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HOPF ALGEBRAS OVER DEDEKIND DOMAINS AND TORSION IN H-SPACES

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The main purpose of this note is to show that if the loop space ΩX of a finite dimensional H-space is free of torsion, then X itself can have p-torsion of at most order p.

§ 1 is devoted to proving a generalization to Dedekind domains of the decomposition theorems Hopf-Leray, and Borel, and § 2 is devoted to recalling the structure of quasimonogenic Hopf algebras over the integers as described by Halpern. § 3 gives the proof of the main theorem which relies somewhat on the statement and proof of Theorem 4.1 of [4].

Theorem 1.5 was included in the author's dissertation (Princeton University, 1961) done under the direction of Professor John Moore.

1. Hopf algebras over Dedekind domains.

Unless further specified K will denote an arbitrary integral domain. By *standard field associated* with K we shall mean any residue class field of K. A K-algebra will be called *monogenic* if it is generated by a single element.

In this section we prove a generalization (Theorem 1.5) for torsion-free algebras over a Dedekind domain with perfect quotient field of the following well known theorem:

THEOREM 1.1. (*Hopf-Leray-Borel*). *If B is a connected, commutative, and associative Hopf algebra of finite type over a perfect field K, then B is isomorphic as a K-algebra to a tensor product of monogenic Hopf algebras over K.*

Proof of the separate cases char $K = 0$ (Hopf-Leray) and char $K \neq 0$ (Borel) may be found in Milnor and Moore [6].

DEFINITIONS. A *closed submodule* of a K-module B is a submodule such that for all $x \in B$ and all $k \in K$, $kx \in A$ implies $x \in A$ or $k = 0$. If A is any submodule of B, then $\bar{A}$, the *closure of A in B*, is given by

$$\bar{A} = \{x \in B \mid kx \in A \text{ for some } k \in K,\ k \neq 0\}$$

REMARKS. A is closed in B is equivalent to $\bar{A} = A$ and to B/A is torsion-free. Note that the intersection of closed submodules is closed

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and $\bar{A}$ is the minimal closed submodule of B which contains A. If Q denotes the field of fractions of K and $j: B \to B \otimes Q$ is the map given by $j(b) = b \otimes 1$, then $\bar{A} = j^{-1}[j(A)]$ where $[j(A)]$ denotes the Q-submodule generated by $j(A)$.

PROPOSITION 1.2. If B is a K-algebra and A is a subalgebra of B, then the closure of A in B is a subalgebra of B. If B is a torsion-free K-coalgebra and A is a subcoalgebra of B, then the closure of A is a subcoalgebra of B.

Proof. The first statement is obvious. For the second let $0 \to A \to B \to C \to 0$ and $0 \to \bar{A} \to B \to C' \to 0$ be exact sequences of K-modules defining C and C'. Let $\tilde{X}$ denote $X \otimes Q$ where Q is the field of fractions of K. Let $j: B \to B \otimes Q = \tilde{B}$ as above. Then j is a monomorphism since B is torsion-free. Furthermore $j(\bar{A}) = [j(A)] \approx \tilde{A}$ and $C' = B/\bar{A}$ is torsion-free so that $C' \to \tilde{C}'$ is a monomorphism. Consider the commutative diagram

$$\begin{array}{ccccc} \bar{A} & \xrightarrow{\alpha} & B \otimes B & \xrightarrow{\beta} & (B \otimes C') \oplus (C' \otimes B) \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_3} \\ \tilde{A} & \xrightarrow{\gamma} & \tilde{B} \otimes \tilde{B} & \xrightarrow{\delta} & (\tilde{B} \otimes \tilde{C}') \oplus (\tilde{C}' \otimes \tilde{B}) \end{array}$$

where α and γ are induced by the coproduct of B. Obviously j_1, j_2, and j_3 are monomorphisms.

$\delta\gamma = 0$: A is a subcoalgebra of B, and tensoring with Q the exact sequence $0 \to \bar{A} \to B \to C' \to 0$ we obtain an exact sequence $0 \to \tilde{A} \to \tilde{B} \to \tilde{C} \to 0$.

