

ALGEBRAS OF BOUNDED SEQUENCES

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Let l^∞ be the algebra of all bounded sequences of complex numbers. The primary purpose of this paper is to settle, by means of a counter-example, a conjecture about subalgebras of l^∞ : If A is a subalgebra of l^∞ which is closed under uniform convergence and separates the points of βN , then $A = l^\infty$.

The main tool used in the construction of the example is a positive result about Boolean algebras (Proposition 1), which seems interesting in its own right.

Our interest in subalgebras of l^∞ stems from recent work on interpolation problems. In 1957, R. C. Buck inquired as to which sequences $\{z_n\}$ in the unit disc of the plane have this property: If $\{w_n\}$ is an arbitrary bounded sequence of complex numbers, there exists a function f , bounded and analytic in the unit disc, such that $f(z_n) = w_n$ for each n . This question was answered very effectively by L. Carleson [2]. It was also answered in a slightly weaker form by D. J. Newman [7]; and, it was partially answered by W. Hayman [5].

Badé and Curtis [1] have obtained strong results on subalgebras of $C(X)$, the algebra of continuous complexvalued functions on a compact Hausdorff space X , in case X has the property that any two disjoint open F_σ subsets have disjoint closures. The smallest nonfinite space with that property is βN , the Čech compactification of the integers. Since l^∞ is isomorphic to $C(\beta N)$, the work of Badé and Curtis has shed some light on a general class of interpolation problems of the type which we previously mentioned [see 6; page 205].

Technical aspects of the work of Badé and Curtis, as well as similar aspects of the interpolation problem for bounded analytic functions, have led to the aforementioned conjecture. Were this conjecture true, it would be a powerful tool in the study of interpolation problems.

We shall settle the conjecture in the negative. We show that, if the continuum hypothesis is valid, there exists a subalgebra A of l^∞ such that

- (i) A is closed under uniform convergence;
- (ii) every nonnegative sequence in l^∞ is the modulus of a sequence in A ;
- (iii) $A \neq l^\infty$.

The example is constructed as follows. Let \mathcal{B} be the Boolean

algebra of measurable subsets of the unit circle, modulo null sets. Let B be the Boolean algebra of subsets of the integers modulo finite sets. As a lattice, B has an unusual property (property (ii) of Lemma 4). We exploit that property and the continuum hypothesis, to produce an isomorphism of \mathcal{B} into B . We extend the inverse of the isomorphism to a homomorphism from B onto \mathcal{B} . The adjoint of such a homomorphism is a homeomorphism of the Stone space $S(\mathcal{B})$ into the Stone space $S(B)$. The space $S(\mathcal{B})$ is the maximal ideal space of L^∞ , the Banach algebra of essentially bounded measurable functions on the unit circle. The space $S(B)$ is $\beta N - N$. From the algebra of bounded analytic functions in the unit disc, we construct a subalgebra of $L^\infty = C(S(\mathcal{B}))$, and carry that algebra with the imbedding of $S(\mathcal{B})$ in $\beta N - N$, to obtain the desired subalgebra of l^∞ .

Homomorphisms of Boolean algebras. In a Boolean algebra B , denote least upper bounds by \vee , greatest lower bounds by \wedge , and complements by $'$. Denote the corresponding set operations by \cup , \cap , and $'$. If $a \leq b$, let $[a, b]$ be the set of all x in B such that $a \leq x \leq b$. If E is a subalgebra of B and x is an element of B , let $E[x]$ be the subalgebra generated by E and x :

$$E[x] = \{(e \wedge x) \vee (f \wedge x'); e, f \in E\}.$$

We shall be concerned with the problem of extending Boolean algebra homomorphisms. The basic extension lemma is the following, implicitly given by Sikorski in [8].

