. Since x is a unipotent adherent of y, x is an adherent of the unipotent part of y [1]; but $H \cap N$, being a connected group of unipotent elements, is algebraic, so $x \in H \cap N$. Thus $H \cap N = \tilde{H} \cap N$. Let M_i be a maximal reductive subgroup of $\tilde{H}$ containing the maximal reductive subgroup A_i of H, $i = 1, 2$. By Theorem 7.1 in [7], there is $n \in \tilde{H} \cap N$ with $nM_1n^{-1} = M_2$; and $\tilde{H} = M_i \cdot (\tilde{H} \cap N)$ (semi-direct). Since $A_i = M_i \cap H$, and $n \in H$, it follows that $nA_1n^{-1} = A_2$, and $H = (H \cap M_i) \cdot (\tilde{H} \cap N)$. We have thus shown that (1) implies (2), and that (1) implies that maximal reductive subgroups of H are conjugate by inner automorphism from $H \cap N$.

Proof that (2) implies (1). Suppose $H = A_H \cdot (N \cap H)$ (semi-direct). If $\tilde{A}_H$ is the algebraic group hull of A_H, then by Proposition 5.5 in [1], $\tilde{H} = \tilde{A}_H \cdot (N \cap H)$ is algebraic, since $N \cap H$, being a connected group of unipotent elements, is algebraic. If $x \in H$, and $x = s \cdot u$ is the Jordan decomposition for x, then $u \in \tilde{H}$, so u, being unipotent, is in $N \cap \tilde{H} = N \cap H$. Thus $s = xu^{-1}$ is in H, and it follows that H is Jordan-splittable.

Definitions (1) and (2) are not equivalent unless $H \cap N$ is connected, as counterexamples of Togo show ([10], [11]). It is obvious, however, that (3) always implies (2), and the first part of the proof did not use the assumption that $H \cap N$ is connected, so (2) and (3) are equivalent for subgroups of connected splittable linear groups.

We now make analogous definitions for simply connected abstract groups.

DEFINITION. Let G be a connected, simply connected solvable Lie group, and H a subgroup.

(1′) H is said to be Jordan *Ad*-splittable if AdH, the image of H in the adjoint group of G, is Jordan-splittable;

(2′) H is said to be splittable if $H = A_H \cdot (H \cap N)$ (semi-direct) where A_H is an *Ad*-reductive subgroup of H, and N is the maximum analytic nilpotent normal subgroup of G.

(3′) H is said to split in G if $G = A \cdot N'$, $H = (A \cap H) \cdot (H \cap N')$ (both semi-direct), where A is Ad-reductive and N' is an analytic subgroup of N normal in G.

For a suitable faithful representation of G, Def. 3 is equivalent to Def. 3′. We deal in this paper with spaces G/H where G is a solvable linear group and H is a closed subgroup. When we refer to H splitting in G it will always be in the sense of Def. 3.

Taking $H = G$ in definitions (1′), (2′), we obtain equivalent conditions on G. To see this, observe that if G satisfies (1′), then AdG, being Jordan-splittable, can be written $AdG = A' \cdot AdN$, where A' is a maximal reductive subgroup of AdG. Letting $\mathfrak{A}$ be a complementary subspace to $\mathfrak{N}$ in $\mathfrak{G}$ invariant under A', it follows easily that $\mathfrak{G} = \mathfrak{A} + \mathfrak{N}$ (semi-direct), and that the analytic group A determined by $\mathfrak{A}$ is Ad-reductive, so G is splittable. Conversely, if $G = A \cdot N$ (semi-direct), with A and N being as in Definition (2′), then $AdG = AdA \cdot AdN$ (semi-direct), so by Theorem (3.1), AdG is Jordan-splittable.

We now prove a partial analogue of Theorem 3.1 for abstract groups.

