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ON SUB-ALGEBRAS OF A C*-ALGEBRA

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The following noncommutative extension of the Stone-Weierstrass approximation theorem has been obtained by Glimm.

Theorem. Let $\mathscr{A}$ be a C^*-algebra with identity I, and let $\mathscr{B}$ be a C^*-sub-algebra containing I. Suppose that $\mathscr{B}$ separates the pure state space of $\mathscr{A}$. Then $\mathscr{B} = \mathscr{A}$.

In the present paper, we apply Glimm's theorem to obtain the following noncommutative generalisation of another result of Stone.

Let $\mathscr{A}$ be a C^*-algebra with identity I and pure state space $\mathscr{P}$. Let $\mathscr{B}$ be a C^*-sub-algebra of $\mathscr{A}$, and defne

$$\mathscr{N} = \{f:\ f \text{ is a pure state of } \mathscr{A} \text{ and } f(B) = 0\ (B \in \mathscr{B})\}\ ,$$
$$\mathscr{E} = \{(g, h):\ g, h \in \mathscr{P} \text{ and } g(B) = h(B)\ (B \in \mathscr{B})\}\ ,$$
$$\mathscr{H}_{\mathscr{B}} = \{A:\ A \in \mathscr{A},\ f(A) = 0\ \ (f \in \mathscr{N}) \text{ and } g(A) = h(A)\ ((g, h) \in \mathscr{E})\}\ .$$

Then $\mathscr{B} = \mathscr{H}_{\mathscr{B}}$.

We will refer to this as Theorem 2 in the sequel. Glimm's theorem is to be found in [1]; Stone's, in [3].

Once it is known that $\mathscr{H}_{\mathscr{B}}$ is a C^*-sub-algebra of $\mathscr{A}$, Theorem 2 is an almost immediate consequence of Glimm's theorem (see § 4). It is clear that $\mathscr{H}_{\mathscr{B}}$ is a closed self-adjoint linear subspace of $\mathscr{A}$; accordingly, most of this paper is devoted to proving that $\mathscr{H}_{\mathscr{B}}$ is closed under multiplication (see § 3).

We remark that, if $\mathscr{A}$ is commutative, then $\mathscr{P}$ consists exactly of all homomorphism from $\mathscr{A}$ on to the complex plane $\boldsymbol{C}$; so in this case, it is immediate from its definition that $\mathscr{H}_{\mathscr{B}}$ is a C^*-sub-algebra. However, this seems not to be obvious in the general case. Indeed, for a *general* set $\mathscr{N}$ of pure states of $\mathscr{A}$ and a *general* subset $\mathscr{E}$ of $\mathscr{P} \times \mathscr{P}$, the class

$$\{A:\ \ A \in \mathscr{A},\ f(A) = 0\ \ (f \in \mathscr{N}) \text{ and } g(A) = h(A)\ \ ((g, h) \in \mathscr{E})\}$$

need not be a sub-algebra of $\mathscr{A}$; for example, let $\mathscr{A}$ consist of all bounded linear operators on a Hilbert space H, let $\mathscr{N}$ be void, and let $\mathscr{E}$ consist of a single pair of vector states arising from orthogonal unit vectors.

2. Notation.

Throughout, $\mathscr{A}$ is a C^*-algebra-by which we shall mean a uniformly closed self-adjoint algebra of operators acting on a (complex) Hilbert space H. We shall always assume that $\mathscr{A}$ contains

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the identity operator I on H. A *state* of $\mathscr{A}$ is a linear functional f on $\mathscr{A}$ such that $f(A^*A) \geqq 0$ $(A \in \mathscr{A})$ and $f(I) = 1$. The set of all states is convex and weak * compact; the Krein-Milman theorem ensures the existence of extreme points, and these are called *pure states*. The *pure state space* of $\mathscr{A}$, denoted by $\mathscr{P}$ (or $\mathscr{P}(\mathscr{A})$ if $\mathscr{A}$ has to be specified), is the weak * closure of the set of all pure states.