LEMMA 1. *Let E be a subalgebra of the Boolean algebra B , and let φ be a homomorphism of E into a Boolean algebra C . If $x \in B$, then φ can be extended to a homomorphism of $E[x]$ into C if and only if there exists an element y in C such that*

- (i) $\varphi([0, x] \cap E) \subseteq [0, y] \cap \varphi(E)$
- (ii) $\varphi([x, 1] \cap E) \subseteq [y, 1] \cap \varphi(E)$.

For each such y , there is a unique extension φ^+ which takes x into y ; and then $\varphi^+(E[x]) = \varphi(E)[y]$. If φ is an imbedding and y is such that equality holds in (i) and (ii), then the corresponding extension φ^+ is also an imbedding.

Proof. We content ourselves with the remark that, if y satisfies (i) and (ii), the extension φ^+ is given by

$$\varphi^+((e_1 \wedge x) \vee (e_2 \wedge x')) = (\varphi(e_1) \wedge y) \vee (\varphi(e_2) \wedge y').$$

LEMMA 2. (Sikorski) *If B is a Boolean algebra and φ is a homomorphism of a subalgebra of B into a complete Boolean algebra C , then φ can be extended to a homomorphism of B into C .*

Proof. In the context of Lemma 1, we have

$$\sup \varphi([0, x] \cap E) \leq \inf \varphi([x, 1] \cap E) .$$

Therefore, the element y can always be found. Apply Zorn's lemma.

Of course, Lemma 2 says that complete Boolean algebras are injective in the category of Boolean algebras and Boolean homomorphisms.

An F -space is a topological space in which any two disjoint open F_σ subsets have disjoint closures. A Boolean algebra is an F -algebra if its Stone space is an F -space. An F -algebra can also be described as one which satisfies property (I) of Badé and Curtis [1]: If $x_1, y_1, x_2, y_2, \dots$ is an orthogonal sequence in B , then there exist orthogonal elements x and y in B , such that x is an upper bound of $\{x_1, x_2, \dots\}$ and y is an upper bound of $\{y_1, y_2, \dots\}$.

LEMMA 3. *A homomorphic image of an F -algebra is an F -algebra. If B is an F -algebra in which orthogonal sets are (at most) countable, then B is complete.*

Proof. Let B and C be Boolean algebras, and let φ be a homomorphism from B onto C . If c_1, c_2, \dots is an orthogonal sequence in C , there can be defined (by induction) an orthogonal sequence of pre-images in B . Choose any b_1 such that $\varphi(b_1) = c_1$. If b_1, \dots, b_k are orthogonal with $\varphi(b_j) = c_j, 1 \leq j \leq k$, choose any x in B with $\varphi(x) = c_{k+1}$ and set $b_{k+1} = (b_1 \vee \dots \vee b_k)' \wedge x$. From the alternative description of F -algebras, it is then clear that C is an F -algebra if B is.

Now, suppose that B is an F -algebra and that orthogonal sets in B are at most countable. To prove that B is complete, it suffices to show that every orthogonal set in B has an upper bound. Evidently, the only case of interest is that of an orthogonal sequence x_1, x_2, \dots which is part of a maximal orthogonal set $\{x_1, y_1, x_2, y_2, \dots\}$. Since B is an F -algebra, there exist orthogonal elements x, y in B such that x is an upper bound of $\{x_1, x_2, \dots\}$ and y is an upper bound of $\{y_1, y_2, \dots\}$. Then x must be the least upper bound of $\{x_1, x_2, \dots\}$. For, if u is another upper bound of $\{x_1, x_2, \dots\}$, then $z = x \wedge u$ is also such an upper bound. If z were less than x , we could adjoin $z' \wedge x$ to $\{x_1, y_1, x_2, y_2, \dots\}$ and obtain a properly larger orthogonal set.

Examples of F -algebras abound. Evidently, any Boolean σ -algebra is an F -algebra. Consequently, any homomorphic image of a σ -algebra is an F -algebra. In particular, if K is a closed subset of the Stone space of a σ -algebra, then the algebra of all (relatively) open-closed subsets of K is an F -algebra. Of course, the same is true of any closed subset of the Stone space of an F -algebra. In other words, a closed subset of a (compact and totally disconnected) F -space is an F -space.