THEOREM 3.2. *Let G be a connected, simply connected solvable splittable Lie group, and H a subgroup of G containing no proper analytic subgroup central in G. Assume that $H \cap N$ is connected, and that $Ad(H \cap N) = AdH \cap AdN$, where N is the maximum analytic nilpotent normal subgroup of G. Then H satisfies* (1′) *if and only if it satisfies* (2′), *and in this case maximal Ad-reductive subgroups of H are conjugate by inner automorphism from $H \cap N$.*

Proof that (1′) implies (2′). Let $h \in H$. Since AdH is Jordan-splittable, we can find $s', u \in H$ with Ads' semisimple, $Adu \in AdH \cap AdN$, and $Adh = Ads'Adu = AduAds'$. Since $AdH \cap AdN = Ad(H \cap N)$, we can assume $u \in H \cap N$. Now $Ad(h^{-1}s'u) = e$, so $h^{-1}s'u = z \in H \cap Z$, where Z is the center of G. Letting $s = s'z^{-1}$, we have $h = su$. Now $Ad(sus^{-1}u^{-1}) = e$, so $sus^{-1}u^{-1} \in Z \cap H \cap N$, which is an analytic subgroup of H central in G, and therefore reduces to the identity. Thus $su = us$. A similar argument shows that s and u are unique, if we require $u \in H \cap N$.

Let A_H be a maximal Ad-reductive subgroup of H. Then $Ad(A_H)$ is a maximal reductive subgroup of AdH. For suppose T is a maximal reductive subgroup of AdH containing $Ad(A_H)$, and let $t \in T$. There is $h \in H$ with $Adh = t$; for any $a \in A_H$, $Ad(aha^{-1}h^{-1}) = (Ada)t(Ada)^{-1}t^{-1} = e$, so $aha^{-1}h^{-1} \in H \cap N \cap Z = (e)$. Thus h commutes with all of A_H, and Adh is semisimple, so $h \in A_H$, and $t \in Ad(A_H)$, whence $T \subset Ad(A_H)$.

If A_H and A'_H are maximal Ad-reductive subgroups of H, then there is $n = Adu$, with $u \in H \cap N$, such that $n(AdA_H)n^{-1} = Ad(A'_H)$, so $Ad(uA_Hu^{-1}) = Ad(A'_H)$. Thus for any $a \in A_H$, there is $a' \in A'_H$ with $Ad(uau^{-1}a') = e$, so $(uau^{-1}a') \in Z \cap H$. Now any element of $Z \cap H$ is contained in every maximal Ad-reductive subgroup of H, so $uau^{-1}a' \in A'_H$, and hence $uau^{-1} \in A'_H$. It follows that $uA_Hu^{-1} \subset A'_H$, and, by maximality of A_H, $uA_Hu^{-1} = A'_H$. We have shown that maximal Ad-reductive subgroups of H are conjugate by inner automorphism from $H \cap N$.

To show that H is a semi-direct product of the form $A'_H(H \cap N)$, we observe that if $h = s \cdot u$ is the unique decompostion given above for h, then since s is contained in a maximal Ad-reductive subgroup of H, there is $n \in H \cap N$ with $nsn^{-1} \in A_H$, so $h = (nsn^{-1}) \cdot (ns^{-1}n^{-1}su)$ is a decomposition of the desired form. Since $A_H \cap (H \cap N)$ is contained in $H \cap N \cap Z$, which consists of the identity alone, the decompostion is semi-direct.

Proof that (2′) implies (1′). If $H = A_H \cdot (H \cap N)$, then $AdH = AdA_H \cdot Ad(H \cap N)$, and since $Ad(H \cap N)$ is connected, it follows from Theorem 3.1 that AdH is Jordan-splittable.

The definitions obtained by requiring that AdH satisfy (2) or (3) are not in general equivalent with (2′) and (3′) as stated, as simple counterexamples show.

That the assumption $Ad(H \cap N) = AdH \cap AdN$ is necessary for the theorem to hold, is shown by the following example: Let G be the simply connected Lie group whose Lie algebra is spanned by X, Y, Z, with bracket relations $[X, Y] = Z$; $[X, Z] = -Y$; $[Y, Z] = 0$; let $h = \exp(2\pi X) \cdot \exp Y$, and let H be the cyclic subgroup of G generated by h. Then AdH is a discrete group of unipotent elements, so H is Jordan Ad-splittable, and $H \cap N = (e)$; thus if H were splittable in the sense of (2′), H would be an Ad-reductive subgroup of G, which is not the case.

4. A lemma on nilpotent groups. We prove here a generalization of Case 2a in Theorem 2.1 of [5] which will be used to apply the results of the preceding section to fiberings of solvmanifolds.

