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NORM DECREASING HOMOMORPHISMS OF GROUP ALGEBRAS

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The homomorphisms φ of the group algebra $L^1(F)$ into the algebra $M(G)$ of measures, where F and G are locally compact groups, has been completely determined when both groups are abelian by P. J. Cohen, and when G is compact and the homomorphism is norm decreasing and order-preserving by Glicksberg. In this paper the structure of norm decreasing homomorphisms φ is determined for arbitrary locally compact F and G. As an application the special structure of all norm decreasing monomorphisms is determined, along with the rather elegant structure of all norm decreasing homomorphisms mapping $L^1(F)$ onto $L^1(G)$.

The analysis is effected by finding all multiplicative subgroups of the unit ball of measures on a locally compact group for, as we show, each φ extends to a norm decreasing homomorphism $\varphi: M(F) \to M(G)$, and is determined by the image under φ of the group of point masses on G, a multiplicative subgroup of the unit ball in $M(G)$.

This paper completes a study of norm decreasing homomorphisms on group algebras initiated by Glicksberg in [4] and [5]. If G is a locally compact group we will denote its group algebra by $L^1(G)$ and its convolution algebra of bounded regular Borel measures by $M(G)$. We present a complete structural analysis of the subgroups of the unit ball in $M(G)$, and a structure theory classifying all norm decreasing homomorphisms $\varphi: L^1(F) \to M(G)$ where F and G are locally compact groups. As an application we determine the special structure of all monomorphisms φ mapping $L^1(F)$ into $M(G)$ and all norm decreasing homomorphisms which map $L^1(F)$ onto $L^1(G)$.

Let $C_0(G)$ be the sup norm algebra of all continuous complex valued functions on G which vanish at infinity, and recall that $C_0(G)^* = M(G)$. If $\mu \in M(G)$ its support $s(\mu)$ is defined so that $x \in s(\mu) \Leftrightarrow$ for each neighborhood U of x there is some $\psi \in C_0(G)$, vanishing outside of U, with $\langle \mu, \psi \rangle \neq 0$. Then $s(\mu)$ is a Borel set. If Γ is a subset in $M(G)$ we define $\operatorname{supp}(\Gamma) = \cup\{s(\mu): \mu \in \Gamma\}$. The convolution of $\mu, \lambda \in M(G)$ is given as an element of $C_0(G)^*$ by defining

$$\langle \mu * \lambda, \psi \rangle = \int_G \left[\int_G \psi(st) d\mu(s) \right] d\lambda(t)$$

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for all $\psi \in C_0(G)$. If $M(G)$ is given the total variation norm it becomes a Banach algebra under this multiplication.

We first show that if $\mu, \lambda \in M(G)$ then

(1) $\|\mu*\lambda\| = \|\mu\|\cdot\|\lambda\| \Rightarrow s(\mu*\lambda) = (s(\mu)s(\lambda))^-$

(2) $\|\mu*\lambda\| = \|\mu\|\cdot\|\lambda\| \Rightarrow |\mu*\lambda| = |\mu|*|\lambda|$.

These facts were first pointed out, in somewhat less general form, by Wendel [11] and Glicksberg [4]. These results suffice for the analysis of the subgroups of the unit ball in $M(G)$.

In order to determine the norm decreasing homomorphisms $\varphi: L^1(F) \rightarrow M(G)$ we use an important observation that such a map always extends to a norm decreasing homomorphism $\bar{\varphi}: M(F) \rightarrow M(G)$ which is continuous on norm bounded sets as a mapping of $(M(F), (so))$ into $(M(G), (\sigma))$. Here (σ) is the usual weak $*$ topology on $M(G)$, and (so) is the strong operator topology on $M(F)$ gotten by letting $M(F)$ act by left convolution on the ideal $L^1(F) \subset M(F)$.

The author is greatly indebtted to the earlier work of Glicksberg presented in [4], [5]. He is also pleased to acknowledge Professor Glicksberg's helpful commentary in private correspondence. It will be clear to the reader familiar with [4] that the proof of the fundamental relation $\|\mu*\lambda\| = \|\mu\|\cdot\|\lambda\| \Rightarrow |\mu*\lambda| = |\mu|*|\lambda|$ is a simple adaptation of a Glicksberg theorem dealing with compact groups. The simpler proof given here was suggested by Glicksberg.

1. Preliminaries. Throughout this paper we will find it convenient to write convergence of a net $\{x_j\}$ to a point x in a topological space (X, τ) as $x_j \xrightarrow{(\tau)} x$ or $x \xleftarrow{(\tau)} x_j$, interchangeably. To avoid confusion in discussing homomorphisms we will use the terms homomorphism (epimorphism, monomorphism) for into (onto, $1:1$) homomorphisms; we reserve the term isomorphism for $1:1$ onto homomorphisms.

Most measure theoretic notions are taken from Halmos [3], including definition of Baire and Borel sets. In the following discussion let $\boldsymbol{B} = \boldsymbol{B}(G)$ ($\boldsymbol{B}_0 = \boldsymbol{B}_0(G)$) be the collection of Borel (Baire) sets in G. If a function f is defined on G and if $\boldsymbol{H}$ is a σ-ring of sets in G, we say that f is $\boldsymbol{H}$ measurable on all $\boldsymbol{H}$ sets of G if $\chi_E f$ is $\boldsymbol{H}$ measurable for each set $E \in \boldsymbol{H}$ (χ_E = characteristic function of E). It is clear that $\boldsymbol{B}_0$ measurability on $\boldsymbol{B}_0$ sets implies $\boldsymbol{B}$ measurability on $\boldsymbol{B}$ sets in G.

If $\mu \in M(G)$ define its Baire contraction μ' by restricting its domain of definition to be $\boldsymbol{B}_0(G)$. A regular Borel measure is uniquely determined by its Baire contraction (see [3], 54. D). If $E \in \boldsymbol{B}(G)$ it must be σ-bounded, and hence there is a Baire set $A \supset E$; this applies in particular to $s(\mu)$ where $\mu \in M(G)$. If $E \in \boldsymbol{B}$ and f is $\boldsymbol{B}$ measurable on G, we let $\int_E f\,d\mu$ denote the integral $\int_G \chi_E f\,d\mu$.

In applying the Fubini theorem, Borel functions have rather pathological properties when compared to those of Baire functions. These difficulties arise from the fact that the product σ-ring $\boldsymbol{B}_0 \times \boldsymbol{B}_0 = \boldsymbol{B}_0(G \times G)$, while we only know $\boldsymbol{B}_0 \times \boldsymbol{B}_0 \subset \boldsymbol{B} \times \boldsymbol{B} \subset \boldsymbol{B}(G \times G)$ for the corresponding Borel sets. If f is $\boldsymbol{B}_0$ measurable on G and if $A, B \in \boldsymbol{B}_0$, then the function $\chi_{A \times B}(s, t)f(st)$ is $\boldsymbol{B}_0 \times \boldsymbol{B}_0$ measurable on $G \times G$ (hence $\boldsymbol{B} \times \boldsymbol{B}$ measurable) and we can apply Fubini to the convolution-like integral

$$(*) \qquad \int_{G \times G} \chi_{A \times B}(s, t) f(st) d\mu \times \lambda(s, t) .$$

If f is $\boldsymbol{B}$ measurable, the best we can say is that $\chi_{A \times B}(s, t)f(st)$ is $\boldsymbol{B}(G \times G)$ measurable, but this does not give the $\boldsymbol{B} \times \boldsymbol{B}$ measurability required to make $(*)$ well defined. To avoid these difficulties we will rely on the following well known observations.

(R1) If f is bounded and $\boldsymbol{B}_0$ measurable, and if $E, F \in \boldsymbol{B}_0$, then $\chi_{E \times F}(s, t)f(st)$ is $\boldsymbol{B}_0 \times \boldsymbol{B}_0$ measurable on $G \times G$.

(R2) If f is bounded and $\boldsymbol{B}_0$ measurable, and if $\mu, \lambda \in M(G)$, let us choose any sets $E, F \in \boldsymbol{B}_0$ such that $E \supset s(\mu)$, $F \supset s(\lambda)$. Then

$$\int_G f(x) d\mu * \lambda(x) = \int_{G \times G} \chi_{E \times F}(s, t) f(st) d\mu \times \lambda(s, t) .$$

(R3) If $\mu \in M(G)$ there exists a unimodular function f_μ which is $\boldsymbol{B}_0$ measurable on $\boldsymbol{B}_0$ sets in G, such that $\mu = f_\mu |\mu|$. Thus if $E \in \boldsymbol{B}$,

$$\mu(E) = \int_G \chi_E(s) d\mu(s) = \int_G \chi_E(s) f_\mu(s) d|\mu|(s) .$$

Notice that $\Psi(t) = \int \psi(st) d\mu(s)$ is in $C_0(G)$ if $\psi \in C_0(G)$, and all $C_0(G)$ functions are $\boldsymbol{B}_0$ measurable on G, so the definition of convolution is meaningful for $\mu, \lambda \in M(G)$. Convolution is actually independent of the order of iteration of the integrals used to define it, in fact the above remarks show that

$$\begin{aligned}\int_G \left[\int_G \psi(st) d\mu(s)\right] d\lambda(t) &= \int_{G \times G} \chi_{E \times F}(s, t) \psi(st) d\mu \times \lambda(s, t) \\ &= \int_G \left[\int_G \psi(st)\, d\lambda(t)\right] d\mu(s) ,\end{aligned}$$

if $E, F \in \boldsymbol{B}_0$ are such that $E \supset s(\mu)$, $F \supset s(\lambda)$.

If G is a locally compact group and if $Q \subset G$ we let $\mathscr{E}_Q = \{\delta_x : x \in Q\}$ where δ_x is the point mass at x for $x \in G$. Let $co[X]$ be the convex span of a set $X \subset M(G)$ and if γ is a vector space topology on $M(G)$, denote the (γ)-closed convex span of X as $co[X : \gamma]$. We will need the following lemmas about the (σ) and (so) topologies on $M(G)$ (see introduction).

LEMMA 1.1.1. *If* $\mu_j \xrightarrow{(so)} \mu$ *with* $\|\mu_j\| \leqq M < \infty$ *in* $M(F)$, *and if* ψ *is right uniformly continuous and bounded on* F, *then we have* $\langle \mu_j, \psi \rangle \rightarrow \langle \mu, \psi \rangle$.

Proof. Since ψ is uniformly continuous, there exists $f \in L^1(F)$ corresponding to $\varepsilon > 0$ such that $\|f\| = 1$ and $\left| \int_F \psi(st)f(t)dt - \psi(s) \right| < \varepsilon/M$ for all $s \in F$. Then we have $|\langle \mu_j * f, \psi \rangle - \langle \mu_j, \psi \rangle| < \varepsilon$ for $j \in J$ and likewise for μ, so that $|\langle \mu_j, \psi \rangle - \langle \mu, \psi \rangle| < 3\varepsilon$ for $j \geqq j_0$.

LEMMA 1.1.2. *If* Q *is a compact set in locally compact group* G, *and if* S *is the circle group, then* $co[S\mathscr{E}_Q : so] = co[S\mathscr{E}_Q : \sigma] = \{\mu \in M(G) : \|\mu\| \leqq 1,\ s(\mu) \subset Q\}$, *and on these sets the* (σ) *and* (so) *topologies coincide.*

Proof. Clearly $S\mathscr{E}_Q$ is both (σ) and (so) compact, thus from [2, p. 511] we see that $co[S\mathscr{E}_Q : so]$ is compact in the (so) topology, as is $co[S\mathscr{E}_Q : \sigma]$ in the (σ) topology. On the unit ball the identity $j : (M(G), so) \rightarrow (M(G), \sigma)$ is continuous by 1.1.1, so $co[S\mathscr{E}_Q : so]$ is (σ) compact and hence must contain $co[S\mathscr{E}_Q : \sigma]$. Since Q is compact it is known that $co[S\mathscr{E}_Q : \sigma] = \{\mu \in M(G) : \|\mu\| \leqq 1,\ s(\mu) \subset Q\}$. But $\mu \in co[S\mathscr{E}_Q : so] \Rightarrow s(\mu) \subset Q$ and $\|\mu\| \leqq 1$, which gives the reverse containment. It is obvious that the topologies are the same on these compact sets, once they are known to coincide.