One can use Lemma 3 to show that certain homomorphic images of σ -algebras are complete. We are interested in some homomorphic images of σ -algebras which might be called anti-complete, because, they contain no strictly increasing sequence which has a least upper bound. One such algebra is the algebra of subsets of the integers modulo finite sets, i.e., the algebra of open-closed subsets of the space $\beta N - N$. The following lemma provides us with a class of algebras of the type which we have in mind.

LEMMA 4. *Let \mathcal{B} be an infinite Boolean σ -algebra, with Stone space S . Let K be a nonempty closed G_δ in S such that the interior of K is empty. Let B be the Boolean algebra of (relatively) open-closed subsets of K .*

(i) *Each maximal orthogonal set in B is either finite or uncountable.*

(ii) *If $a_1 \leq a_2 \leq \dots$ and $b_1 \geq b_2 \geq \dots$ in B , and if $a_n < b_n$ for every n , then there is an element c in B such that $a_n < c < b_n$ for every n .*

Proof. If x is in \mathcal{B} , let $[x]$ be the corresponding open-closed subset of the Stone space S . The fact that K is a nonempty closed G_δ without interior means that

$$K = \bigcap_n [e_n]$$

where $e_1 > e_2 > \dots$ is a strictly decreasing sequence in \mathcal{B} and $\inf_n e_n = 0$. The map

$$\gamma(x) = [x] \cap K$$

is a homomorphism from \mathcal{B} onto B , and, $\gamma(x) = \gamma(y)$ if and only if there exists an n such that $x \wedge e_n = y \wedge e_n$.

For the proof of (i), suppose that $\{a_1, a_2, \dots\}$ is a countable orthogonal set in B . We shall show that there exists a nonzero element of B which is orthogonal to each a_n . We may assume that each a_n is nonzero. Let x_1, x_2, \dots be an orthogonal sequence in B with $\gamma(x_n) = a_n$. If k is a positive integer, then $\gamma(x_k) = a_k \neq 0$; hence, $x_k \wedge e_n \neq 0$ for all n . Thus, we must have $x_k \wedge (e_n - e_{n+1}) \neq 0$ for infinitely many values of n . From this it is clear that there is a sequence of integers $n_1 < n_2 < \dots$ such that the elements

$$y_k = x_k \wedge (e_{n_k} - e_{n_{k+1}})$$

are nonzero, $k = 1, 2, \dots$. Let $y = \sup_k y_k$. For each k , the element $y \wedge e_{n_{k+1}}$ is orthogonal to x_k . Therefore, $\gamma(y)$ is orthogonal to each a_k . Since

$$y \wedge e_k \geq y \wedge e_{n_k} \geq y_k > 0,$$

we see that $\gamma(y) \neq 0$.

The proof of (ii) is essentially the same as the proof that every nonempty closed G_δ in the space $\beta N - N$ has a nonempty interior [4; pages 98, 99]. First note that B has no atoms. Such an atom would correspond to a point of K which is isolated in K . The point would then be a G_δ in S . Any such point is isolated in S and must be in the interior of K . Since B has no atoms, (ii) is trivial if both sequences are eventually constant. Otherwise, the sequence $a_1, b_1, a_2 - a_1, b_1 - b_2, \dots$ has infinitely many nonzero terms. By (i), there exist nonzero elements a, b in B such that the sequence $a, b, a_1, b'_1, a_2 - a_1, b_1 - b_2, \dots$ is orthogonal. Since B is an F -algebra, there is an element c in B which is $\{b, b'_1, b_1 - b_2, \dots\}$ and which is an upper bound of $\{a, a_1, a_2 - a_1, \dots\}$. Then, for each n

$$a_n < a_n \vee a \leq c \leq b_n - b < b_n.$$

That completes the proof.