Then $j_3\beta\alpha = \delta\gamma j_1 = 0$. j_3 is a monomorphism and thus $\beta\alpha = 0$, $\operatorname{Im} \alpha \subset \operatorname{Ker} \beta = \bar{A} \otimes \bar{A}$, and $\alpha(\bar{A}) \subset \bar{A} \times \bar{A}$.

COROLLARY 1.3. *If A is a sub Hopf algebra of a torsion-free Hopf algebra over K, then the closure of A is a sub Hopf algebra of B.*

PROPOSITION 1.4. Let A be a sub Hopf algebra of a torsion-free Hopf algebra over a Dedekind domain K. Then A is closed in B if and only if B is a flat A-module.

Proof. The following statements are equivalent.

(a) A is closed in B.

(b) B/A is a torsion-free K-module.

(c) $0 \to A \otimes L \to B \otimes L$ is exact for every standard field L associated with K.

(d) $0 \to P(A \otimes L) \to P(B \otimes L)$ is exact for every standard field

L associated with K.

(e) B is a flat A-module.

It is obvious that (a), (b), and (c) are equivalent. (c) and (d) are equivalent by Proposition 3.8 of Milnor and Moore [6]. That (d) and (e) are equivalent is Proposition 6 of Moore [7].

PROPOSITION 1.5. Let B be a torsion-free Hopf algebra over a Dedekind ring K, let A be a sub Hopf algebra of B, and suppose that C is a normal closed sub Hopf algebra of A and B. Then A is closed in B if and only if $A//C$ is closed in $B//C$.

Proof. In view of Proposition 1.4 it is sufficient to show that $\mathrm{Tor}_n^A(X, B) \approx \mathrm{Tor}_n^{A//C}(X, B//C)$ for any $A//C$-module X.

Proposition 1.4 implies that A and B are flat C-modules and $\mathrm{Tor}_n^C(K, A) = \mathrm{Tor}_n^C(K, B) = 0$ for all $n > 0$. A change of rings $C \to A$ yields $\mathrm{Tor}_n^A(A//C, B) \approx \mathrm{Tor}_n^C(K, B) = 0$ since $A//C = A \otimes_C K$. (Cf. Cartan and Eilenberg [3, p. 117].) A second change of rings $A \to A//C$ yields $\mathrm{Tor}_n^A(X, B) \approx \mathrm{Tor}_n^{A//C}(X, A//C \otimes_A B)$. Since $B//C \approx A//C \otimes_A B$ as an $A//C$-module, the proof is complete.

DEFINITIONS. An integral domain K is *quasiperfect* if the field of fractions Q is perfect. A torsion-free Hopf algebra B over K is *quasimonogenic* if $B \otimes Q$ is monogenic.

THEOREM 1.6. *Suppose B is a torsion-free, associative, and commutative Hopf algebra of finite type over a quasiperfect Dedekind domain K. Then as an algebra B is isomorphic to a tensor product of quasimonogenic Hopf algebras.*

Proof. Since B is torsion-free, the map $j: B \to B \otimes Q$ is a monomorphism. $B = B \otimes Q$ satisfies the hypotheses of Theorem 1.1 and we write $B \approx \bigotimes_{i \in I} B_i$ where x_i generates B_i and the indexing is arranged so that $\deg x_i \leqq \deg x_{i+1}$.

There are nonzero elements $k_i \in K$ such that $k_i x_i \in Im j$ and we let B_i denote the closure of the subalgebra generated by $j^{-1}(k_i x_i)$. Let $f\colon \bigotimes_{i \in I} B_i \to B$ be the map induced by the injections $B_i \to B$. f is a monomorphism since the diagram

$$\begin{array}{ccc} \bigotimes_{i \in I} B_i & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ \bigotimes_{i \in I} \tilde{B}_i & \longrightarrow & \tilde{B} \end{array}$$

is commutative and the other maps are monomorphisms.

Let $B^n = \bigotimes_{i=1}^n B_i$. We show by induction that B^n is a closed sub Hopf algebra of B. By Corollary 1.3 $B^1 = B_1$ is a closed sub Hopf algebra of B. Suppose B^{n-1} is a closed sub Hopf algebra of B. We see directly from the definition of B_n that $B_n \approx B^n//B^{n-1}$ is closed in $B//B^{n-1}$ and by Proposition 1.5 B^n is closed in B.