Given a state f of $\mathscr{A}$, there is a *-representation ϕ_f of $\mathscr{A}$ on a Hilbert space H_f, and a unit vector x_f in H_f, such that $\phi_f(\mathscr{A})x_f$ is dense in H_f, and

$$f(A) = \langle \phi_f(A)x_f,\ x_f \rangle \qquad (A \in \mathscr{A})\ .$$

To within unitary equivalence, ϕ_f is unique. Furthermore, ϕ_f is irreducible if and only if f is a pure state (see, for example, [2] 245, 265, 266). We shall always use the symbols ϕ_f, H_f, x_f in the sense just described.

3. Some lemmas.

Throughout this section we shall assume that $\mathscr{B}$ is a C^*-sub-algebra of $\mathscr{A}$, and that $I \in \mathscr{B}$. We use the notations introduced in the statement of Theorem 2; note that, since $I \in \mathscr{B}$, $\mathscr{N}$ is empty and

$$\mathscr{H}_{\mathscr{B}} = \{A:\ \ A \in \mathscr{A} \text{ and } g(A) = h(A)\ \ ((g, h) \in \mathscr{E})\}\ .$$

For completeness, we give a proof of the following simple result.

LEMMA 1. (i) *Let $f \in \mathscr{P}$, $S \in \mathscr{A}$ and suppose that $f(S^*S) = 1$. Define $g(A) = f(S^*AS)$ $(A \in \mathscr{A})$. Then $g \in \mathscr{P}$.*

(ii) *Let $f \in \mathscr{P}$, $x \in H_f$, $\|x\| = 1$, and define $g(A) = \langle \phi_f(A)x, x\rangle$ $(A \in \mathscr{A})$. Then $g \in \mathscr{P}$.*

Proof. (i) Clearly g is a state. Suppose first that f is a pure state, and let $x = \phi_f(S)x_f$. Then for each $A \in \mathscr{A}$,

$$(1) \qquad \langle \phi_f(A)x, x\rangle = \langle \phi_f(S^*AS)x_f, x_f\rangle = f(S^*AS) = g(A)\ .$$

With $A = I$ we obtain $\|x\| = 1$; and since f is a pure state, ϕ_f is irreducible, so $\phi_f(\mathscr{A})x$ is dense in H_f. This, with (1), implies that ϕ_f and ϕ_g are unitarily equivalent. Thus ϕ_g is irreducible, so g is pure.

Now suppose only that $f \in \mathscr{P}$. There is a net (f_i) of pure states which converges to f in the weak * topology. Since $f_i(S^*S) \rightarrow f(S^*S) = 1$, we may suppose that $f_i(S^*S) > 0$ for each i. Let $k_i = [f_i(S^*S)]^{-1/2}$, $S_i = k_iS$, and define $g_i(A) = f_i(S_i^*AS_i)$ $(A \in \mathscr{A})$. Then $f_i(S_i^*S_i) = 1$, and the argument of the preceding paragraph shows that g_i is a pure state. For each $A \in \mathscr{A}$,

$$g_i(A) = \frac{f_i(S^*AS)}{f_i(S^*S)} \to f(S^*AS) = g(A) .$$

Hence (g_i) is a net of pure states which converges to g in the weak * topology, so $g \in \mathscr{P}$.

(ii) Since $\phi_f(\mathscr{A})x_f$ is dense in H_f, we may choose $S_n \in \mathscr{A}$ $(n = 1, 2, \cdots)$ such that

$$\| \phi_f(S_n)x_f \| = 1 , \qquad \| \phi_f(S_n)x_f - x \| \to 0 .$$

Thus $f(S_n^* S_n) = 1$, and by part (i) of this lemma, we may define g_n in $\mathscr{P}$ by $g_n(A) = f(S_n^* A S_n)$ $(A \in \mathscr{A})$. Then for each $A \in \mathscr{A}$,

$$g_n(A) = \langle \phi_f(A)\phi_f(S_n)x_f,\ \phi_f(S_n)x_f \rangle \to \langle \phi_f(A)x, x \rangle = g(A) .$$

Thus $g \in \mathscr{P}$.