We now focus our attention on a Boolean algebra B , which has property (ii) of Lemma 4. As we have indicated, the algebra of open-closed subsets of $\beta N - N$ is one such algebra. It is easy to see that a Boolean algebra B has property (ii) if and only if its Stone space $S(B)$ has these three topological properties:

- (a) $S(B)$ is an F -space;
- (b) every nonempty (closed) G_δ in $S(B)$ has a nonempty interior;
- (c) $S(B)$ has no isolated points.

We might remark that, if $S(B)$ satisfies (a), (b), and (c), then the Stone space for the algebra of Borel sets in $S(B)$ modulo first category sets is a Stonian space in which every first category set is nowhere dense and on which every measure has nowhere dense support. See Dixmier [3].

LEMMA 5. *Let B be a Boolean algebra which has property (ii) of Lemma 4. Let I be a countably generated ideal in B , and let F be a countable subset of B which is disjoint from I . If y is a strict upper bound of I , there exists an element z in B such that*

- (a) $z < y$
- (b) z is a strict upper bound of I
- (c) $F \cap [0, z]$ is empty.

Proof. First let us show that, if u is an element of B which is not in I , there is an $x \leq y$ such that $u \not\leq x$ and x is a strict upper bound of I . We may suppose that $u \leq y$; otherwise we may take $x = y$. Since u is not in I , the ideal generated by I and u' is proper. Let J be that ideal. Since J is countably generated, we can find $t < 1$ such

that t is a strict upper bound of J . Choose t_1 with $0 < t_1 < t'$, and let $x = y - t_1$. It is easy to see that x is a strict upper bound of I and that $u \not\leq x$.

If $F = \{f_1, f_2, \dots\}$, the preceding paragraph shows us how to construct a sequence $y_1 \geq y_2 \geq \dots$ such that each y_n is a strict upper bound of I and such that, for each n , we have $f_n \not\leq y_n$. Since B has property (ii) of Lemma 4, there exists an element z in B which is a strict upper bound of I and is a strict lower bound of the dual ideal generated by y_1, y_2, \dots . Any such z has properties (a), (b), and (c).

PROPOSITION 1. Let B be a Boolean algebra which has property (ii) of Lemma 4. Any Boolean algebra of cardinality at most \aleph_1 can be imbedded in B .

Proof. Let C be a Boolean algebra, the cardinality of which does not exceed \aleph_1 . Then C is the union of a well-ordered chain of countable subalgebras. We can arrange that the chain starts with the subalgebra $\{0, 1\}$ and that each member of the chain is generated by the preceding subalgebras and one additional element. By Lemma 1 and induction, it suffices to prove the following. If E is a countable subalgebra of C , if φ is an isomorphism of E onto the subalgebra F of B , and if x is in $C - E$, then there exists an element y in B such that

$$(a) \quad \varphi([0, x] \cap E) = [0, y] \cap F$$

$$(b) \quad \varphi([x, 1] \cap E) = [y, 1] \cap F.$$

Let I be the ideal in B which is generated by the left-hand member of (a), and let D be the dual ideal in B which is generated by the left-hand member of (b). Then, both I and D are countably generated, and each element of D is a strict upper bound of I . If we apply property (ii) of Lemma 4 and then Lemma 5, we obtain an element u_0 of B which is a strict upper bound of I and is a strict lower bound of D , while $[0, u_0]$ and $F - I$ are disjoint.

Now, we show that, if v is any element of $F - D$, then there exists u such that $u \geq u_0$; $v \not\leq u$; u is a strict lower bound of D ; and $[0, u] \cap (F - I)$ is empty. If $v \not\geq u_0$, we may take $u = u_0$. If $v \geq u_0$, observe that the dual ideal D_1 , which is generated by $D \cup \{v'\}$, is both countably generated and proper. Hence, there exists a nonzero element w in B which is a strict lower bound of D_1 . Then $0 < w < v'$, and $u_0 \vee w$ is a lower bound of D , with $u_0 < u_0 \vee w$. We apply Lemma 5 to the ideal $[0, u_0]$ and the subset $F - I$. We obtain an element u such that $u_0 < u < u_0 \vee w$ and $(F - I) \cap [0, u]$ is empty. Then u is a strict lower bound of D , and $v \not\leq u$, because $u \wedge v' \geq u \wedge w > 0$.