LEMMA 4.1. *Let G be a connected, simply connected nilpotent Lie group, R a fully reducible group of automorphisms of G (i.e., the group dR of induced automorphisms of $\mathfrak{G}$ is fully reducible, considering $\mathfrak{G}$ as vector space), and H an analytic subgroup of G invariant under R. Then there is a closed subset E of G, invariant under R and homeomorphic to a euclidean space, such that the mapping of $E \times H$ onto G given by $(e, h) \rightarrow e \cdot h$ is a homeomorphism.*

Proof. We prove the lemma first in the case that H is normal in G. Since $\mathfrak{H}$ is invariant under dR, there is a complementary subspace $\mathfrak{E}$ in $\mathfrak{G}$ invariant under dR. Since $\mathfrak{G} = \mathfrak{E} + \mathfrak{H}$, $G = \exp(\mathfrak{E} + \mathfrak{H})$. Let $g \in G$. Then there is a unique element $X \in \mathfrak{G}$ with $g = \exp X$, and $X = Y_1 + Y_2$ with $Y_1 \in \mathfrak{E}$, $Y_2 \in \mathfrak{H}$. Let $h = \exp(-Y_1)\exp(Y_1 + Y_2)$. By the Campbell-Hausdorff formula, $h = \exp(Y_2 + Q)$, where Q is a finite linear combination of terms of the form $ad^{p_1}(Y_1 + Y_2)\cdot ad^{q_1}Y_1 \cdots ad^{p_k}(Y_1 + Y_2)$. $ad^{q_k}Y_1(Y_1 + Y_2)$, with $\Sigma(p_i + q_i) \leqq \dim \mathfrak{G}$. Since $adY_1(Y_1 + Y_2) = [Y_1, Y_2] \in \mathfrak{H}$, and $adZ(\mathfrak{H}) \subset \mathfrak{H}$ for all $Z \in \mathfrak{G}$, we see that $Q \in \mathfrak{H}$.

Consequently $Y_2 + Q \in \mathfrak{H}$, and $h \in H$. Thus, we have $g = (\exp Y_1)h = e \cdot h$, with $e \in E = \exp \mathfrak{E}$. To show uniqueness, we suppose $e_1h_1 = e_2 \cdot h_2$ with $e_1, e_2 \in \exp \mathfrak{E}$, $h_1, h_2 \in H$. Then $e_1 = e_2h_2h_1^{-1} = e_2h'$, so $\exp(\log e_1) = \exp(\log e_2)\cdot\exp(\log h')$. Again using the Campbell-Hausdorff formula, we have $\exp(\log e_2)\cdot\exp(\log h') = \exp(\log e_2 + Y)$, where $Y \in \mathfrak{H}$. Then $\exp(\log e_1) = \exp(\log e_2 + Y)$, whence $\log e_1 = \log e_2 + Y$, so $Y \in \mathfrak{H} \cap \mathfrak{E} = (0)$, and consequently $e_1 = e_2$, $h_1 = h_2$.

The map of $(E \times H) \to G$ given by $(e, h) \to eh$ is continuous, by continuity of group multiplication; let π denote the projection (as vector space) of $\mathfrak{G}$ onto $\mathfrak{E}$ given by $\pi(Y_1 + Y_2) = Y_1$, where $Y_1 \in \mathfrak{E}$, $Y_2 \in \mathfrak{H}$. Since π is linear, it is continuous. It is clear from the construction that, to obtain the representation $e \cdot h$ for an element $g \in G$, we define e by $e = \exp(\pi(\log g))$. Since log, π, and exp are continuous, e depends continuously on g, and hence $h = e^{-1}g$ does also. Thus φ: $(e, h) \to eh$ is a homeomorphism. That E is homeomorphic to euclidean space follows from the fact that exp: $\mathfrak{E} \to E$ is a homeomorphism.

In the case that H is not normal in G, let H_1 be the normalizer of H in G, and define inductively H_{i+1} to be the normalizer of H_i in $G(i = 1, 2, \cdots)$. For some integer K, $H_K = G$. That this is true is seen by observing that for all i, H_i is connected, so either $\dim H_i > \dim H_{i-1}$ or $H_i = H_{i-1}$. But if $H_i = H_{i-1}$, then the $\mathfrak{G}/\mathfrak{H}_i$ part of $ad\mathfrak{H}_i$ has no nonzero eigenvector corresponding to the eigenvalue zero, and this contradicts the nilpotency of $\mathfrak{G}$. Therefore, $\dim H_i > \dim H_{i-1}$, so $H_K = G$ for some $K \leqq \dim G$.