LEMMA 2. *Let $T \in \mathscr{H}_{\mathscr{B}}$, $S \in \mathscr{B}$. Then $S^*TS \in \mathscr{H}_{\mathscr{B}}$.*

Proof. Let $(f_1, f_2) \in \mathscr{E}$. We have to show that $f_1(S^*TS) = f_2(S^*TS)$. Since $S^*S \in \mathscr{B}$, we have $f_1(S^*S) = f_2(S^*S)$; and after multiplying S by a suitable scalar, we may clearly suppose that $f_1(S^*S)$ is either 0 or 1.

If $f_i(S^*S) = 0$, then S is in the left kernel of f_i $(i = 1, 2)$, and $f_1(S^*TS) = f_2(S^*TS) = 0$.

If $f_i(S^*S) = 1$, define $g_i(A) = f_i(S^*AS)$ $(A \in \mathscr{A})$. By Lemma 1 (i), $g_i \in \mathscr{P}$. If $B \in \mathscr{B}$, then $S^*BS \in \mathscr{B}$, so $f_1(S^*BS) = f_2(S^*BS)$; that is, $g_1(B) = g_2(B)$. Hence $(g_1, g_2) \in \mathscr{E}$, and since $T \in \mathscr{H}_{\mathscr{B}}$, it follows that $g_1(T) = g_2(T)$; that is, $f_1(S^*TS) = f_2(S^*TS)$. This completes the proof.

LEMMA 3. *Let $T \in \mathscr{H}_{\mathscr{B}}$ and $R, S \in \mathscr{B}$. Then $R^*TS \in \mathscr{H}_{\mathscr{B}}$.*

Proof. This follows from Lemma 2 since

$$\begin{aligned} 4\,R^*TS &= (R + S)^*T(R + S) - (R - S)^*T(R - S) \\ &\quad - i(R + iS)^*T(R + iS) + (R - iS)^*T(R - iS) . \end{aligned}$$

LEMMA 4. *Let $f \in \mathscr{P}$ and let M be a closed subspace of H_f which is invariant under $\phi_f(\mathscr{B})$. Then M is a invariant under $\phi_f(\mathscr{H}_{\mathscr{B}})$.*

Proof. Suppose that the lemma is false. Then we may choose $T \in \mathscr{H}_{\mathscr{B}}$ and $x \in M$ such that $\phi_f(T)x \notin M$. Let $y = (I - E)\phi_f(T)x$, where E is the projection from H_f on to M. Given t in $[0, 2\pi)$, define $y_t = x + \exp(it)y$, $z_t = ky_t$, where

$$k = [\| x \|^2 + \| y \|^2]^{-1/2} = \| y_t \|^{-1} .$$

Thus $z_t \in H_f$, $\| z_t \| = 1$, and by Lemma 1 (ii) we may define $g_t \in \mathscr{P}$ by $g_t(A) = \langle \phi_f(A) z_t, z_t \rangle$ $(A \in \mathscr{A})$. Since $\phi_f(\mathscr{B})$ leaves both M and $H_f \ominus M$ invariant, it follows that for each $B \in \mathscr{B}$,

$$\begin{aligned} g_t(B) &= k^2 \langle \phi_f(B)(x + e^{it}y),\ x + e^{it}y \rangle \\ &= k^2[\langle \phi_f(B)x, x\rangle + \langle \phi_f(B)y, y\rangle] , \end{aligned}$$