Consequently $\bigotimes_{i\in I} B_i$ is a closed sub Hopf algebra of B and Coker f is torsion-free. But $(\bigotimes_{i\in I} B_i)\otimes Q \approx B\otimes Q$ so that $(\text{Coker} f)\times Q = 0$. Consequently Coker $f = 0$ and f is an epimorphism.

2. Quasimonogenic Hopf algebras. Here we recall for the special case $K = Z$ the description due to Halpern [5] of the quasimonogenic Hopf algebras which appear as factors in the splitting guaranteed by Theorem 1.6. We include some homological computation useful in applications of the main theorem.

THEOREM 2.1. (*Halpern*). *Every quasimonogenic Hopf algebra B over the integers has one of the following forms:*

(a) $B = E(x, 2m-1)$, *the exterior algebra on a generator of odd degree. x is called a quasigenerator as well as a generator.*

(b) *B has a series of generators $x_1, \cdots, x_n, \cdots$ of even degree where* $\deg x_n = n$ *degree x_1 which satisfy relations of the form $x_1 x_{n-1} = \beta_n x_n$ where the β_n are positive integers. Furthermore the β_n satisfy the conditions* (i)–(iii).

(i) $\beta_1 = 1$

(ii) *There are integers β_n^* for each n such that $\beta_n\beta_n^* = n$.*

(iii) $\beta_n \mid \beta_{kn}$ *and* $\beta_n^* \mid \beta_{kn}^*$ *for all positive integers k and n.*

B will be called a partially divided polynomial algebra. Any sequence of integers satisfying (i)–(iii) *will be called a fundamental sequence. x_1 is called the quasigenerator of B.*

Halpern shows that every sequence of positive integers $\{\beta_n\}$ satisfying (i)–(iii) gives rise to a quasimonogenic Hopf algebra whose dual Hopf algebra comes from $\{\beta_n^*\}$. When $\beta_n = 1$ for all n, the resulting Hopf algebra is the polynomial algebra on the primitive generator x_1. If $\beta_n = n$ for all n, the Hopf algebra associated with the sequence $\{\beta_n\}$ is the divided polynomial algebra on the primitive generator x_1. If we set

$$\beta_m ? = \beta_m \cdots \beta_1 \text{ and } \beta_{m,n} = \beta_{m+n}?/\beta_m ? \beta_n ? ,$$

then the product and coproduct in the algebra associated with $\{\beta_n\}$ are given by the rules

$$x_m x_n = \beta_{m,n} x_{m+n}$$
$$\Delta(x_n) = \sum_{i+j=n} \beta_{i.j}^* x_i x x_j$$

where $\beta^*_{m,n} = \beta^*_{m+n}?/\beta^*_m?\beta^*_n?$. Note that $\beta_n?\beta^*_n? = n!$ and therefore $\beta_{m,n}\beta^*_{m,n} = [m, n]$ the binomial coefficient.

To illustrate how one might construct a fundamental sequence we observe that if $(m, n) = 1$ (m and n are relatively prime) it follows that $\beta_{mn} = \beta_m\beta_n$. To see this note that (ii) implies that $(\beta_m, \beta_n) = 1 = (\beta^*_m, \beta^*_n)$. By (iii) $\beta_m\beta_n \mid \beta_{mn}$ and $\beta^*_m\beta^*_n \mid \beta^*_{mn}$. But $\beta_m\beta^*_m\beta_n\beta^*_n = mn = \beta_{mn}\beta^*_{mn}$ consequently $\beta_m\beta_n = \beta_{mn}$. As a result if $m = m_1 \cdots m_g$ is the decomposition of m into primary factors, we can write $\beta_m = \beta_{m_1} \cdots \beta_{m_g}$.

LEMMA 2.2. *Let α_m denote the greatest common divisor of the quasibinomial coefficients $\beta_{k,m-k}$, $0 < k < m$. Then $\alpha_m = 1$ unless $m = p^n$ and $\beta_m = p\beta_{m/p}$ in which case $\alpha_m = p$.*

Proof. $\beta_{k,m-k}$ divides $\binom{m}{k}$ and therefore α_m divides q_m, the greatest common divisor of the binomial coefficients $\binom{m}{k}$ for $0 < k < m$. By Lucas's theorem (Adem [1, Theorem 25.1]) we see that if $m \neq p^n$ for a given prime p, then $\binom{m}{k} \neq 0 \bmod p$ for some k, $0 < k < m$. Consequently if $m \neq p^n$ for any prime p, then $q_m = 1$ and $\alpha_m = 1$.