LEMMA 1.1.3. *If* G *is a locally compact group,* $co[S\mathscr{E}_G : so]$ *is the unit ball in* $M(G)$ *if* S *is the circle group.*

Proof. Let $\mu \in M(G)$, $\|\mu\| = 1$, and let K_n be compacta such that $K_{n+1} \supset K_n$ and $\bigcup_{n=1}^\infty K_n \supset s(\mu)$. Then $\mu_n = \mu \mid K_n \in M(G)$ is such that $\|\mu_n\| \leqq 1$, $\mu_n \in co[S\mathscr{E}_{K_n} : so]$, and $\mu_n \xrightarrow{\text{norm}} \mu$. Thus μ is in the norm closure of $\bigcup_{n=1}^\infty co[S\mathscr{E}_{K_n} : so]$, which lies within $co[S\mathscr{E}_G : so]$.

LEMMA 1.1.4. *On the unit ball in* $M(G)$, *convolution is a jointly strong operator continuous operation.*

Proof. Let $\mu_j \xrightarrow{(so)} \mu$ and $\lambda_k \xrightarrow{(so)} \lambda$ in the unit ball. If $f \in L^1(G)$, then because $\|\mu_j\| \leqq 1$ for all $j \in J$ we have $\|\mu_j * \lambda_k * f - \mu * \lambda * f\| \leqq \|\mu_j * (\lambda_k * f) - \mu_j * (\lambda * f)\| + \|\mu_j * (\lambda * f) - \mu * \lambda * f\| \leqq \|\lambda_k * f - \lambda * f\| + \|\mu_j * (\lambda * f) - \mu * (\lambda * f)\| \rightarrow 0$.

LEMMA 1.1.5. *If* G *is a locally compact group, the unit ball in*

$L^1(G) \subset M(G)$ is (so) dense in the unit ball in $M(G)$; in particular, $L^1(G)$ is (so) dense in $M(G)$.

Proof. Clearly there exists a left approximate identity $\{e_j\}$ of norm one in $L^1(G)$. If $\mu \in M(G)$ then $\mu * e_j \in L^1(G)$ and $\|\mu * e_j\| \leqq \|\mu\|$; furthermore, if $f \in L^1(G)$ we have

$$\|\mu * f - (\mu * e_j) * f\| = \|\mu * f - \mu * (e_j * f)\| \longrightarrow 0 .$$

2. Idempotent measures of norm one. If G is a locally compact group and $K \subset G$ is a compact subgroup, define $m_K \in M(G)$ to be the normalized Haar measure on K, so that

$$\langle m_K, \psi \rangle = \int_K \psi(s) dm_K(s)$$

for $\psi \in C_0(G)$. Let $K^\wedge$ be the set of all continuous unimodular multiplicative complex valued functions on K, and if $\beta \in K^\wedge$ let βm_K denote the Haar measure on K weighted with the function β. Then βm_K is an idempotent of norm one in $M(G)$; it is our purpose to show that these are the only idempotent measures of norm one in $M(G)$.

THEOREM 2.1.1. *Let G be a locally compact group. Then if $\mu, \lambda \in M(G)$ are such that $\|\mu * \lambda\| = \|\mu\| \cdot \|\lambda\|$ it follows that $s(\mu * \lambda) = (s(\mu)s(\lambda))^-$, the closure in G of $s(\mu)s(\lambda)$.*

Proof. It is sufficient to consider the case $\|\mu\| = \|\lambda\| = 1$. Clearly $(s(\mu)s(\lambda))^- \supset s(\mu * \lambda)$. If this inclusion is proper we can find a compact Baire set U which is such that $(\operatorname{int} U) \cap (s(\mu)s(\lambda))^- \neq \varnothing$, while at the same time $U \cap s(\mu * \lambda) = \varnothing$. Let $E, F \in \boldsymbol{B}_0(G)$ be such that $E \supset s(\mu)$, $F \supset s(\lambda)$, and define $V = \{(s, s^{-1}u) : u \in U, s \in E\} \subset G \times G$; notice that $V \in \boldsymbol{B}_0 \times \boldsymbol{B}_0$ and is such that $\chi_V(s, t) = \chi_U(st)$ for $s \in E$, $t \in G$, thus

$$\begin{aligned}
\int_{(E \times F) \cap V} & \psi(st) d\mu \times \lambda(s, t) \\
&= \int_{E \times F} \chi_V(s, t) \psi(st) d\mu \times \lambda(s, t) \\
&= \int_{E \times F} \chi_U(st) \psi(st) d\mu \times \lambda(s, t) \\
&= \int_G \chi_U(x) \psi(x) d\mu * \lambda(x) .
\end{aligned}$$

Given $\varepsilon > 0$, there is a function $\psi \in C_0(G)$ such that $\|\psi\|_\infty = 1$ and $|\langle \mu * \lambda, \psi \rangle| > 1 - \varepsilon$. If V is any Baire set in $G \times G$, then

$$
(*)\qquad
\begin{aligned}
1-\varepsilon < |\langle \mu*\lambda, \psi\rangle| &= \left| \int_{G\times G} \chi_{E\times F}(s,t)\psi(st)d\mu\times\lambda(s,t) \right| \\
&= \left| \int_{E\times F} \psi(st)d\mu\times\lambda(s,t) \right| \\
&= \left| \int_{(E\times F)\setminus V} \psi(st)d\mu\times\lambda(s,t) \right. \\
&\qquad \left. + \int_{(E\times F)\cap V} \psi(st)d\mu\times\lambda(s,t) \right| .
\end{aligned}
$$

For V as above, the right hand side of $(*)$ consists of the single term

$$
\begin{aligned}
\left| \int_{(E\times F)\setminus V} \psi(st)d\mu\times\lambda(s,t) \right| &\leqq \int_{(E\times F)\setminus V} |\psi(st)|\, d|\mu|\times|\lambda|(s,t) \\
&\leqq \int_{E\times F} d|\mu|\times|\lambda| - \int_{(E\times F)\cap V} d|\mu|\times|\lambda| \\
&= 1 - \int_{(E\times F)\cap V} d|\mu|\times|\lambda| .
\end{aligned}
$$

But from our definition of U it is clear that $\int_{(E\times F)\cap V} d|\mu|\times|\lambda| = \delta > 0$, and thus for all $\varepsilon > 0$ we get $1-\varepsilon \leqq 1-\delta$, a contradiction.

THEOREM 2.1.2. *If G is a locally compact group and if $\mu, \lambda \in M(G)$ are measures such that $\|\mu*\lambda\| = \|\mu\|\cdot\|\lambda\|$, then $|\mu*\lambda| = |\mu|*|\lambda|$.*

Proof. Again it suffices to consider the case $\|\mu\| = \|\lambda\| = 1$. If $F \supset s(\mu)$ is a Baire set, it is σ-bounded and from the Radon-Nikodym theorem we know that there is a Baire measurable function f_μ on F such that $\mu(E) = \int \chi_E(x) f_\mu(x) d|\mu|(x)$ for all $E \in \boldsymbol{B}(G)$ such that $E \subset F$. clearly $|f_\mu(x)| = 1$ $|\mu|$-a.e. on F; we define a new function

$$
\rho_\mu(x) = \begin{cases} f_\mu(x) \text{ if } x \in F \text{ and } |f_\mu(x)| = 1 \\ 1 \text{ for all other } x \in G . \end{cases}
$$

Then ρ_μ is a unimodular function on G Baire measurable on Baire sets in G.

We will show $|\mu*\lambda| \leqq |\mu|*|\lambda|$. Since these are positive measures, both of norm one (since $\|\mu*\lambda\| = \|\mu\|\cdot\|\lambda\|$ for any positive measures $\mu, \lambda \in M(G)$), our result must follow. If $\psi \in C_0(G)$ and $\psi \geqq 0$, then

$$
\begin{aligned}
\langle |\mu*\lambda|, \psi\rangle &= \int_G \frac{\psi(x)}{\rho_{\mu*\lambda}(x)} d\mu*\lambda(x) \\
&= \int_G \left[\int_G \frac{\psi(st)}{\rho_{\mu*\lambda}(st)} d\mu(s) \right] d\lambda(t) \\
&= \int_G \left[\int_G \psi(st) \frac{\rho_\mu(s)\rho_\lambda(t)}{\rho_{\mu*\lambda}(st)} d|\mu|(s) \right] d|\lambda|(t) .
\end{aligned}
$$

Now the last integral is positive and the integrand is a unimodular multiple of $\psi(st)$, so it must be less than or equal to

$$\int_G\left[\int_G \psi(st)d\,|\,\mu\,|\,(s)\right]d\,|\,\lambda\,|\,(t) = \langle|\,\mu\,|*|\,\lambda\,|,\ \psi\rangle\,.$$

The following lemma is given in Loynes [6] and Pym [8], and is also a simple consequence of 2.1.1 and 2.1.2.

PROPOSITION 2.1.3. If G is a locally compact group and if $\mu\in M(G)$ is a positive idempotent of norm one, then there is a compact subgroup $K\subset G$ such that $\mu = m_K$.

We can now prove the main assertion of this section.

THEOREM 2.1.4. *If G is a locally compact group and $\mu\in M(G)$ is an idempotent of norm one, then there is a compact subgroup $K\subset G$ and a function $\rho\in K^\wedge$ such that $\mu = \rho m_K$.*

Proof. Write $\mu = \rho\,|\,\mu\,|$ where ρ is a unimodular function on G, Baire measurable on Baire sets in G. From 2.1.2 we see that $|\,\mu\,|$ is a positive idempotent of norm one, so that $|\,\mu\,| = m_K$ for some compact subgroup $K\subset G$ from 2.1.3.

Now ρ is a bounded Borel measurable function on K since $\boldsymbol{B}(K) = \{E\cap K : E\in \boldsymbol{B}(G)\}$, so the function $\rho \star \rho(t) = \int_K \rho(s^{-1}t)\rho(s)dm_K(s)$ is continuous on K (we are taking $\star$ as the convolution of two functions on K here). If $\psi\in C_0(G)$ and if $F\in \boldsymbol{B}_0(G)$ is such that $F\supset s(\mu) = K$, we have

$$\begin{aligned}\int_K \psi(x)\rho(x)\,dm_K(x) &= \langle\mu,\ \psi\rangle = \langle\mu*\mu,\ \psi\rangle\\ &= \int_G\left[\int_G \psi(st)\,\rho(t)dm_K(t)\right]\rho(s)dm_K(s)\\ &= \int_K\left[\int_K \psi(t)\rho(s^{-1}t)dm_K(t)\right]\rho(s)dm_K(s)\\ &= \int_{G\times G}\chi_{F\times F}(s,\ t)\rho(s^{-1}t)\cdot\psi(t)\rho(s)\chi_{K\times K}(s,\ t)dm_K\times m_K(s,\ t)\ .\end{aligned}$$

Clearly $\psi(t)\rho(s)\chi_{K\times K}(s,\ t)$ is $\boldsymbol{B}\times\boldsymbol{B}$ measurable and a slight modification of $R1$ gives the $\boldsymbol{B}_0\times\boldsymbol{B}_0$ measurability of $\rho(s^{-1}t)\chi_{F\times F}(s,\ t)$ on $G\times G$. Thus Fubini applies and we get

$$\begin{aligned}&= \int \chi_{K\times K}(s,\ t)\psi(t)\rho(s)\rho(s^{-1}t)dm_K\times m_K(s,\ t)\\ &= \int_K\left[\int_K \rho(s)\rho(s^{-1}t)dm_K(s)\right]\psi(t)dm_K(t)\\ &= \int_K \psi(t)\rho \star \rho(t)dm_K(t)\ ,\end{aligned}$$

so ρ is $|\mu|$-a.e. identical to a continuous function on K. Taking ρ to be continuous on K, it is clear that $(s, t) \to \rho(st)$ is continuous on $K \times K$. But we can apply the argument of 2.1.2:

$$1 = \|\mu * \mu\| = \int_K \left[\int_K \frac{\rho_\mu(s)\rho_\mu(t)}{\rho_{\mu*\mu}(st)} \, d|\mu|(s) \right] d|\mu|(t)$$

$$= \int_K \left[\int_K \frac{\rho(s)\rho(t)}{\rho(st)} \, dm_K(s) \right] dm_K(t) \, ,$$

which means that ρ is a multiplicative function on K.