Suppose $F - D = \{v_1, v_2, \dots\}$. We employ the argument which we have just concluded, to construct an increasing sequence $u_0 \leq u_1 \leq u_2 \leq \dots$ such that, for each n , we have the following: $v_n \not\leq u_n$; $[0, u_n] \cap (F - I)$

is empty; u_n is a strict lower bound of D . Once again we apply property (ii) of Lemma 4 and then Lemma 5, to conclude that there exists an element y in B such that $[0, y] \cap (F - I)$ is empty, y is a strict lower bound of D , and y is a strict upper bound of the ideal generated by u_1, u_2, \dots . This is the y which we seek.

COROLLARY. *Let B be an infinite Boolean F -algebra. Let C be a Boolean algebra of cardinality at most \aleph_1 in which orthogonal sets are at most countable. If C^+ is the completion of C by cuts, then C^+ is a homomorphic image of B . Hence, the Stone space $S(C^+)$ can be imbedded in the Stone space $S(B)$.*

Proof. Select any (infinite) discrete sequence in the Stone space $S(B)$. Its closure is homeomorphic to the space βN . Hence, the algebra of open-closed subsets on $\beta N - N$ is a homomorphic image of B .

We see that we need only prove the Corollary when B is the Boolean algebra of open-closed subsets of the space $\beta N - N$. In that case, Proposition 1 provides us with an isomorphism φ , from a subalgebra of B onto the algebra C . We can extend φ to a homomorphism ψ , from B into the completion C^+ . Since orthogonal sets in C are at most countable, the same is true in the algebra C^+ . By Lemma 3, the range of the homomorphism ψ is complete. Hence, ψ maps B onto C^+ .

Suppose C_0 and C_1 are the free Boolean algebras on \aleph_0 and \aleph_1 generators, respectively. Each of these algebras satisfies the conditions on the algebra C in the Corollary. Therefore, the Stone spaces $S(C_0^+)$ and $S(C_1^+)$ can be imbedded in $\beta N - N$. In particular, $\beta N - N$ contains closed subsets which are Stonian but not hyper-Stonian [3].

PROPOSITION 2. Assume the continuum hypothesis. Let B be the Boolean algebra of subsets of the integers, modulo finite sets. If C is a complete Boolean algebra of cardinality at most 2^{\aleph_0} , then C is a homomorphic image of B . Hence, the Stone space $S(C)$ can be imbedded in the space $\beta N - N$.

Proof. Since $2^{\aleph_0} = \aleph_1$, Proposition 1 tells us that C can be imbedded in B . Then, we have an isomorphism φ , from a subalgebra of B onto the algebra C . Since C is complete, φ can be extended to a homomorphism of B onto C .

The example. Let L^∞ be the algebra of essentially bounded measurable functions on the unit circle in the plane. Identify functions which agree almost everywhere. Then, with the essential supremum as norm, L^∞ is a commutative Banach algebra which is isometrically isomorphic

to $C(X)$, the algebra of continuous functions on its maximal ideal space X . The isomorphism is the Gelfand representation $f \rightarrow \hat{f}$. We let H^∞ be the subalgebra of L^∞ which consists of the functions h whose negative Fourier coefficients vanish. If $h \in H^\infty$ then the Poisson formula

$$h(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(e^{i\theta}) Re \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta$$

extends h to a bounded analytic function in the unit disc; also, each bounded analytic function arises as the Poisson integral of its boundary values

$$h(e^{i\theta}) = \lim_{r \rightarrow 1} h(re^{i\theta})$$

which exist almost everywhere. For a summary of these basic facts, see [6].