Let $\varepsilon(n)$ denote the number of factors of p in $p^n!$ Then a simple counting argument shows that $\varepsilon(n) = p\varepsilon(n-1) + 1$. We know that $q_{p^n} = p^r$ for $0 < r$ by the argument above. Writing

$$\left(\frac{p^n!}{(p^{n-1}!)^p}\right) = \binom{p^n}{p^{n-1}}\binom{(p-1)p^{n-1}}{p^{n-1}} \cdots \binom{2p^{n-1}}{p^{n-1}}$$

and nothing that Lucas's theorem implies $\binom{ap^{n-1}}{p^{n-1}} \equiv \binom{a}{1} \not\equiv 0 \bmod p$, we see that $\binom{p^n}{p^{n-1}}$ has as many factors of p as $(p^n!)/(p^{n-1}!)^p$, namely just one. Consequently $q_{p^n} = p$ and $\alpha_{p^n} = 1$ or $\alpha_{p^n} = p$.

Writing $\beta[a, b]$ for $\beta_{a,b}$ we find that, for $(a-1)p^{n-1} \leqq k < ap^{n-1}$ $(0 < a < p)$,

$$\beta[k, p^n - k] = \beta[k, ap^{n-1} - k]\beta[ap^{n-1}, (p-a)p^{n-1}]/\beta[ap^{n-1} - k, (p-a)p^{n-1}]\,.$$

Then $\beta[k, ap^{n-1} - k]$ divides $\binom{ap^{n-1}}{k} \not\equiv 0 \bmod p$ (by Lucas's theorem) and similarly $\beta[ap^{n-1} - k, (p-a)p^{n-1}] \not\equiv 0$, so that p divides $\beta[k, p^n - k]$ if and only if p divides $\beta[ap^{n-1}, (p-a)p^{n-1}]$. Taking $k = p^{n-1}$, it follows that $\alpha_m = p$ if and only if $\beta[p^{n-1}, (p-1)p^{n-1}] \equiv 0 \bmod p$. However

$$\beta[p^{n-1}, (p-1)p^{n-1}] = (\beta_{p^n}/\beta_{p^{n-1}})\beta[p^{n-1} - 1, (p-1)p^{n-1}]$$

and $\beta[p^{n-1} - 1, (p-1)p^{n-1}] \not\equiv 0 \bmod p$ by the usual argument using the theorem of Lucas cited above. Thus $\alpha_m = p$ if and only if p divides $\beta_{p^n}/\beta_{p^{n-1}}$, or in other words, if and only if $\beta_{p^n} = p\beta_{p^{n-1}}$.

THEOREM 2.3. *Let B denote a torsion-free, even dimensional, quasimonogenic Hopf algebra over the integers with generators $\{x_n\}$ and with fundamental sequence $\{\beta_n\}$. Let $S_p = \{p^n \mid \beta_{p^n} = p\beta_{p^{n-1}}\}$ for a given prime p. Then:*

(1) *Among the generators $\{x_n\}$ the indecomposable ones are x_1 and the x_m's for which $m \in S_p$ for some prime p.*

(2) *The relations among these indecomposable generators may all be derived from the relations of the form*

$$(\beta_n ?)x_n = (\beta_m ?x_m)^{n/m}$$

where m and n are consecutive elements of some S_p or else $m = 1$ and n is the smallest element of some S_p.

Proof. (1) If $m > 1$ and $m \notin S_p$ for any prime p, then by 2.2, $\alpha_m = 1$. Consequently there exist integers λ_k for $0 < k < m$ such that $\sum_{k=1}^{m-1} \lambda_k \beta_{k,m-k} = 1$, and therefore

$$x_m = \left(\sum_{k=1}^{m-1} \lambda_k \beta_{k,m-k}\right) x_m = \sum_{k=1}^{m-1} \lambda_k x_k x_{m-k}$$

and x_m is decomposable.