3. Subgroups of the unit ball in a measure algebra. In this section we consider a locally compact group G and let Γ be a subgroup in the unit ball of $M(G)$. We will denote this unit ball by $\Sigma_{M(G)}$ and refer to the weak $*$ topology on $M(G)$ as the (σ) topology. Given Γ we denote $H_0 = \operatorname{supp}(\Gamma) = \cup\{s(\mu) : \mu \in \Gamma\}$.

LEMMA 3.1.1. *Both H_0 and its closure in G are subgroups of G, and if the unit of Γ is denoted i ($i = \rho m_K$ for some compact subgroup $K \subset G$ and some $\rho \in K^\wedge$), then K is a normal subgroup of both H_0 and its closure. Furthermore, if $\mu \in \Gamma$ then $s(\mu)$ is a single coset of the group K in H_0.*

Proof. If $\mu \in \Gamma$ then $s(\mu)$ is a union of right (or of left) cosets of K because $i * \mu = \mu * i = \mu \Rightarrow (s(\mu)s(i))^- = s(\mu)\cdot K = K \cdot s(\mu) = s(\mu)$ from 2.1.1. If $\mu \in \Gamma$ then $s(\mu^{-1}) = s(\mu)^{-1}$. In fact, if $x \in s(\mu)$, $y \in s(\mu^{-1})$ then $xy = k \in (s(\mu)s(\mu^{-1}))^- = s(\mu * \mu^{-1}) = K$, so that $x^{-1} = yk^{-1} \in s(\mu^{-1})K = s(\mu^{-1})$, and vice versa.

If $g_1 \in s(\mu)$, $g_2 \in s(\mu^{-1})$ we have the relations

$$(*) \qquad K = s(i) = (s(\mu)s(\mu^{-1}))^- \supset s(\mu)s(\mu^{-1}) \supset g_1K^2g_2 \supset g_1Kg_2$$

$$(**) \qquad K = s(i) = (s(\mu)s(\mu^{-1}))^- \supset s(\mu)s(\mu^{-1}) \supset Kg_1g_2K \supset \{g_1g_2\} \, .$$

Thus $s(\mu)$ is a single coset of K; otherwise we could find $g_1, g_2 \in s(\mu)$ with $g_1 \notin Kg_2$, and this would $\Rightarrow g_1g_2^{-1} \notin K$. But $g_2^{-1} \in s(\mu)^{-1} = s(\mu^{-1})$ and $(**) \Rightarrow g_1g_2^{-1} \in K$, a contradiction. We see now that all supports are compact and hence $s(\mu)s(\lambda) = s(\mu * \lambda)$ for all $\mu, \lambda \in \Gamma$.

Clearly H_0 is a subgroup of G since $s(\mu * \lambda) = s(\mu)s(\lambda)$ and $s(\mu)^{-1} = s(\mu^{-1})$; hence its closure is also a subgroup in G. We get normality of K by considering $g \in H_0$ and taking any $\mu \in \Gamma$ such that $g \in s(\mu)$. Then if we take $g_1 = g$, $g_2 = g^{-1} \in s(\mu^{-1})$ in $(*)$, we get $K \supset gKg^{-1}$.

The following theorem gives the structure of a subgroup Γ; it gives only a necessary condition on the structure of a collection of measures Γ in the unit ball of $M(G)$ in order that Γ be a subgroup. Necessary and sufficient conditions will be given later.

PROPOSITION 3.1.2. If Γ is a subgroup of $\Sigma_{M(G)}$ for locally compact group G, let $i = \rho m_K$ be its unit and let $H_0 = \operatorname{supp}(\Gamma)$. Then there exists a subgroup $\Omega \subset S \times G$, with the property

$$H_0 = \{g \in G : (\alpha, g) \in \Omega \text{ for some } |\alpha| = 1\} ,$$

such that $\Gamma = \{\alpha\delta_g * \rho m_K : (\alpha, g) \in \Omega\}$.

REMARK. Here S is the circle group and $S \times G$ is the usual product group. In 2.1.4 we have already shown that the unit is $i = \rho m_K$ where K is normal in H_0 and $\rho \in K^\wedge$.

Proof. Let ρ_μ be a unimodular function on G, Baire measurable on Baire sets in G, such that $\mu = \rho_\mu |\mu|$ for $\mu \in \Gamma$. If $g \in s(\mu)$ we have shown that $s(\mu) = gK$ and we know that ρ_μ is determined $|\mu|$-a.e. on $s(\mu)$. But if $s(\mu) = gK$, then $|\mu| = \delta_g * m_K$; in fact, we have $\mu * i = \mu$, which $\Rightarrow |\mu| * |i| = |\mu| * m_K = |\mu|$, and this gives $|\mu| = \delta_g * m_K$ since $\|\mu\| = 1$. We first show that ρ_μ is $|\mu|$-a.e. identical to a continuous function on $s(\mu)$, or equivalently that $\rho_\mu{}^g(x) = \rho_\mu(gx)$ is m_K-a.e. equal to a continuous function on K. We have

$$\int_G \psi d\mu = \int_K \psi(gt)\rho_\mu{}^g(t)dm_K(t)$$

for $\psi \in C_0(G)$, while $\mu = i * \mu \Rightarrow$

$$\begin{aligned}\int_G \psi d\mu &= \int_{gK}\left[\int_K \psi(st)\rho(s)\,dm_K(s)\right]d\mu(t)\\ &= \int_K\left[\int_K \psi(sgt)\rho(s)\rho_\mu(gt)dm_K(s)\right]dm_K(t) .\end{aligned}$$

For $g \in H_0$, the map $\pi_g : s \rightarrow gsg^{-1}$ is an automorphism of K such that $m_K(\pi_g E) = m_K(E)$ for Borel sets $E \subset K$; thus if we define $\pi_g{}^*\beta(s) = \beta(gsg^{-1})$ for $s \in K$, $g \in H_0$, and $\beta \in K^\wedge$, then $\pi_g{}^*\beta \in K^\wedge$ and the last expression above is

$$\begin{aligned}&= \int_K\left[\int_K \psi(gst)\pi_g{}^*\rho(s)\rho_\mu(gt)dm_K(s)\right]dm_K(t)\\ &= \int_K \psi(gt)\left[\int_K \pi_g{}^*\rho(s)\rho_\mu{}^g(s^{-1}t)dm_K(s)\right]dm_K(t)\\ &= \int_K \psi(gt)[\pi_g{}^*\rho \star \rho_\mu{}^g](t)dm_K(t)\end{aligned}$$

where $\star$ gives the convolution of two functions on K rather than functions on G. Since $\pi_g{}^*\rho$ and $\rho_\mu{}^g$ are bounded and $\boldsymbol{B}(K)$ measurable functions on K, their convolution on K is a continuous function, and the above equalities $\Rightarrow \rho_\mu{}^g = \pi_g{}^*\rho \star \rho_\mu{}^g$ m_K-a.e. on K.

Take each function ρ_μ to be continuous on $s(\mu)$ for $\mu \in \Gamma$. Then we have $\rho_\mu(x)\rho_\lambda(y) = \rho_{\mu*\lambda}(xy)$ for all $(x, y) \in s(\mu) \times s(\lambda)$ in $G \times G$, because

$$1 = \int_G d\,|\,\mu*\lambda\,| = \int_G \frac{1}{\rho_{\mu*\lambda}(z)} d\mu*\lambda(z)$$
$$= \int_{s(\mu)}\left[\int_{s(\lambda)} \frac{\rho_\mu(s)\rho_\lambda(t)}{\rho_{\mu*\lambda}(st)} d\,|\,\mu\,|\,(s)\right] d\,|\,\mu\,|\,(t)\ ,$$

and since the last integrand is continuous and unimodular. If $s \in K$ and $g \in s(\mu)$ for $\mu \in \Gamma$, then we have

$$\rho_\mu{}^g(s) = \rho_\mu(gs) = \rho_{\mu*i}(gs) = \rho_\mu(g)\rho_i(s)$$
$$= \rho_\mu(g)\rho(s) = c_g\cdot\rho(s)$$

which means that $\rho_\mu{}^g = c_g\cdot\rho$ on K where $c_g = \rho_\mu(g)$ is a scalar of modulus one. Clearly $\mu = \rho_\mu(g)\cdot(\delta_g*\rho m_K)$ if $g \in s(\mu)$; i.e. if $g \in s(\mu)$, then for some scalar α with $|\,\alpha\,| = 1$ we have $\mu = \alpha\delta_g*\rho m_K$.

Let $\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g*\rho m_K \in \Gamma\}$. We have shown that for each $g \in H_0$ we can find a scalar $|\,\alpha\,| = 1$ such that $(\alpha, g) \in \Omega$, so we only have to show that

$$(\alpha_1\delta_{g_1}*\rho m_K)*(\alpha_2\delta_{g_2}*\rho m_K) = \alpha_1\alpha_2\delta_{g_1g_2}*\rho m_K\ .$$

Since the left side is in Γ we get $(\alpha_1\alpha_2, g_1g_2) \in \Omega$, and this will give the group property. But $\alpha_2\delta_{g_2}*\rho m_K \in \Gamma$ and $i = \rho m_K$ is the unit of Γ; hence

$$\alpha_1\delta_{g_1}*\rho m_K*(\alpha_2\delta_{g_2}*\rho m_K) = \alpha_1\delta_{g_1}*\alpha_2\delta_{g_2}*\rho m_K = \alpha_1\alpha_2\delta_{g_1g_2}*\rho m_K$$

as required. Clearly $\Gamma = \{\alpha\delta_g*\rho m_K : (\alpha, g) \in \Omega\}$.

COROLLARY 3.1.3. *If $\mu, \lambda \in \Gamma$ we have $s(\mu) = s(\lambda) \Leftrightarrow \mu = \alpha\lambda$ for some scalar α with $|\,\alpha\,| = 1$.*

Proof. If $s(\mu) = s(\lambda) = gK$ then there are scalars α, β of unit modulus such that $\mu = \alpha\delta_g*\rho m_K$ and $\lambda = \beta\delta_g*\rho m_K$.

COROLLARY 3.1.4. *If $Si = \{\alpha i : |\,\alpha\,| = 1\}$ and if $\Gamma \cap Si = \{i\}$, then for $\mu, \lambda \in \Gamma$ we have $\mu = \lambda$ whenever $s(\mu) = s(\lambda)$.*

PROPOSITION 3.1.5. If G is a locally compact group and Γ is a subgroup of $\Sigma_{M(G)}$ let us write its unit as $i = \rho m_K$, where $\rho \in K^\wedge$, and let $H_0 = \text{supp}(\Gamma)$. Then $K_0 = \{x \in K : \rho(x) = 1\}$ is a compact subgroup of G which is normal in both K and H_0.