which is independent of t. Hence, for each s, t in $[0, 2\pi)$, we have $(g_s, g_t) \in \mathscr{E}$. Since $T \in \mathscr{H}_{\mathscr{B}}$, it follows that $g_s(T) = g_t(T)$; so $g_t(T)$ is independent of $t \in [0, 2\pi)$. However,

$$\begin{aligned} g_t(T) &= k^2 \langle \phi_f(T)(x + e^{it}y),\ x + e^{it}y \rangle \\ &= p + qe^{it} + re^{-it} , \end{aligned}$$

where p, q, r are independent of t and

$$r = k^2 \langle \phi_f(T)x, y \rangle = k^2 \| y \|^2 \neq 0 .$$

Thus $g_t(T)$ is not independent of $t \in [0, 2\pi)$, and we have obtained a contradiction. This proves the lemma.

LEMMA 5. *$\mathscr{H}_{\mathscr{B}}$ is a C^*-sub-algebra of $\mathscr{A}$.*

Proof. Suppose that $(g, h) \in \mathscr{E}$. Let M_g be the closed subspace of H_g which is generated by $\phi_g(\mathscr{B})x_g$. It follows from Lemma 4 that M_g is invariant under $\phi_g(\mathscr{H}_{\mathscr{B}})$. When $T \in \mathscr{H}_{\mathscr{B}}$, we shall write $\phi_g(T) \mid M_g$ for the operator (from M_g into M_g) obtained by restricting $\phi_g(T)$ to M_g. Similar notations will be used with h in place of g.

Given $T \in \mathscr{H}_{\mathscr{B}}$ and $R, S \in \mathscr{B}$, we have (Lemma 3) $R^* TS \in \mathscr{H}_{\mathscr{B}}$. Since $(g, h) \in \mathscr{E}$, it follows that $g(R^* TS) = h(R^* TS)$, or equivalently that

$$\langle \phi_g(T)\phi_g(S)x_g,\ \phi_g(R)x_g \rangle = \langle \phi_h(T)\phi_h(S)x_h,\ \phi_h(R)x_h \rangle . \tag{2}$$

By taking $T = I$, we deduce the existence of a unitary operator U from M_g on to M_h such that

$$U\phi_g(S)x_g = \phi_h(S)x_h \qquad (S \in \mathscr{B}) . \tag{3}$$

Equation (2) then implies that

$$\langle \phi_g(T)v, w \rangle = \langle \phi_h(T)Uv, Uw \rangle \qquad (T \in \mathscr{H}_{\mathscr{B}})$$

for all $v, w \in \phi_g(\mathscr{B})x_g$, hence for all $v, w \in M_g$. The last equation is equivalent to

$$\phi_g(T) \mid M_g = U^*[\phi_h(T) \mid M_h]U \qquad (T \in \mathscr{H}_{\mathscr{B}}) . \tag{4}$$

Now suppose that $T_1, T_2 \in \mathscr{H}_{\mathscr{B}}$. Given $(g, h) \in \mathscr{E}$, construct U as above. Since $\phi_g(T_i)$ leaves M_g invariant ($i = 1, 2$), so does $\phi_g(T_1T_2)$, and

$$\phi_g(T_1T_2) \mid M_g = [\phi_g(T_1) \mid M_g][\phi_g(T_2) \mid M_g] ;$$

similar considerations apply with h in place of g. From (4), with $\mathrm{T} = T_1, T_2$, we deduce that

$$\phi_g(T_1T_2) \mid M_g = U^*[\phi_h(T_1T_2)\, M_h]U .$$

Since $x_g \in M_g$ and $Ux_g = x_h$, the last equation implies that

$$\langle \phi_g(T_1T_2)x_g,\ x_g\rangle = \langle \phi_h(T_1T_2)x_h,\ x_h\rangle ;$$

that is, $g(T_1T_2) = h(T_1T_2)$. This holds whenever $(g, h) \in \mathscr{E}$, so $T_1T_2 \in \mathscr{H}_{\mathscr{B}}$.