(2) Clearly the relations given do hold between the generators involved. On the other hand from the relations given we may easily obtain the relations $(\beta_n ?)x_n = (x_1)^n$ for $n \in S_p$ for some prime p. From these we may obtain the relations $(\beta_n ?)x_n = (x_1)^n$ for any integer n, x_n being written in terms of indecomposable generators. Finally we can obtain the relations $x_1 x_{m-1} = \beta_m x_m$ which characterize the algebra B.

REMARK 2.4. The relations in B are all derived from relations of the form $x^{pq} = pRy$ where x and y are indecomposable, p is prime, and $R \not\equiv 0$ since for m and n consecutive elements of S_p we have that p divides $(\beta_n ?)/(\beta_m ?)^{n/m}$ precisely once which is easily proved by an argument similar to that of 2.2.

3. Application to torsion in *H*-spaces.

THEOREM 3.1. *Let X be a pathwise connected and simply connected H-space of finite homological type and dimension whose loop space is torsion-free. Then X can have p-torsion of at most order p.*

Proof. The hypotheses imply that $H_*(\Omega X; Z)$ is a torsion-free Hopf algebra of finite type, and furthermore $H_*(\Omega X; Z)$ has no elements of odd degree since $H_*(\Omega X; Q)$ is even dimensional by [8, § 7, Theorem III]. Therefore 1.6 and 1.7 imply that $H_*(\Omega X; Z)$ is a tensor product of partially divided polynomial Hopf algebras. In each factor the relations among generators are of the form $x^{p^q} = pRy$ where p is prime and $R \not\equiv 0 \mod p$, as remarked in 2.4. It follows easily that $H_*(\Omega X; Z_p)$ is a tensor product of polynomial and truncated polynomial algebras—either from a direct computation using the structure of $H_*(\Omega X; Z)$ or applying the decomposition theorem of Borel to $H_*(\Omega X; Z_p)$ which has no elements of odd degree by the remark above.

The classifying space $B_{\Omega X}$ has the same homotopy type as X by Corollary 9.2 of [9], and there is a homotopy equivalence of chain complexes $B(C(\Omega X)) \to C(B_{\Omega X})$ from the bar construction on the chains of ΩX to the chains of the classifying space $B_{\Omega X}$. Therefore the homology of X may be computed as the homology of the bar construction on the chains of ΩX.

Let $x \in H_{2m}(\Omega X; Z)$ and suppose that $x^{p^q} = pRy$ is one of the relations in $H_*(\Omega X; Z)$, x and y being indecomposable elements. Let a and b denote chains of ΩX which represent x and y respectively. Then $a^{p^q} = pRb + \partial c$. Then the element $[a^{p^q-1} | a] + [c]$ is a cycle mod p and its homology class z is called the transpotence of x', where x' denotes the reduction of x mod p. Clearly $\beta z = Rs_*y'$, where β denotes the Bockstein, and s_* denotes the suspension homomorphism in homology mod p, and y' denotes the reduction of y mod p.

From the proof of Theorem 4.1 of [4] it follows clearly that every primitive generator of even degree in $H_*(X; Z_p)$ is such a transpotence z of some generator x' in $H_*(\Omega X; Z_p)$.[1] See also, S. Gitler, Nota Sobre La Transpotencia de Cartan, Bol. Soc. Mex. 1963, 85–91. Furthermore by Theorem 4.1 of [4] it follows that $s_*y' \neq 0$ since $\deg y' \equiv 0 \mod p$. (For $p = 2$ it is true because $\deg y' \equiv 0 \mod 4$). Therefore the primitive generators of even degree in $H_*(\Omega X; Z_p)$ have nonzero Bockstein while those of odd degree have zero Bockstein since ΩX is torsion free and suspension commutes with the Bockstein.

The remainder of the proof is a simple exercise in spectral sequences of Hopf algebras showing that in the Bockstein spectral sequence, E^2 is an exterior algebra on generators of odd degree, and that therefore $E^2 = E^\infty$. Since the higher differentials in the Bockstein spectral sequence may by identified with the higher order Bockstein operations, it is immediate that X has p-torsion of order p at most.

Bott [2] shows that a compact, connected, and simply connected Lie group satisfies the hypotheses of 3.1.

[1] See also, S. Gitler, Nota Sobre La Transpotencia de Cartan, Bol. Soc. Mex. 1963, 85-91.

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