Proof. If $\mu \in \Gamma$ and ρ_μ is the unimodular function, Baire measurable on Baire sets in G, such that $\mu = \rho_\mu |\mu|$, then we know that ρ_μ is a translate of ρ to $s(\mu)$, and we also know that $\rho_\mu(x)\rho_\lambda(y) = \rho_{\mu*\lambda}(xy)$ for all $x \in s(\mu)$, $y \in s(\lambda)$, from 3.1.2. Obviously K_0 is normal in K; normality in H_0 is more troublesome.

If $y \in K_0$, $x \in H_0$, then $xyx^{-1} \in K$ and if $x \in s(\mu)$ we get

$$\begin{aligned}\rho(xyx^{-1}) &= \rho_\mu(x)\rho_i(y)\rho_\mu^{-1}(x^{-1}) = \rho_\mu(x)\cdot 1\cdot \rho_\mu^{-1}(x^{-1})\\ &= \rho_{\mu*\mu}^{-1}(xx^{-1}) = \rho(e) = 1\ ,\end{aligned}$$

which $\Rightarrow xyx^{-1} \in K_0$.

PROPOSITION 3.1.6. Let G be a locally compact group, let H_0 be an arbitrary subgroup, let $K \supset K_0$ be a pair of compact subgroups of G which lie within H_0 and are normal therein, and assume that $\rho \in K^\wedge$ is a function such that $K_0 = \text{Ker}\,\rho$. Then we have $\rho m_K * \delta_g * \rho m_K = \delta_g * \rho m_K$ for all $g \in H_0 \Leftrightarrow K$ is central in $H_0 \bmod K_0$ (i.e. K/K_0 is a central subgroup of H_0/K_0).

Proof. If K is central in $H_0 \bmod K_0$ and $\psi \in C_0(G)$, then

$$\begin{aligned}\langle \rho m_K * \delta_g * \rho m_K, \psi\rangle &= \int_G\Big[\int_G\Big[\int_G \psi(sxt)\rho(s)\rho(t)dm_K(s)\Big]d[\delta_g](x)\Big]dm_K(t)\\ &= \int_G\Big[\int_G \psi(sgt)\rho(st)dm_K(s)\Big]dm_K(t)\ .\end{aligned}$$

But $sg = gs \bmod K_0$, so that $sg = gsk$ for some $k \in K_0$, and the last expression becomes

$$\begin{aligned}&= \int_G\Big[\int_G \psi(gskt)\rho(st)dm_K(s)\Big]dm_K(t)\\ &= \int_G\Big[\int_G \psi(gst)\rho(st)dm_K(s)\Big]dm_K(t)\\ &= \int_G \psi(gs)\rho(s)dm_K(s) = \langle \delta_g * \rho m_K, \psi\rangle\ ,\end{aligned}$$

since $(\rho m_K)^2 = \rho m_K$.

If, conversely, $\rho m_K * \delta_g * \rho m_K = \delta_g * \rho m_K$ for all $g \in H_0$, we show that K is central in $H_0 \bmod K_0$ as follows. Let

$$\psi_j = \frac{\chi_{gU_j}}{m_K(U_j \cap K)}$$

where $\{U_j : j \in J\}$ is a basis of compact symmetric neighborhoods of the unit in G, and make $\{\psi_j : j \in J\}$ a net of functions in $L^1(G)$ under the obvious partial ordering. Then we have

$$\begin{aligned}
1 = \rho(e) \longleftarrow & \int_K \frac{\chi_{U_j}(s)}{m_K(U_j)} \rho(s) dm_K(s) \\
&= \int_K \frac{\chi_{gU_j}(gs)}{m_K(U_j)} \rho(s) dm_K(s) \\
&= \int_K \psi_j(gs)\rho(s) dm_K(s) \\
&= \int_K \psi_j d[\delta_g * \rho m_K] = \int_K \psi_j d[\rho m_K * \delta_g * \rho m_K] \\
&= \int_K \Big[\int_K \psi_j(sgt)\rho(s)\rho(t) dm_K(s)\Big] dm_K(t) \\
&= \int_K \Big[\int_K \frac{\chi_{gU_j}(gst)}{m_K(U_j)} \rho(gsg^{-1})\rho(t) dm_K(s)\Big] dm_K(t) \\
&= \int_K \Big[\int_K \frac{\chi_{U_j}(st)}{m_K(U_j)} \rho(gsg^{-1})\rho(t) dm_K(s)\Big] dm_K(t) .
\end{aligned}$$

But ρ is uniformly continuous on K and hence, given $\varepsilon > 0$ there is an index $j(\varepsilon)$ such that $j > j(\varepsilon)$ in the partial ordering of $J \Rightarrow |\rho(t) - \rho(t')| < \varepsilon$ if $t \in t'U_j$. Hence if $j > j(\varepsilon)$ we get: $\chi_{U_j}(st) \neq 0 \Rightarrow t \in s^{-1}U_j$, which $\Rightarrow |\rho(t) - \rho(s^{-1})| < \varepsilon$. Some trivial computations then show that the last integral is always within ε of the following expression if $j > j(\varepsilon)$.

$$\int_K \rho(gsg^{-1})\rho(s^{-1}) dm_K(s) = \int_K \rho(gsg^{-1})\overline{\rho(s)} dm_K(s) .$$

But $s \to \rho(gsg^{-1})$ is a function in $K^\wedge$, and from the known orthogonality of one dimensional representations of K, this integral can be nonzero $\Leftrightarrow \rho(gsg^{-1}) = \rho(s)$ for all $s \in K$. This means that $gs = sg \bmod K_0$ for all $s \in K$, $g \in H_0$.

COROLLARY 3.1.7. *If G is a locally compact group and Γ is a subgroup of $\Sigma_{M(G)}$, let $H_0 = \operatorname{supp}(\Gamma)$, and let us write the unit of Γ as $i = \rho m_K$ as in 2.1.4, where $K \subset G$ is a compact subgroup and $\rho \in K^\wedge$. Then if $K_0 = \operatorname{Ker} \rho$, K must be central in $H_0 \bmod K_0$.*

Proof. From 3.1.5 we know that K_0 must be normal in H_0. Furthermore, $i * \mu = \mu \Rightarrow \rho m_K * \delta_g * \rho m_K = \delta_g * \rho m_K$ for all $g \in H_0$ (see 3.1.2).

THEOREM 3.1.8. (*Structure Theorem for Subgroups*). *Let G be locally compact group and let Γ be a subgroup of $\Sigma_{M(G)}$ with unit i. Then we have*

(1) $H_0 = \cup\{s(\mu) : \mu \in \Gamma\}$ *is a subgroup of* G.
(2) $i = \rho m_K$ *where* $K \subset G$ *is a compact subgroup and* $\rho \in K^\wedge$.
(3) K *and* $K_0 = \operatorname{Ker}\rho$ *lie within* H_0 *and are normal in* H_0.
(4) K *is central in* $H_0 \bmod K_0$.
(5) $\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K \in \Gamma\}$ *is a subgroup of* $S \times G$ *with* $H_0 = \{g \in G : (\alpha, g) \in \Omega$ *for some* $|\alpha| = 1\}$.

and we have $\Gamma = \{\alpha\delta_g * \rho m_K : (\alpha, g) \in \Omega\}$.

Conversely, let H_0 *be a subgroup in* G, *let* $K \subset G$ *be a compact subgroup lying within* H_0, *and let* $\rho \in K^\wedge$ *be chosen such that*
(1) K *and* $K_0 = Ker\ \rho$ *are both normal in* H_0.
(2) K *is central in* H_0 *mod* K_0.

and let Ω *be any subgroup of* $S \times G$ *with* $H_0 = \{g \in G : (\alpha, g) \in \Omega$ *for some* $|\alpha| = 1\}$. *Then* $\Gamma = \{\alpha\delta_g * \rho m_K : (\alpha, g) \in \Omega\}$ *is a subgroup of* $\Sigma_{M(G)}$ *with* $H_0 = \cup\{s(\mu) : \mu \in \Gamma\}$, *with* $i = \rho m_K$ *as a unit, and with* $\Omega \subset \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K \in \Gamma\} = \Omega \cdot \{(\rho(k), k) : k \in K\}$.

Proof. The first part follows from 3.1.2, 3.1.5, and 3.1.7. Conversely, if K is central in $H_0 \bmod K_0 = \operatorname{Ker}\rho$ in a scheme of this sort we must have $\rho m_K * \delta_g * \rho m_K = \delta_g * \rho m_K$ for all $g \in H_0$. This means that Γ is a group, since the only difficulty in showing this lies in the verification that Γ is closed under convolution. It follows immediately that $H_0 = \cup\{s(\mu) : \mu \in \Gamma\}$ and that $\tau_0 : (\alpha, g) \to \alpha\delta_g * \rho m_K$ is a homomorphism of Ω onto Γ with kernel $\Omega \cap \{(\rho(k), k) : k \in K\}$. Notice that Ω and $\Omega' = \Omega \cdot \{(\rho(k), k) : k \in K\}$ give rise to the same group of measures Γ.

The classical example of a subgroup in $\Sigma_{M(G)}$ is a group of translates of normalized Haar measure $\Gamma = \{\delta_x * m_Q : x \in G_0\}$, where $Q \subset G$ is a compact subgroup, normal in the subgroup G_0. Theorem 3.1.8 can be stated in a form which shows that every subgroup $\Gamma \subset \Sigma_{M(G)}$ corresponds to a subgroup of this type in $\Sigma_{M(S \times G)}$ rather than $\Sigma_{M(G)}$.

Let π_S, π_G be the projection homomorphisms in $S \times G$ and let $\Omega \supset \Omega_0$ be subgroups in $S \times G$ satisfying the conditions
(1) Ω_0 is a compact subgroup of $S \times G$ normal in Ω.
(2) $S \cap \Omega_0 = (1, e)$, so $\pi_G^{-1}(x) \cap \Omega_0$ is a single point if $x \in \pi_G(\Omega_0)$.

If we are given a compact subgroup $K \subset G$ and a function $\rho \in K^\wedge$, we define the mappings

$$\tau_0 : S \times G \longrightarrow M(G)$$
$$\tau^* : C_0(G) \longrightarrow C_0(S \times G)$$
$$\tau^{**} : M(S \times G) \longrightarrow M(G)$$

such that $\tau_0(\alpha, g) = \alpha\delta_g * \rho m_K$, $\tau^*\psi(\alpha, g) = \langle \tau_0(\alpha, g), \psi \rangle$, and $\langle \tau^{**}\mu, \psi \rangle = \langle \mu, \tau^*\psi \rangle$. Clearly $\tau^*\psi \in C_0(S \times G)$ since K is compact, and $\tau^{**}\delta_{(\alpha, g)} =$

$\alpha\delta_g * \rho m_K$ for $(\alpha, g) \in S \times G$. Furthermore, $\tau^{**} : (M(S \times G), (\sigma)) \rightarrow (M(G), (\sigma))$ is a linear map which is continuous on norm bounded sets since $\mu_j \xrightarrow{(\sigma)} \mu$ in $\Sigma_{M(S\times G)}$, $\psi \in C_0(G) \Longrightarrow$

$$\langle \tau^{**}\mu_j, \psi\rangle = \langle \mu_j, \tau^*\psi\rangle \longrightarrow \langle \mu, \tau^*\psi\rangle = \langle \tau^{**}\mu, \psi\rangle .$$

Also, τ^{**} is a norm decreasing linear map.