Since H_{i-1} is normal in H_i (letting $H_0 = H$), $H_i = E_i \cdot H_{i-1}$, by the first part of the proof, where E_i is a euclidean subspace of H_i invariant under R (clearly H_i is invariant under R for all i). Then the map of $E_K \times E_{K-1} \times \cdots \times E_1 \times H \to G$ given by $(e_K, \cdots, e_1, h) \to e_K \cdot e_{K-1} \cdots e_1h$ can be factored into

$$\begin{aligned}(E_K \times E_{K-1} \times \cdots \times E_1 \times H) &\to (E_K \times \cdots \times E_2 \times H_1)\\ &\to (E_K \times \cdots \times E_3 \times H_2) \to \cdots \to (E_K \times H_{K-1}) \to G\ ,\end{aligned}$$

each of which is a homeomorphism, and, letting $E = E_KE_{K-1}\cdots E_1$, the image of $E_K \times E_{K-1} \times \cdots \times E_1$ under this map, we see that E is homeomorphic to euclidean space, and is closed in G, since $E \times (id.)$ is closed in $E \times H$.

It is clear from the construction that E is an R-invariant exp-set.

5. Applications to fiberings of solvmanifolds. Before considering more general types of solvmanifolds, we wish to single out one special case, in which the fibers are permuted transitively by a reductive subgoup of G.

THEOREM 5.1. *Let G be a connected splittable solvable linear group, H a closed subgroup which splits in G, and assume $H \cap N$ is connected. Then G/H has a euclidean fibering such that the fibers are permuted transitively by a reductive subgroup of G.*

Proof. Let $G = A \cdot N$, $H = (H \cap A) \cdot (H \cap N)$ be the simultaneous semi-direct decompositions of G and H. $A/H \cap A$, being a connected abelian group, may be written as the direct product of a toroidal group and a vector group. Let P denote the inverse image in A of the toroidal group, and let V denote a complementary subgroup to P in A, so V is a vector group, and an A-invariant exp-set, and $A = P \cdot V$ (direct).

By Lemma 4.1, there is an A_H-invariant exp-set E such that $N = E \cdot (H \cap N)$, since $H \cap N$ is analytic. Now we may write $G = P \cdot V \cdot E \cdot (H \cap N)$, or letting $W = \{v \cdot e \mid v \in V, e \in E\}$, $G = P \cdot W \cdot (H \cap N)$, $H = (P \cap H) \cdot (H \cap N)$, where $P \cap H = A_H$, and W and $H \cap N$ are A_H-invariant exp-sets. By Lemma 4.1 in [5], it follows that G/H is a fiber bundle over $P/H \cap P$, which is a toroidal group, with fiber W, which is a euclidean space, and the group P permutes the fibers transitively.

We now prove a lemma on fiber bundles which we shall need later.

LEMMA 5.2. *Let B be the total space of a fiber bundle with base space X, fiber $F_1 \times F_2$, and group G, where F_1, F_2 are topological spaces on which the Lie group G operates, and the operation of G on $F_1 \times F_2$ is given by $g \cdot (f_1, f_2) = (g \cdot f_1, g \cdot f_2)$. Then B is the total space of a fiber bundle with fiber F_2 and group G, whose base space Y is itself a fiber bundle with base space X, fiber F_1, and group G.*

Proof. We can regard the space B as the quotient space of the space

$$T_1 = \bigcup_{j \in J} \{(x, f_1, f_2, j) \mid x \in U_j, f_1 \in F_1, f_2 \in F_2\}$$

by the relation

$$R_1\colon\ (x, f_1, f_2, j)R_1(x', f_1', f_2', k) \quad \text{if } x = x'\ ,$$
$$f_1' = g_{kj}(x)\cdot f_1\ , \quad f_2' = g_{kj}(x)\cdot f_2\ ,$$

since G operates componentwise on $F_1 \times F_2$; here $\{U_j\}$ is the family of coordinate neighborhoods on X, and $g_{kj}(x)\colon U_k \cap U_j \to G$ are the coordinate transformations.