LEMMA 6. *Let K be a closed subset of X , the maximal ideal space of L^∞ , and suppose the interior of K is empty. If u is any nonnegative continuous function on K , there exists a function h in H^∞ such that the modulus of \hat{h} agrees with u on K .*

Proof. Let m be the measure on X which corresponds to normalized Lebesgue measure on the unit circle. In other words, if ψ is the linear functional

$$\psi(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) d\theta$$

on L^∞ , the Riesz representation theorem, together with the isomorphism between L^∞ and $C(X)$, provides us with a unique probability measure m on X such that

$$\psi(f) = \int_X \hat{f} dm, \quad f \in L^\infty.$$

Then X is a hyperstonian space and m is a normal measure on X , i.e., if S is a measurable subset of X , m assigns the same measure to S and the interior of S :

$$m(S) = m(\text{int } S).$$

Consider the given closed set K in X . Since K has no interior, $m(K) = 0$. Hence, we can construct a nonnegative continuous function on X which vanishes on K and has its logarithm integrable with respect to m . That is, there exists an f in L^∞ such that

- (a) $f \geq 0$
- (b) $\hat{f} = 0$ on K
- (c) $\int \log f(e^{i\theta})d\theta > -\infty$.

Let u be a nonnegative continuous function on K . We can extend u to a nonnegative continuous function on X . Thus, we have a function g in L^∞ such that $g \geq 0$ and $\hat{g} = u$ on K . With the function f of the previous paragraph define

$$h(z) = \exp \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log [f(e^{i\theta}) + g(e^{i\theta})]d\theta .$$

Since $\log (f + g)$ is integrable and bounded above, h is a bounded analytic function in the unit disc. The boundary values

$$h(e^{i\theta}) = \lim_{r \rightarrow 1} h(re^{i\theta})$$

then satisfy

$$|h| = f + g, \text{ almost everywhere .}$$

So $|\hat{h}| = \hat{f} + \hat{g}$. In particular, since $\hat{f} = 0$ on K , we have

$$|\hat{h}|_K = |\hat{g}|_K = u .$$

The closed set K of the last lemma may be sufficiently small so that the restriction of \hat{H}^∞ to K is all of $C(K)$. If K is a nonempty G_δ , that does not happen; however, we shall not stop to prove that here. We shall choose a particular K on which it does not happen.

Let K be the set of maximal ideals of L^∞ which contain the function $(1 - z)$. Then K is a closed G_δ without interior in X , and the restriction of \hat{H}^∞ to K is a proper closed subalgebra of $C(K)$. See [6; page 187].

Let \mathcal{B} be the Boolean algebra of measurable sets on the circle, modulo null sets. The Stone space of \mathcal{B} is the space X . The Stone representation for \mathcal{B} is simply the restriction of the Gelfand representation of L^∞ to the collection of characteristic functions of measurable sets. According to Proposition 2, there exists a homeomorphism τ , of X into the space $\beta N - N$. We use τ and the set K to define an algebra of continuous functions on βN . Let A be the algebra of continuous functions f on βN such that, on the set K , the composition $f \circ \tau$ agrees with the restriction of a function in \hat{H}^∞ . We assert that

- (i) A is a uniformly closed subalgebra of $C(\beta N)$;
- (ii) every nonnegative continuous function on βN is the modulus of a function in A ;
- (iii) $A \neq C(\beta N)$.

It is clear that (i) and (iii) are satisfied. Property (ii) follows from this observation. Suppose g is a nonnegative continuous function on

a totally disconnected compact Hausdorff space Y . Let M be a closed subset of Y , and let h be a continuous complex-valued function on Y such that the modulus of h agrees with g on M . Then, there exists f in $C(Y)$ such that $f|_M = h|_M$ and $|f| = g$ on all of Y .

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