Now take $K = \pi_G(\Omega_0)$, where $\Omega \supset \Omega_0$ satisfy (1) and (2) above, and define the function $\rho(k)$ on K such that $(\rho(k), k) \in \Omega_0$ for each $k \in K$. It is clear that $\rho \in K^\wedge$ since Ω_0 is compact. Let us also define $H_0 = \pi_G(\Omega)$, $K_0 = \pi_G(\Omega_0 \cap G)$. Then $K_0 = \operatorname{Ker} \rho$ and it is easily verified that K_0 is normal in both H_0 and K, and that K is central in $H_0 \bmod K_0$, from conditions (1) and (2). Thus $\Gamma = \{\alpha\delta_g * \rho m_K : (\alpha, g) \in \Omega\}$ is a subgroup in $\Sigma_{M(G)}$ since 3.1.8 applies to the system of objects H_0, K, K_0, ρ.

The mapping $\tau^{**}\delta : (\alpha, g) \rightarrow \alpha\delta_g * \rho m_K$ is a homomorphism on Ω since

$$\begin{aligned}\tau^{**}\delta_{(\alpha_1\alpha_2,\, g_1g_2)} &= \alpha_1\alpha_2\delta_{g_1g_2} * \rho m_K = \alpha_1\delta_{g_1} * \rho m_K * \alpha_2\delta_{g_2} * \rho m_K \\ &= \tau^{**}\delta_{(\alpha_1,\, g_1)} * \tau^{**}\delta_{(\alpha_2,\, g_2)} .\end{aligned}$$

From normality of Ω_0 in Ω it follows that $\Gamma^\sim = \{\delta_x * m_{\Omega_0} : x \in \Omega\}$ is a subgroup of $\Sigma_{M(S\times G)}$.

LEMMA 3.1.9. *If $\Omega \supset \Omega_0$ satisfy conditions* (1), (2) *above, and if $\Gamma = \{\alpha\delta_g * \rho m_K : (\alpha, g) \in \Omega\}$ then $\tau_0 : \Omega \rightarrow \Gamma$ is an epimorphism with kernel Ω_0.*

Proof. We have indicated that τ_0 is an epimorphism. If $\tau_0(\alpha, g) = \rho m_K$, then $g \in K$ and we have $\alpha\delta_g * \rho m_K = \alpha\overline{\rho(g)}\rho m_K = \rho m_K$; hence $\alpha = \rho(g)$ and $(\alpha, g) = (\rho(g), g)$ with $g \in K$, so $(\alpha, g) \in \Omega_0$ by definition of ρ.

THEOREM 3.1.10. *Given subgroups $\Omega \supset \Omega_0$ in $S \times G$ satisfying* (1) *and* (2) *let $K = \pi_G(\Omega_0)$, define $\rho = \pi_S \circ (\pi_G \mid \Omega_0)^{-1}$ on K, and define $\tau^{**} : M(S \times G) \rightarrow M(G)$ as above. Then $\rho \in K^\wedge$ (so τ_0 and τ^{**} are well defined), $\Gamma = \{\alpha\delta_g * \rho m_K : (\alpha, g) \in \Omega\}$ and $\Gamma^\sim = \{\delta_x * m_{\Omega_0} : x \in \Omega\}$ are subgroups in $\Sigma_{M(G)}$ and $\Sigma_{M(S\times G)}$ respectively, and τ^{**} is an isomorphism between $\Gamma^\sim$ and Γ. Conversely, if $\Gamma \subset \Sigma_{M(G)}$ is a subgroup with unit $i = \rho m_K$, it arises from a pair of subgroups $\Omega \supset \Omega_0$ in $S \times G$ which satisfy conditions* (1) *and* (2) *by means of the above construction if we take $\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K \in \Gamma\}$ and $\Omega_0 = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K = \rho m_K\}$.*

Proof. To establish the first part we will show that $\tau^{**}(\delta_x * m_{\Omega_0}) = \tau^{**}\delta_x = \tau_0(x)$ for any $x \in \Omega$; then from 3.1.9 it is clear that τ^{**} is an isomorphism between $\Gamma^\sim$ and Γ. But Ω_0 is compact, so there exists a net $\{\lambda_j\}$ in $\operatorname{co}[\mathscr{E}_{\Omega_0}]$ with $\lambda_j \xrightarrow{(\sigma)} m_{\Omega_0}$; hence $\delta_x * \lambda_j \xrightarrow{(\sigma)} \delta_x * m_{\Omega_0}$ and

$\tau^{**}(\delta_x * m_{\Omega_0}) \xleftarrow{(\sigma)} \tau^{**}(\delta_x * \lambda_j) \equiv \tau^{**}(\delta_x)$, as required. Conversely, if $\Gamma \subset \Sigma_{M(G)}$ is a subgroup, and if $\Omega \supset \Omega_0$ are formed as indicated, then properties (1) and (2) hold as a consequence of the following lemma, which will be of interest later on. Once this is shown, it is easy to check (see 3.1.8) that $\pi_G(\Omega) = \operatorname{supp}(\Gamma)$, $\pi_G(\Omega_0) = K$, and $\rho = \pi_S \circ (\pi_G \mid \Omega_0)^{-1}$ on K. In the first part we showed that τ^{**} must be an isomorphism of $\{\delta_x * m_{\Omega_0} : x \in \Omega\}$ onto $\{\alpha\delta_g * \rho m_K : (\alpha, g) \in \Omega\}$ and we know that Γ coincides with the latter subgroup of $\Sigma_{M(G)}$ from 3.1.2.

LEMMA 3.1.11. *Let G be a locally compact group and let $\Gamma \subset \Sigma_{M(G)}$ be a subgroup with unit $i = \rho m_K$. Form the pair of subgroups in $S \times G$:*

$$\Omega = \{(\alpha, g) \in S \times G :\ \alpha\delta_g * \rho m_K \in \Gamma\} \supset$$
$$\Omega_0 = \{(\alpha, g) \in S \times G :\ \alpha\delta_g * \rho m_K = \rho m_K\} .$$

Then we have $\Omega_0 = \{(\rho(k), k) : k \in K\}$ and this is a compact subgroup of $S \times G$, normal in Ω. If we define the map $\tau_0 : (\alpha, g) \to \alpha\delta_g * \rho m_K$ for $(\alpha, g) \in S \times G$, then $\tau_0 : S \times G \to (M(G), (\sigma))$ is continuous and $\tau_0 : \Omega \to \Gamma$ is an epimorphism with kernel Ω_0.

Proof. If $\tau_0(\alpha, g) = i$ then $g \in K$ and we have $\alpha\delta_g * \rho m_K = \alpha \cdot \overline{\rho(g)} \cdot \rho m_K$. Hence $\alpha = \rho(g)$ and $(\alpha, g) = (\rho(g), g)$ with $g \in K$. Since $\rho \in K^\wedge$, Ω_0 is a compact subgroup of $S \times G$. Let $H_0 = \operatorname{supp}(\Gamma)$, $K_0 = \operatorname{Ker} \rho$; from 3.1.8, K is central in $H_0 \bmod K_0$, so $\delta_g * \rho m_K = \rho m_K * \delta_g * \rho m_K$ for $g \in H_0$ (see 3.1.6). Thus τ_0 is a homomorphism on Ω (that it is onto is clear from 3.1.2) since

$$\begin{aligned} \tau_0(\alpha_1\alpha_2, g_1g_2) &= \alpha_1\alpha_2\delta_{g_1} * \delta_{g_2} * \rho m_K \\ &= \alpha_1\alpha_2\delta_{g_1} * \rho m_K * \delta_{g_2} * \rho m_K = \tau_0(\alpha_1, g_1) * \tau_0(\alpha_2, g_2) \end{aligned}$$

if $g_1, g_2 \in H_0$. Obviously $\Omega_0 = \operatorname{Ker} \tau_0 \mid \Omega$, so Ω_0 is normal in Ω. The continuity of τ_0 is clear.

4. Norm decreasing homomorphisms on locally compact groups. Let G be a locally compact group and consider on $M(G)$ the (σ) and (so) topologies defined in § 1. Every norm decreasing homomorphism on $L^1(G)$ extends naturally to a norm decreasing homomorphism on $M(G)$. To appreciate the usefulness of this extension theorem it is helpful to recall 1.1.3.

THEOREM 4.1.1. *Let F, G be locally compact groups and let $\varphi : L^1(F) \to M(G)$ be any norm decreasing homomorphism. Then φ extends uniquely to a norm decreasing homomorphism $\bar{\varphi} : M(F) \to M(G)$ which is continuous on norm bounded sets as a map of $(M(F), (so))$*

into $(M(G), (\sigma))$. *If* $\{e_j : j \in J\}$ *is a left approximate identity of norm one in* $L^1(F)$ *then the extension is given explicitly by the formula*

$$\bar{\varphi}(\mu) = \lim \{\varphi(e_j * \mu) : j \in J\} \qquad \text{all } \mu \in M(F) ,$$

where the limit is in the (σ) *topology. A similar result holds for right approximate identities.*

Proof. Let $B = \varphi(L^1(F))$ and let A be the (σ) closure of B in $M(G)$, so that A and B are subalgebras of $M(G)$.

LEMMA 4.1.2. *Let* $\{e_j{}^l : j \in J\}$ *be a left approximate identity of norm one and let* $\{e_k{}^r : k \in K\}$ *be a right approximate identity of norm one in* $L^1(F)$. *Then in* $M(G)$ *the* (σ) *limit points of the nets* $\{\varphi(e_j{}^l)\}$ *and* $\{\varphi(e_k{}^r)\}$ *all coincide in a single idempotent* $\iota \in M(G)$, *so we must have convergence*

$$\varphi(e_j{}^l) \xrightarrow{(\sigma)} \iota$$
$$\varphi(e_k{}^r) \xrightarrow{(\sigma)} \iota$$

in the (σ) *topology. If* $\varphi \neq 0$ *then* $\iota \neq 0$ *and we have* $a = \iota * a = a * \iota$ *for all* $a \in A$.

Proof. Since $\|\varphi(e_j{}^l)\| \leqq 1$ there is at least one (σ) limit point λ for this net, and for an appropriate subnet we get $\varphi(e_{j(p)}{}^l) \xrightarrow{(\sigma)} \lambda$. Thus if $f \in L^1(F)$ we have

$$\lambda * \varphi f \xleftarrow{(\sigma)} \varphi(e_{j(p)}{}^l) * \varphi f = \varphi(e_{j(p)}{}^l * f) \xrightarrow{\text{norm}} \varphi(f) .$$

Hence if $\{\mu_i : i \in I\}$ is a net in B with $\mu_i = \varphi(f_i)$ and $\mu_i \xrightarrow{(\sigma)} \mu$ in A, we have

$$\lambda * \mu \xleftarrow{(\sigma)} \lambda * \mu_i = \lambda * \varphi f_i = \varphi f_i = \mu_i \xrightarrow{(\sigma)} \mu$$

so that $\lambda * a = a$ for all $a \in A$. In particular we have $\lambda \in A$ so $\lambda * \lambda = \lambda$. Similarly if ν is a (σ) limit point of $\{\varphi(e_k{}^r)\}$ then $\nu * \nu = \nu$ and $a * \nu = a$ for all $a \in A$.

If λ, ν are (σ) limit points as above, we have λ, ν in A, which $\Rightarrow$ $\lambda = \lambda * \nu = \nu$; hence $\lambda = \nu$ and all limit points (left or right) coincide in a single idempotent ι such that $\iota * a = a * \iota = a$ if $a \in A$. If $\varphi \neq 0$, clearly $\iota \neq 0$.

The main step in our proof is to show that, if $\{f_j : j \in J\}$ is a norm bounded net in $L^1(F)$ which is (so) convergent to some $\mu \in M(F)$, then the net $\{\varphi(f_j)\}$ converges to a limit λ_μ in the (σ) topology, and this limit depends only on μ, rather than on the particular choice of the net $\{f_j\}$. This can be done for any $\mu \in M(F)$, in view of 1.1.5.