Let Y be the fiber bundle with base space X, fiber F_1, group G, coordinate neighborhoods $\{U_j\}$ and coordinate transformations g_{kj}, whose existence and uniqueness up to equivalence is proved in Theorem 3.2 in [8], and let $\psi_j\colon U_j \times F_1 \to p^{-1}(U_j)$ be the coordinate functions, where $p\colon Y \to X$ is the projection. Y may be regarded as the quotient space of

$$T_2 = \bigcup_{j\in J}\{(x, f_1, j) \mid x \in U_j, f_1 \in F_1\}$$

by the relation

$$R_2\colon\ (x, f_1, j)R_2(x', f_1', k) \quad \text{if } x = x',\ f_1' = g_{kj}(x)\cdot f_1\ .$$

Denote by q_1 the map of T_1 onto B given by mapping a point into its R_1-equivalence class, and by q_2 the map of T_2 onto Y given by mapping a point into its R_2-equivalence class, so q_1 and q_2 are open and continuous.

Define $\pi\colon B \to Y$ by $\pi(q_1(x, f_1, f_2, j)) = q_2(x, f_1, j)$. π is well-defined, for if $q_1(x, f_1, f_2, j) = q_1(x', f_1', f_2', k)$, then $x = x'$, $f_1' = g_{kj}(x)\cdot f_1$, so $q_2(x, f_1, j) = q_2(x', f_1', k)$. It is clear that π is open and continuous, since q_1 and q_2 are both open and continuous.

Let $V_j = p^{-1}(U_j)$, so $\{V_j\}$ is an open covering of Y, and define $\varphi_j\colon V_j \times F_2 \to \pi^{-1}(V_j)$ by $\varphi_j(y, f_2) = q_1(\psi_j^{-1}(y), f_2, j)$. Since $p\pi(q_1(\psi_j^{-1}(y), f_2, j)) = pq_2(\psi_j^{-1}(y), j) = p(y) \in U_j$, $\varphi_j(V_j \times F_2) \subset \pi^{-1}(V_j)$. To see that φ_j is onto, observe that if $b = q_1(x, f_1, f_2, k)$, then $b = q_1(x, g_{jk}(x)\cdot f_1, g_{jk}(x)\cdot f_2, j)$, so $b = \varphi_j(\psi_j(x, g_{jk}(x)\cdot f_1), g_{jk}(x)\cdot f_2, j)$. The openness and continuity of φ_j follow from that of q_1 and ψ_j^{-1}.

To show that φ_j is one-to-one, suppose that $\varphi_j(y, f_2) = \varphi_j(y', f_2')$. Then $q_1(\psi_j^{-1}(y), f_2, j) = q_1(\psi_j^{-1}(y'), f_2', j)$, hence $f_2 = f_2'$, and $\psi_j^{-1}(y) = \psi_j^{-1}(y')$. Since ψ_j is one-to-one, $y = y'$, so φ_j is one-to-one.

We have thus shown that B has a fibering of the desired type, and we need only show that G is the group of the bundle. Suppose $y \in V_j \cap V_k$, and consider $f_2' = \varphi_{k,y}^{-1}\cdot\varphi_{j,y}(f_2)$. Since $\varphi_j(y, f_2) = \varphi_k(y, f_2')$ we have $q_1(\psi_j^{-1}(y), f_2, j) = q_1(\psi_k^{-1}(y), f_2', k)$, or, denoting $\psi_j^{-1}(y)$ by (x, f_1), $\psi_k^{-1}(y)$ by (x, f_1'), $q_1(x, f_1, f_2, j) = q_1(x, f_1', f_2', k)$, so $f_2' = g_{kj}(x)\cdot f_2$. Since $g_{kj}(x) \in G$, the lemma is proved.

We now apply our previous results to prove two theorems on the existence of euclidean fiberings.

THEOREM 5.3. *Let G be a connected solvable linear Lie group, and H a closed subgroup which splits in G. Then G/H has a euclidean fibering.*

Proof. Let N denote the analytic subgroup of unipotent elements of G, let A_H be a maximal reductive subgroup of H such that $H = A_H \cdot (H \cap N)$, and let A be a maximal reductive subgroup of G containing A_H, so $G = A \cdot N$, $A_H = H \cap A$. If we denote by F the minimum analytic subgroup of N containing $H \cap N$, then H is contained in the normalizer of F, and the subgroup HF is closed in G, since $HF/H = F/F \cap H$ is compact. HF splits in G, since $HF = A_H \cdot (H \cap N)F = A_H \cdot F$ (semi-direct), and $HF \cap N = F$.