We have now shown that $\mathscr{H}_{\mathscr{B}}$ admits multiplication; since $\mathscr{H}_{\mathscr{B}}$ is clearly a closed self-adjoint linear subspace of $\mathscr{A}$, the lemma is proved.

4. Proof of Theorem 2. We shall use the notations introduced in the statement of Theorem 2. It is immediate from the definition of $\mathscr{H}_{\mathscr{B}}$ that $\mathscr{B} \subseteq \mathscr{H}_{\mathscr{B}}$.

We first consider the case in which $I \in \mathscr{B}$, so that the theory developed in § 3 applies to show that $\mathscr{H}_{\mathscr{B}}$ is a C^*-algebra. We remark that each element f of the pure state space $\mathscr{P}(\mathscr{H}_{\mathscr{B}})$ can be extended to an element $\bar{f}$ of $\mathscr{P}(\mathscr{A})$. For there is a net (f_i) of pure states of $\mathscr{H}_{\mathscr{B}}$, converging to f in the weak * topology. Each f_i can be extended to a pure state $\bar{f}_i$ of $\mathscr{A}$ (see, for example, [2] 304). Since $\mathscr{P}(\mathscr{A})$ is compact, the net $(\bar{f}_i)$ has at least one weak * limit point $\bar{f} \in \mathscr{P}(\mathscr{A})$, and $\bar{f}$ is an extension of f.

Suppose that $\mathscr{B} \neq \mathscr{H}_{\mathscr{B}}$. Then by Glimm's theorem there exist distinct $g, h \in \mathscr{P}(\mathscr{H}_{\mathscr{B}})$ such that $g(B) = h(B)$ ($B \in \mathscr{B}$). We may extend g, h to elements, $\bar{g}, \bar{h}$ respectively of $\mathscr{P}(\mathscr{A})$. Clearly $(\bar{g}, \bar{h}) \in \mathscr{E}$. Thus, by the definition of $\mathscr{H}_{\mathscr{B}}$, $\bar{g}(T) = \bar{h}(T)$ whenever $T \in \mathscr{H}_{\mathscr{B}}$; that is, $g = h$, contrary to hypothesis. This proves Theorem 2 for the case in which $\mathrm{I} \in \mathscr{B}$.

If $I \notin \mathscr{B}$, let $\mathscr{B}_1 = \mathscr{B} + \boldsymbol{C}I$ be the C^*-algebra generated by I, $\mathscr{B}$ ($\boldsymbol{C}$ denotes the complex field). With an obvious modification of the notation introduced in Theorem 2, it is clear that $\mathscr{N}(\mathscr{B}_1)$ is empty and that $\mathscr{E}(\mathscr{B}_1) = \mathscr{E}(\mathscr{B})$. Thus $\mathscr{H}_{\mathscr{B}} \subseteq \mathscr{H}_{\mathscr{B}_1}$; since $I \in \mathscr{B}_1$, the first part of this proof shows that $\mathscr{B}_1 = \mathscr{H}_{\mathscr{B}_1}$, so $\mathscr{H}_{\mathscr{B}} \subseteq \mathscr{B}_1$.

Now let f be the pure state of $\mathscr{B}_1$ defined by $f(\lambda I + B) = \lambda$ ($\lambda \in \boldsymbol{C}$, $B \in \mathscr{B}$), and let g be any extension of f to a pure state of $\mathscr{A}$. Clearly $g \in \mathscr{N}(\mathscr{B})$. Hence $g(\mathscr{H}_{\mathscr{B}}) = (0)$, and

$$\mathscr{H}_{\mathscr{B}} \subseteq \mathscr{B}_1 \cap g^{-1}(0) = f^{-1}(0) ;$$

that is, $\mathscr{H}_{\mathscr{B}} \subseteq \mathscr{B}$. The reverse inclusion has already been noted, so $\mathscr{B} = \mathscr{H}_{\mathscr{B}}$.

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