First consider any $f \in L^1(F)$ and notice that $\|f_j * f - \mu * f\| \to 0$, which $\Rightarrow \|\varphi f_j * \varphi f - \varphi(\mu * f)\| \to 0$. Let λ be any (σ) limit point of the norm bounded net $\{\varphi(f_j)\}$; there exists a convergent subnet $\varphi(f_{j(i)}) \xrightarrow{(\sigma)} \lambda$. Then

$$\varphi(\mu * f) \xleftarrow{\text{norm}} \varphi(f_{j(i)} * f) = \varphi f_{j(i)} * \varphi f \xrightarrow{(\sigma)} \lambda * \varphi f$$

for all $f \in L^1(F)$, which means that $\varphi(\mu * f) = \lambda * \varphi f$ for all $f \in L^1(F)$. Clearly $\lambda \in A$ since each $\varphi(f_j) \in B$, and this means that

$$\lambda = \lambda * \iota \xleftarrow{(\sigma)} \lambda * \varphi(e_k{}^r) = \varphi(\mu * e_k{}^r) .$$

Thus we get

$$\lambda = \lim \{\varphi(\mu * e_k{}^r) : k \in K\}$$

in the (σ) topology, and this formula doesn't depend on anything but the choice of $\mu \in M(F)$. Hence if $f_j \xrightarrow{(so)} \mu$, then λ is the only possible limit point of $\{\varphi(f_j)\}$, so if we take $\lambda_\mu = \lim \{\varphi(\mu * e_k{}^r)\}$, we always have $\varphi f_j \xrightarrow{(\sigma)} \lambda_\mu$.

Notice that if $f \in L^1(F)$ we have $\|f * e_k{}^r - f\| \to 0$, which gives

$$\lambda_f \xleftarrow{(\sigma)} \varphi(f * e_k{}^r) \xrightarrow{\text{norm}} \varphi(f)$$

so that $\varphi f = \lambda_f$ for all $f \in L^1(F)$. Now define $\bar{\varphi}(\mu) = \lambda_\mu$ for $\mu \in M(F)$, and verify the properties required. Clearly $\bar{\varphi}(f) = \varphi(f)$ for all $f \in L^1(f)$, so $\bar{\varphi}$ extends φ.

If (γ) is a locally convex topology on $M(F)$ we define the bounded (γ) topology $(b\gamma)$ by taking as a basis of neighborhoods about zero all sets $X \cap Y$ where X is a (γ) neighborhood of zero, and Y is a fixed norm neighborhood of zero. From the discussion above we know that if $\mu \in M(F)$ and if W is a (σ) neighborhood of zero in $M(G)$, then there is an open (bso) neighborhood V of zero in $M(F)$ such that $\bar{\varphi}((\mu + V) \cap L^1(F)) \subset \bar{\varphi}\mu + W$. Now let $W' \subset W$ be a (σ) neighborhood of zero such that $W' = -W'$ and $W' + W' \subset W$, and let U be an open (bso) neighborhood of zero in $M(F)$ such that

$$\bar{\varphi}((\mu + U) \cap L^1(F)) \subset \bar{\varphi}\mu + W' .$$

If $\lambda \in \mu + U$ we can find a (bso) neighborhood U_λ of zero such that

(1) $\bar{\varphi}((\lambda + U_\lambda) \cap L^1(F)) \subset \bar{\varphi}\lambda + W'$

(2) $\lambda + U_\lambda \subset \mu + U$.

Then we have $\bar{\varphi}((\lambda + U_\lambda) \cap L^1(F)) \subset \bar{\varphi}((\mu + U) \cap L^1(F)) \subset \bar{\varphi}\mu + W'$ and $\bar{\varphi}((\lambda + U_\lambda) \cap L^1(F)) \subset \bar{\varphi}\lambda + W'$, which together imply that

$$(\bar{\varphi}\lambda + W') \cap (\bar{\varphi}\mu + W') \neq \varnothing$$

(see 1.1.5), which means that $\bar{\varphi}\lambda \in \bar{\varphi}\mu + W$. Hence $\bar{\varphi}(\mu + U) \subset \bar{\varphi}\mu + W$ as required for continuity.

Clearly $\|\bar{\varphi}\mu\| \leqq \sup\{\|\varphi(\mu * e_k^r)\|\} \leqq \|\mu\|$ for $\mu \in M(F)$, and if $\mu, \lambda \in M(F)$ we have

$$\bar{\varphi}(\mu * \lambda) \xleftarrow{(\sigma)} \bar{\varphi}(\mu * (\lambda * e_k^r)) = \bar{\varphi}\mu * \varphi(\lambda * e_k^r) \xrightarrow{(\sigma)} \bar{\varphi}\mu * \bar{\varphi}\lambda$$

since $\lambda * e_k^r \xrightarrow{(so)} \lambda$. Hence $\bar{\varphi}$ is a norm decreasing homomorphism.

Example. In 4.1.1 we cannot replace the (so) topology with the (σ) topology in $M(F)$. Indeed, if $Z =$ integers, $S =$ circle, and if β is some irrational number, then $\varphi(\sum_{n=1}^\infty \alpha_n \delta_{x_n}) = \sum_{n=1}^\infty \alpha_n \delta_{(e^{ix_n\beta})}$ gives a norm decreasing homomorphism $\varphi : L^1(Z) \to M(S)$. This map coincides with its extension $\bar{\varphi}$. The sequence $\{\mu_n = \delta_{(n)} : n = 1, 2 \cdots\}$ is (σ) convergent to zero, while $\varphi(\mu_n)$ is not (σ) convergent in $M(S)$.

Remark. The proof of 4.1.1 is also valid for any *bounded* homomorphism $\varphi : L^1(F) \to M(G)$, which means that the structure of a bounded homomorphism is determined once we know the structure of the bounded group of measures $\bar{\varphi}(\mathscr{E}_F)$; however, the structure of the bounded subgroups in $M(G)$ is generally not known unless G is abelian or the subgroup lies within $\Sigma_{M(G)}$.

4.2. The structure of norm decreasing homomorphisms. If $\bar{\varphi}$ extends the norm decreasing homomorphism $\varphi : L^1(F) \to M(G)$, as in 4.1.1, then $\Gamma = \bar{\varphi}(\mathscr{E}_F)$ is a subgroup of the unit ball in $M(G)$. Using the continuity properties of $\bar{\varphi}$ demonstrated in 4.1.1 and our knowledge of the structure of Γ we can determine φ completely (see 1.1.3).

Let $i = \rho m_K$ be the unit of Γ and denote $H_0 = \operatorname{supp}(\Gamma)$, $\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_k \in \Gamma\} \supset \Omega_0 = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K = \rho m_K\}$, and $K_0 = \operatorname{Ker} \rho$. Let $\pi : S \times G \to (S \times G/\Omega_0)_r$ be the canonical map onto the space of right cosets of Ω_0, so π is a homomorphism when restricted to $S \times H_0$, and let $\tau_0 : (\alpha, g) \to \alpha\delta_g * \rho m_K$ for $(\alpha, g) \in S \times G$. Then define $\theta : F \to (\Omega/\Omega_0) \subset (S \times G/\Omega_0)_r$ to be $\theta = \pi \circ \tau_0^{-1} \circ \bar{\varphi} \circ \delta$, so that $\theta(x) = \pi(\alpha, g)$ if and only if $\bar{\varphi}(\delta_x) = \alpha\delta_g * \rho m_K$ in $M(G)$. The mappings involved are shown in the following (commutative) diagram.

$$\begin{tikzcd} S \times G \supset \Omega \arrow[d, "\pi"'] & \Gamma \arrow[l, "\tau_0^{-1}"'] \\ (S \times G/\Omega_0)_r \supset \Omega/\Omega_0 & F \arrow[l, "\theta"'] \arrow[u, "\bar{\varphi}\circ\delta"'] \end{tikzcd}$$

Figure 1

PROPOSITION 4.2.1. The map $\theta : F \rightarrow \Omega/\Omega_0$ is an epimorphism and is continuous as a mapping $\theta : F \rightarrow (S \times G/\Omega_0)_r$.

Proof. Let $x_1, x_2 \in F$ and let $(\alpha_1, g_1), (\alpha_2, g_2) \in \Omega$ be chosen such that $\pi(\alpha_i, g_i) = \theta(x_i)$, and let $(\alpha, g) \in \Omega$ be chosen such that $\pi(\alpha, g) = \theta(x_1 x_2)$. Thus $\bar{\varphi}(\delta_{x_i}) = \alpha_i \delta_{g_i} * \rho m_K$. We have $\theta(x_1 x_2) = \theta x_1 \cdot \theta x_2$ if $\pi(\alpha, g) = \pi(\alpha_1, g_1)\pi(\alpha_2, g_2)$, which happens if $(\alpha_1\alpha_2, g_1 g_2) \in (\alpha, g)\Omega_0$. This follows since

$$\begin{aligned}\bar{\alpha}\alpha_1\alpha_2\delta_{(g^{-1}g_1g_2)} * \rho m_K &= (\bar{\alpha}\delta_{g^{-1}} * \rho m_K) * (\alpha_1\delta_{g_1} * \rho m_K) * (\alpha_2\delta_{g_2} * \rho m_K) \\ &= \bar{\varphi}(\delta_{x_1x_2})^{-1} * \bar{\varphi}(\delta_{x_1}) * \bar{\varphi}(\delta_{x_2}) = \rho m_K .\end{aligned}$$

We want to show $\theta : F \rightarrow (S \times G/\Omega_0)_r$ is continuous. Because $\bar{\varphi} \circ \delta$ and π are continuous it suffices to show that $\tau_0 : S \times G \rightarrow N = \tau_0(S \times G)$ is an open map when N has the restricted (σ) topology. Let $(\alpha_0, g_0) \in S \times G$ and let $U \times V$ be a product of open sets in S, G with $\alpha_0 \in U$, $g_0 \in V$. It suffices to show that $\tau_0(U \times V)$ is always a (σ) neighborhood of $\tau_0(\alpha_0, g_0)$ in N. If this set fails to be a neighborhood there is a net $\{(\alpha_j, g_j)\}$ such that $\mu_j = \tau_0(\alpha_j, g_j) = \alpha_j\delta_{g_j} * \rho m_K \xrightarrow{(\sigma)} \alpha_0\delta_{g_0} * \rho m_K$, while $\mu_j \notin \tau_0(U \times V)$. We can assume $g_j \in g_0 WK$ for some compact neighborhood W of g_0, and, by taking subnets, we get $g_j \rightarrow g_1 \in g_0 K$, $\alpha_j \rightarrow \alpha_1 \in S$.

If we let $g_j{}^* = g_j(g_1{}^{-1}g_0)$, then $g_j{}^* \rightarrow g_0$. Let $\alpha_j{}^* = \alpha_j\rho(g_1{}^{-1}g_0)$; this makes sense because $g_0 g_1{}^{-1} \in K$. Then we have

$$\begin{aligned}\tau_0(\alpha_j{}^*, g_j{}^*) &= \alpha_j{}^*\delta_{(g_j{}^*)} * \rho m_K = \alpha_j\rho(g_1{}^{-1}g_0)\delta_{g_j} * \delta_{(g_1{}^{-1}g_0)} * \rho m_K \\ &= \alpha_j\delta_{g_j} * \rho m_K = \tau_0(\alpha_j, g_j) \xrightarrow{(\sigma)} \alpha_0\delta_{g_0} * \rho m_K .\end{aligned}$$

Since $g_j{}^* \rightarrow g_0$ we must have $\alpha_j{}^* \rightarrow \alpha_0$ and $\alpha_j{}^*$ is eventually in U; hence $\tau_0(\alpha_j{}^*, g_j{}^*) = \tau_0(\alpha_j, g_j)$ is eventually in $\tau_0(U \times V)$, a contradiction.