Let $\tilde{A}_H$ be the minimum analytic subgroup of A containing A_H and the maximum compact subgroup of A, and let V be a subgroup of A complimentary to $\tilde{A}_H$. Then V is a vector group, $A = \tilde{A}_H \cdot V$, and $G = \tilde{A}_H \cdot V \cdot N$. Letting $S = V \cdot N$, we have $HS = A_H \cdot S$ (semi-direct), HS is closed, $A_H S/A_H = S/S \cap A_H = S$ is euclidean, and $G/HS = \tilde{A}_H \cdot S/A_H \cdot S$ is homeomorphic to $\tilde{A}_H/A_H$, which is compact.

Now G/H is a fiber bundle with base space G/HS, fiber HS/H, and group HS, acting on HS/H by left translation. Since $HS/A_H = A_H S/A_H$ is euclidean, and hence solid, we can reduce the group of the bundle to A_H, acting on HS/H by left translation. (Theorem 12.5 in [8]).

Letting E be an A_H-invariant exp-set such that $N = E \cdot F$ (the existence of E is proved in Lemma 4.1), we obtain $HS = A_H \cdot V \cdot E \cdot F = V \cdot E \cdot F \cdot A_H$. Let A_H operate on $V \times E \times F \times A_H$ by $a(v, e, f, a') = (ava^{-1}, aea^{-1}, afa^{-1}, aa')$, so the homeomorphism of HS onto $V \times E \times F \times A_H$ is equivariant with respect to the action of A_H on HS, by left translation, and on the product space as defined above. The map of HS/H onto $V \times E \times (F \cdot A_H/H)$ given by $sH \to (v, e, hH)$, where $s = v \cdot e \cdot h$, $v \in V$, $e \in E$, $h \in F \cdot A_H$, is clearly a homeomorphism, also equivariant with respect to the action of A_H.

Thus the fiber HS/H of the bundle whose total space is G/H is homeomorphic to the Cartesian product of a euclidean space $V \times E$, and the compact connected nilmanifold $F \cdot A_H/(F \cap H) \cdot A_H = F/F \cap H$, and the group A_H of the bundle acts componentwise. Applying Lemma 5.2, we thus obtain the stated result, that G/H is a fiber bundle with compact base space and euclidean fiber, and the action of the group is equivalent to the action of a linear group. Proof of the theorem is complete.

In case the eigenvalues of the elements of $Ad(A_H)$ are all positive, we can obtain Mostow's result of [4] that the fibering is trivial.

COROLLARY. *If the eigenvalues of the elements of $Ad(A_H)$ are positive, then $G/H = Y \times E'$, where E' is a euclidean space, and Y is compact.*

Proof. The action of the group A_H on the fiber $E'(= V \times E)$ is equivalent to the action of $Ad(A_H)$ on the tangent space $\mathfrak{E}'$ of E' at the identity. Taking a basis in $\mathfrak{E}'$ with respect to which the matrices of $Ad(A_H)$ are diagonal, and letting A denote the minimum analytic subgroup of diagonal matrices with positive eigenvalues containing $Ad(A_H)$, we observe that A operates naturally on $\mathfrak{E}'$. Defining $\lambda(a)(\exp e_i) = \exp(a(e_i))$, for $a \in A$, $e_i \in \mathfrak{E}_i$ ($\mathfrak{E}_i$ being the subspaces of $\mathfrak{E}'$ which make E' an A_H-invariant exp-set), we see that λ defines an operation of A on E', and the action of A_H is equivalent to the action of a subgroup of A, so we may regard A as the group of the bundle. A is euclidean, and hence solid, so, by Theorem 12.5 in [8], we may reduce the group of the bundle to the identity. Thus $G/H = Y \times E'$.

The theorem just proved made assumptions about both G and H. By making a somewhat stronger assumption on G, we are able to drop all restrictions on H, except that it be closed.

THEOREM 5.4. *Let G be a connected solvable matrix group, and assume that G is of finite index in its algebraic group hull. Then for any closed subgroup H of G, G/H has a euclidean fibering.*

Proof. Let H' be the intersection of G with the algebraic group hull of H, so H' has a finite number of connected components. Denote by H^0 the identity component of H', by H^* the intersection of H with H^0, and let $F = H \cdot H^0$, so F/H^0, being contained in the finite group H'/H^0, has a finite number of connected components, and F is closed in G. $H^* \cap [H^0, H^0]$ is uniform in $[H^0, H^0]$, since $[H^0, H^0]$ is the minimum analytic group containing $[H, H] \subset H^*$. Thus $H^* \cdot [H^0, H^0]$ is closed, and normal in H^0. The factor group $H^0/H^* \cdot [H^0, H^0]$, being abelian and connected, is the direct product of a toroidal group and a vector group. Let T be the inverse image in H^0 of the toroidal group, so H^0/T is a vector group, and T/H^*, being a fiber bundle over the compact base space $T/H^* \cdot [H^0, H^0]$ with compact fiber $H^* \cdot [H^0, H^0]/H^* = [H^0, H^0]/H^* \cap [H^0, H^0]$, is compact.

We note that, since H^0/H^* is a fiber bundle with euclidean base space H^0/T and compact fiber T/H^*, by Corollary 1.6 in [8], $H^0/H^* = F/H$ is homeomorphic with the product space $(H^0/T) \times (T/H^*)$.

HT, being a finite extension of T (since $H/H \cap T = H/H^*$ is finite), is closed in G, and has a finite number of connected components, so $HT = K \cdot E$, where K is a maximal compact subgroup of HT and E, being a K-invariant exp-set, is euclidean (cf. [5], [6]). Observe that K leaves T invariant under inner automorphism, since $T = HT \cap H^0$, so there is a K-invariant exp-set V in H^0 with $F = V \cdot E \cdot K = V \cdot HT$.

Since F is Jordan-splittable and $F \cap N$ is connected, N being the group of unipotent matrices in G, F splits in G, so FN is closed and normal in G. Then G/FN is a connected abelian group, so is a direct product of a toroidal group and a vector group. Let G_1 be the inverse image in G of the toroidal group, so G/G_1 is euclidean and G_1/FN is compact.

Since $F = V \cdot HT$, and $H \cap N$ is connected, by Lemma 4.1 we can write $FN = W' \cdot V \cdot HP$, where W' is a K-invariant exp-set complementary to $F \cap N$ in N.

Now G_1/H is a bundle with compact base space G_1/FN, fiber FN/H, and group FN acting by left translation on FN/H. Noting that $FN/K = W' \cdot V \cdot E \cdot K/K \approx W' \times V \times E$, we see that FN/K is solid, and we can reduce the group of the bundle to K acting on FN/H by left translation. Letting $W = W' \cdot V$, we see that if $g \in FN$ and $g = w \cdot t$ is the unique decomposition with $w \in W$, $t \in HT$, then the map $gH \to (w, tH)$ is a homeomorphism of FN/H onto $W \times HT/H$, and this map is equivariant with respect to the action of K, where K acts componentwise on $W \otimes HT/H$, by inner automorphism on W and left translation on HT/H. By Lemma 5.2, G_1/H is a fiber bundle over a compact base space X with euclidean fiber and compact group K, whose action on the fiber is equivalent to the action of a linear group.

G/H, being a bundle over the euclidean space G/G_1 with fiber G_1/H, is homeomorphic to the product bundle $(G/G_1) \times (G_1/H)$, by Corollary 1.6 in [8]. It is easily seen that G/H is thus a bundle over X with euclidean fiber $(G/G_1) \times W$, and group K acting componentwise, where K acts trivially on G/G_1 and by inner automorphism on W. Proof of the theorem is complete.

The proof of Theorem 5.4 included an investigation of the factor space of H^0/H^*, which can also be applied to H'/H, where H' is the minimum subgroup of finite index in the algebraic group hull of H, and containing H. If $\tilde{H}$ is the algebraic group hull of H, then $\tilde{H}/H$ is homeomorphic with the cartesian product of H'/H with a finite set S, having the discrete topology, and each component is homeomorphic to H'/H, which is the cartesian product of a compact space C and a euclidean space E. Since the space $C \times S$ is compact, and $\tilde{H}/H \approx (C \times S) \times E$, we have thus proved the following special result:

THEOREM 5.5 *Let G be a solvable algebraic matrix group, and*

H any closed subgroup whose algebraic group hull is G. Then $G/H \approx C \times E$, where C is a compact space and E is euclidean.

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