Let $\tau^*\psi(\alpha, g) = \langle \alpha\delta_g * \rho m_K, \psi \rangle$ for $\psi \in C_0(G)$. Then $\tau^*\psi \in C_0(S \times G)$ since K is compact, and in fact $\tau^*\psi$ is constant on right cosets of Ω_0 in $S \times G$ since $\Omega_0 = \{(\rho(k), k) : k \in K\}$. If $\Psi \in C_0(S \times G)$ and is constant on right cosets of Ω_0, let us identify it with a function

$$\pi^*\Psi \in C_0((S \times G/\Omega_0)_r) .$$

This function vanishes at infinity since Ω_0 is compact. We can give an integral representation for norm decreasing homomorphisms as follows.

THEOREM 4.2.2. *Let F, G be locally compact groups and let $\varphi : L^1(F) \rightarrow M(G)$ be a nonzero norm decreasing homomorphism with extension $\bar{\varphi}$ to $M(F)$, as in* 4.1.1. *Denote*

(1) $\Gamma = \bar{\varphi}(\mathscr{E}_F)$

(2) $i = \rho m_K$ *the unit of* Γ

(3) $\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K \in \Gamma\}$

(4) $\Omega_0 = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K = i\}$.
Define the maps

$$\begin{aligned}\tau_0 &: S \times G \longrightarrow \Gamma \\ \tau^* &: C_0(G) \longrightarrow C_0(S \times G) \\ \pi^*\tau^* &: C_0(G) \longrightarrow C_0((S \times G/\Omega_0)_r) \\ \theta &: F \longrightarrow \Omega/\Omega_0\end{aligned}$$

as indicated above. Then we have the representation

$$(*) \qquad \langle \bar{\varphi}\mu, \psi \rangle = \langle \mu, (\pi^*\tau^*\psi) \circ \theta \rangle$$

for all $\mu \in M(F)$ *and* $\psi \in C_0(G)$.

REMARK. Since $\theta : F \to \Omega/\Omega_0$ is a continuous homomorphism and $\pi^*\tau^*\psi \in C_0((S \times G/\Omega_0)_r)$, it follows that $(\pi^*\tau^*\psi) \circ \theta$ is a uniformly continuous and bounded function on F. Thus the right hand side of $(*)$ is uniquely determined. We will want to make use of 1.1.1 in the following discussion.

Proof. We have $\langle \bar{\varphi}(\delta_x), \psi \rangle = \langle \delta_{\theta x}, \pi^*\tau^*\psi \rangle = \langle \delta_x, (\pi^*\tau^*\psi) \circ \theta \rangle$ if $x \in F$. If $\mu \in M(F)$ is of norm one then there exists a net $\{\sigma_j : j \in J\}$ in the convex span of the extreme points of $\Sigma_{M(F)}$ such that $\|\sigma_j\| = 1$ and $\sigma_j \xrightarrow{(so)} \mu$ (see 1.1.3). If we write $\sigma_j = \sum \lambda(j, x)\delta_x$ (finite sum), we can apply 1.1.1. to get

$$\begin{aligned}\langle \bar{\varphi}\mu, \psi \rangle \longleftarrow \langle \bar{\varphi}(\sigma_j), \psi \rangle &= \sum \lambda(j, x) \langle \bar{\varphi}(\delta_x), \psi \rangle \\ &= \sum \lambda(j, x) \langle \delta_x, (\pi^*\tau^*\psi) \circ \theta \rangle \\ &= \langle \sigma_j, (\pi^*\tau^*\psi) \circ \theta \rangle \longrightarrow \langle \mu, (\pi^*\tau^*\psi \rangle \circ \theta \rangle .\end{aligned}$$

Thus $\langle \bar{\varphi}\mu, \psi \rangle = \langle \mu, (\pi^*\tau^*\psi) \circ \theta \rangle$.

As a converse we have the following theorem which classifies all norm decreasing homomorphisms.

THEOREM 4.2.3. *Let F, G be locally compact groups and let Γ be a subgroup of $\Sigma_{M(G)}$ with unit $i = \rho m_K$ and with*

$$\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K \in \Gamma\} ,$$

$\Omega_0 = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K = i\}$. *Then if $\theta : F \to \Omega/\Omega_0$ is any continuous epimorphism (Ω/Ω_0 is given the restricted topology from $(S \times G/\Omega_0)_r$), the relation*

$$(*) \qquad \langle \bar{\varphi}\mu, \psi \rangle = \langle \mu, (\pi^*\tau^*\psi) \circ \theta \rangle$$

for $\mu \in M(F)$, $\psi \in C_0(G)$ *defines a norm decreasing homomorphism*

$\bar{\varphi}: (M(F), (so)) \to (M(G), (\sigma))$ *which is continuous on norm bounded sets, and we have* $\bar{\varphi}(\mathscr{E}_F) = \Gamma$.

REMARK. If $\bar{\varphi}$ has the above continuity properties it is clear that $\bar{\varphi}$ is obtained, as in 4.1.1, by extending the norm decreasing homomorphism $\bar{\varphi} \mid L^1(F)$.

Proof. From 3.1.11 we see that Ω_0 is a compact subgroup of $S \times G$ which is normal in Ω, so Ω/Ω_0 is well defined. We have also noted that $\tau^*\psi(\alpha, g) = \langle \alpha\rho_g * \rho m_K, \psi \rangle$ is in $C_0(S \times G)$ and

$$\pi^*\tau^*\psi \in C_0((S \times G/\Omega_0)_r),$$

so $(\pi^*\tau^*\psi) \circ \theta$ is bounded and uniformly continuous on F. Hence $(*)$ is always well defined.

Clearly $\bar{\varphi}: (M(F), (so)) \to (M(G), (\sigma))$ is a norm decreasing linear map, and continuity on norm bounded sets follows from 1.1.1. Now $\bar{\varphi}(\delta_x) = \alpha\delta_g * \rho m_K$ for all $(\alpha, g) \in \pi^{-1}\theta(x)$, so $\bar{\varphi} \circ \delta = (\tau_0 \circ \pi^{-1}) \circ \theta$; thus, $\bar{\varphi}(\mathscr{E}_F) = \Gamma$ and $\bar{\varphi}$ is a continuous homomorphism of $(\mathscr{E}_F, (so))$ into $(M(G), (\sigma))$. Convolution is a jointly (so) continuous operation in $\Sigma_{M(F)}$, so $\bar{\varphi}$ is a norm decreasing homomorphism of $M(F)$ in view of the density theorems 1.1.3, 1.1.4.

A norm decreasing homomorphism $\varphi: L^1(F) \to M(G)$ is order preserving if $\mu \geqq 0 \Rightarrow \varphi(\mu) \geqq 0$. From the continuity properties given in 4.1.1 and the structure theorem 3.1.8 it follows that φ is order preserving $\Leftrightarrow \bar{\varphi}(\mathscr{E}_F)$ is a group of translates of Haar measure $\{\delta_x * m_Q : x \in G_0\}$, where $Q \subset G$ is a compact subgroup, normal in the subgroup G_0. Every norm decreasing homomorphism φ is closely related to an order preserving norm decreasing homomorphism of $L^1(F)$ into $M(S \times G)$.

If $\Omega \supset \Omega_0$ are two subgroups in $S \times G$ satisfying conditions (1) and (2) in the discussion following 3.1.8, define the maps $\tau_0, \cdots, \tau^{**}$ as indicated there.

THEOREM 4.2.4. *If* $\bar{\varphi}: (M(F), (so)) \to (M(G), (\sigma))$ *is a norm decreasing homomorphism, continuous on norm bounded sets, and if* $\Gamma = \bar{\varphi}(\mathscr{E}_F)$ *has unit* $i = \rho m_K$, *then the subgroups*

$$\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K \in \Gamma\} \supset \Omega_0 = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K = \rho m_K\}$$

satisfy conditions (1) *and* (2) *of* 3.1.9 *and we can factor* $\bar{\varphi} = \tau^{**} \circ \Phi$ *where* Φ *is some order preserving norm decreasing homomorphism of* $M(F)$ *into* $M(S \times G)$. *Here* Φ *maps* $\mathscr{E}_F$ *to the group of measures* $\{\delta_x * m_{\Omega_0} : x \in \Omega\}$ *and* τ^{**} *is a homomorphism on the range of* Φ.

Conversely, if $\Phi : M(F) \to M(S \times G)$ *is any order preserving norm decreasing homomorphism, let* $\Omega = \operatorname{supp}(\Phi(\mathscr{E}_F)) \supset \Omega_0 = s(\Phi(\delta_e))$. *If* Ω, Ω_0 *satisfy conditions* (1), (2) *in* 3.1.9, *then* $\varphi = \tau^{**} \circ \Phi : (M(F), (so)) \to (M(G), (\sigma))$ *is a norm decreasing homomorphism, continuous on norm bounded sets.*

Proof. Let $\Omega \supset \Omega_0$ satisfy (1) and (2) of 3.1.9 and define M_Ω to be the subspace of measures in $M(S \times G)$ whose intersection with $\Sigma_{M(S\times G)}$ is $co[S\mathscr{E}_\Omega : \sigma]$. We assert that

(*) M_Ω is a subalgebra in $M(S \times G)$, $\tau^{**}(\delta_x * m_{\Omega_0}) = \tau^{**}(\delta_x) = \tau_0(x)$ for $x \in \Omega$, and τ^{**} is a norm decreasing homomorphism on M_Ω.

Clearly $M_\Omega = \{\mu : s(\mu) \subset \bar{\Omega}\}$, and is a subalgebra. We have already shown (in discussing 3.1.10) that $m_{\Omega_0} \in M_\Omega$ and $\tau^{**}(\delta_x * m_{\Omega_0}) = \tau^{**}(\delta_x) = \tau_0(x)$ for all $x \in \Omega$. Thus τ^{**} is multiplicative on $S\mathscr{E}_\Omega$, and since convolution is separately (σ) continuous we can show that $\tau^{**}(\delta_x * \mu) = \tau^{**}(\delta_x) * \tau^{**}(\mu)$ for $\mu \in M_\Omega$, $x \in \bar{\Omega}$. Then if $\lambda_j \xrightarrow{(\sigma)} \lambda$ for $\|\lambda\| = 1$, where $\lambda_j \in co[S\mathscr{E}_\Omega]$ we use the same idea once more to get

$$\tau^{**}(\lambda * \mu) \xleftarrow{(\sigma)} \tau^{**}(\lambda_j * \mu) = \tau^{**}(\lambda_j) * \tau^{**}(\mu) \xrightarrow{(\sigma)} \tau^{**}(\lambda) * \tau^{**}(\mu) ,$$

so $\tau^{**} \mid M_\Omega$ is a homomorphism.

Now Φ maps $S\mathscr{E}_F$ into M_Ω, and if $\mu \in M(F)$, $\|\mu\| = 1$, there exists a net $\{\mu_j\} \subset co[S\mathscr{E}_F]$ such that $\mu_j \xrightarrow{(so)} \mu$. This means $\Phi\mu_j \xrightarrow{(\sigma)} \Phi\mu$ while $\Phi\mu_j \in \Sigma_{M(S\times G)} \cap M_\Omega$, so $\Phi\mu \in M_\Omega$ and Φ maps $M(F)$ into M_Ω. Thus $\tau^{**}\Phi$ is well defined and is a norm decreasing homomorphism with the desired continuity properties (τ^{**} is (σ) continuous on $M(S \times G)$).

Conversely, let $\bar{\varphi}$ be given; then Ω, Ω_0 defined above satisfy (1) and (2), as shown in 3.1.11. The homomorphism $\theta : F \to \Omega/\Omega_0$, associated with $\bar{\varphi}$ as in 4.2.2, is continuous, so $\theta^*\psi = \psi \circ \theta$ is uniformly continuous and bounded (UCB) on F and we can consider the dual maps.

$$\theta^* : C_0((S \times G/\Omega_0)_r) \longrightarrow UCB(F)$$
$$\theta^{**} : M(F) \longrightarrow M((S \times G/\Omega_0)_r) .$$

For $\psi \in C_0(S \times G)$ define $\pi^*\psi \in C_0((S \times G/\Omega_0)_r)$ by lifting the function $\pi^*\psi(x) = \int \psi(xt)dm_{\Omega_0}(t)$ (constant on right cosets of Ω_0) over to the coset space $(S \times G/\Omega_0)_r$. The desired map Φ is given by

$$\langle \Phi\mu, \psi \rangle = \langle \theta^{**}\mu, \pi^*\psi \rangle = \langle \mu, (\pi^*\psi) \circ \theta \rangle$$

for $\psi \in C_0(S \times G)$. It is easy to verify that $\Phi(\delta_x) = \delta_{(\alpha, g)} * m_{\Omega_0}$ for all $(\alpha, g) \in \pi^{-1}\theta(x)$; therefore, as indicated in 3.1.10, we have $\langle \tau^{**}\Phi(\delta_x), \psi \rangle = \langle \tau^{**}(\delta_{(\alpha, g)} * m_{\Omega_0}), \psi \rangle = \langle \tau^{**}(\delta_{(\alpha, g)}), \psi \rangle = \langle \alpha\delta_g * \rho m_K, \psi \rangle = \langle \bar{\varphi}(\delta_x)\psi \rangle$ for all $x \in F$. Thus $\bar{\varphi} = \tau^{**}\Phi$ on $S\mathscr{E}_F$. But from 1.1.1 we see that Φ

defined above is continuous on norm bounded sets mapping from the (so) to the (σ) topology; clearly, then $\tau^{**}\Phi : (M(F), (so)) \rightarrow (M(G), (\sigma))$ is continuous on norm bounded sets. Now $\bar{\varphi}$ and $\tau^{**}\Phi$ both enjoy this continuity property and coincide on $S\mathscr{E}_F$; from 1.1.3 it follows that they coincide on all of $M(F)$, and this is the desired factorization of $\bar{\varphi}$.

5. Examples and applications. In 5.1 we analyze the special structure of norm decreasing monomorphisms $\varphi : L^1(F) \rightarrow M(G)$ between locally compact groups F and G; then in § 5.2 we give the structure of all norm decreasing homomorphisms φ which map $L^1(F)$ onto $L^1(G)$. Maps in the latter class have very simple structure.

5.1. Norm decreasing monomorphisms. Let us denote $\mathscr{F} = S\mathscr{E}_F = \{\alpha\delta_x : |\alpha| = 1,\ x \in F\}$ and $\mathscr{F}_0 = \mathscr{E}_F$ throughout this discussion.

LEMMA 5.1.1. *If $\varphi : L^1(F) \rightarrow M(G)$ is a norm decreasing monomorphism, and if $\bar{\varphi}$ is its extension to $M(F)$ as in* 4.1.1, *then φ is a monomorphism of $M(F)$ into $M(G)$. Furthermore $\bar{\varphi}(\mathscr{F}_0) \cap Si = \{i\}$, where $i = \bar{\varphi}(\delta_e)$, and $\mu = \lambda$ in $\bar{\varphi}(\mathscr{F}_0)$ whenever $s(\mu) = s(\lambda)$.*

Proof. If $\mu, \lambda \in M(F)$ have $\bar{\varphi}\mu = \bar{\varphi}\lambda = \xi$ and $\mu \neq \lambda$, then there is some $f \in L^1(F)$ such that $\mu * f \neq \lambda * f$ while $\varphi(\mu * f) = \varphi(\lambda * f) = \xi * \varphi f$, a contradiction. Hence $\bar{\varphi}(\mathscr{F}_0) \cap Si = \{i\}$ and the last property follows from 3.1.4.

We propose to study the structure of all norm decreasing homomorphisms φ whose extensions $\bar{\varphi}$ have the special property $\Gamma_0 \cap Si = \{i\}$, where $\Gamma_0 = \bar{\varphi}(\mathscr{F}_0)$ and $i = \bar{\varphi}(\delta_e)$ is the unit in Γ_0. This discussion will apply to norm decreasing monomorphisms as a particular case. Hereafter we will denote $\Gamma = \bar{\varphi}(\mathscr{F})$, $\Gamma_0 = \bar{\varphi}(\mathscr{F}_0)$ (writing the unit of these groups as $i = \rho m_K$), $H_0 = \operatorname{supp}(\Gamma)$, and

$$\Omega = \{(\alpha, g) \in S \times G : \alpha\delta_g * \rho m_K \in \Gamma\} .$$

Let $\pi : G \rightarrow (G/K)_r$ be the canonical map onto the right coset space, so $\pi : H_0 \rightarrow H_0/K$ is the corresponding canonical homomorphism. Let γ_F, γ_G be the topologies on F, G and, if γ is a group topology on G, let γ/π denote the quotient space topology on $(G/K)_r$ (notice $\gamma/\pi = \pi(\gamma)$). The restriction of γ_G to a subset $N \subset G$ is $\gamma_G | N$. We will speak interchangeably of a topology γ and the collection of open sets it specifies.

The following lemma holds for all locally compact groups; notation is chosen so its meaning in the present context is clear.

LEMMA 5.1.2. *Let F, G be locally compact groups and consider any system of subgroups $K \subset H_0 \subset G$ with K a compact subgroup in G which is normal in H_0. Let $\pi : G \to (G/K)_r$ be the canonical map onto the right coset space. If $\zeta : F \to (H_0/K, \gamma_G/\pi)$ is a continuous epimorphism then $\zeta(\gamma_F)$ and $\pi^{-1}\circ\zeta(\gamma_F)$ are topologies in H_0/K and H_0 respectively; moreover, if γ is the common refinement in H_0 of $(\gamma_G \mid H_0)$ and $\pi^{-1}\circ\zeta(\gamma_F)$ then (H_0, γ) is a locally compact topological group, $\gamma/\pi = \zeta(\gamma_F)$, and $\zeta : F \to (H_0/K, \gamma/\pi)$ is an open, continuous epimorphism.*

REMARK. Unless K is trivial, $\pi^{-1}\circ\zeta(\gamma_F)$ will not be a Hausdorff topology, but in all other respects (homogeneity, joint continuity of multiplication, etc.) it is like a group topology.

Proof. The topology axioms for $\pi^{-1}\circ\zeta(\gamma_F)$ follow if we can verify them for $\zeta(\gamma_F)$. Only the finite intersection property is nontrivial. If $V_1, V_2 \in \gamma_F$ let $U_i = V_i\cdot\mathrm{Ker}\,\zeta$ and notice that $\zeta^{-1}(x) \cap U_i \neq \varnothing$ implies that $\zeta^{-1}(x) \subset U_i$. Thus

$$\begin{aligned}\zeta(V_1) \cap \zeta(V_2) = \zeta(U_1) \cap \zeta(U_2) &= \{x : \zeta^{-1}(x) \cap U_i \neq \varnothing,\ i = 1, 2\}\\ &= \zeta(U_1 \cap U_2) \in \zeta(\gamma_F)\ .\end{aligned}$$

Now (H_0, γ) is a Hausdorff space and the collection of sets $\mathscr{U} = \{U \cap V : U = W \cap H_0,\ W \in \gamma_G;\ V = \pi^{-1}\circ\zeta(X),\ X \in \gamma_F\}$ is a base for γ. If $U \cap V \in \mathscr{U}$ then $(U \cap V)^{-1} = U^{-1} \cap V^{-1} \in \mathscr{U}$, so the inverse mapping is bicontinuous. It is quite easy to verify that γ is homogeneous, in the sense that $\gamma = \{xU : U \in \gamma\}$ for any $x \in H_0$, so joint continuity of multiplication will only be proved at the identity $e \in H$. If e lies within $U \cap V \in \mathscr{U}$ there exist $U_0 \in \gamma_G \mid H_0$ and $V_0 \in \pi^{-1}\circ\zeta(\gamma_F)$, which contain e, such that $U_0^2 \subset U$ and $V_0^2 \subset V$; hence $(U_0 \cap V_0) \times (U_0 \cap V_0)$ is an open neighborhood of (e, e) in $(H_0, \gamma) \times (H_0, \gamma)$ which maps into $U \cap V$ under the product mapping.

Clearly $\gamma \supset \pi^{-1}\circ\zeta(\gamma_F)$, so that $\gamma/\pi = \pi(\gamma) \supset \pi\circ\pi^{-1}(\zeta(\gamma_F)) = \zeta(\gamma_F)$. For the converse inclusion, we first make a few simple assertions:

(1) If $A \subset H_0$ is a union of K-cosets and if B is any subset of H_0, then $(A \cap B)\cdot K = A \cap (B\cdot K)$;

(2) If $A_\alpha \subset H_0$ for indices $\alpha \in I$, then $(\bigcup_{\alpha\in I} A_\alpha)\cdot K = \bigcup_{\alpha\in I}(A_\alpha\cdot K)$.

Now a typical element in γ has the form $X = \bigcup_{\alpha\in I} A_\alpha \cap B_\alpha$ where $A_\alpha = \pi^{-1}\circ\zeta(U_\alpha)$ for some $U_\alpha \in \gamma_F$, and $B_\alpha = V_\alpha \cap H_0$ for some $V_\alpha \in \gamma_G$. Evidently $A_\alpha = A_\alpha\cdot K$ and $B_\alpha\cdot K = \pi^{-1}\circ(\pi B_\alpha)$, so we get

$$(*)\qquad \begin{aligned}\pi(X) = \pi(X\cdot K) &= \pi(\textstyle\bigcup_{\alpha\in I}(A_\alpha \cap B_\alpha)\cdot K)\\ &= \pi(\textstyle\bigcup_{\alpha\in I} A_\alpha \cap (B_\alpha\cdot K))\\ &= \pi(\textstyle\bigcup_{\alpha\in I} \pi^{-1}\zeta(U_\alpha) \cap \pi^{-1}\pi(B_\alpha))\\ &= \textstyle\bigcup_{\alpha\in I} \zeta(U_\alpha) \cap \pi(B_\alpha)\ .\end{aligned}$$

But continuity of ζ implies that $\zeta(\gamma_F) \supset (\gamma_G/\pi \mid H_0/K)$ and it is easily verified that the latter collection of sets is just $\pi(\gamma_G \mid H_0)$; hence the last term in $(*)$ is in $\zeta(\gamma_F)$, giving $\pi(\gamma) \subset \zeta(\gamma_F)$. Clearly $\gamma/\pi = \zeta(\gamma_F) \Rightarrow \zeta : F \to (H_0/K, \gamma/\pi)$ is an open mapping; so $(H_0/K, \gamma/\pi)$ is topologically isomorphic to the locally compact quotient group $F/\mathrm{Ker}(\zeta)$. To see the local compactness of (H_0, γ), notice that $K \subset H_0$ is γ compact because $\gamma \mid K = \gamma_G \mid K$. A result due to Mackey gives local compactness (see Montgomery-Zippen [7], p. 52).

If $x \in F$ write $s(x) = s(\bar{\varphi}(\delta_x))$, a coset of K in H_0. The map $\zeta = \pi \circ s : F \to ((G/K)_r, \gamma_G/\pi)$ carries F onto H_0/K, is a homomorphism (see 2.1.1), and is continuous since $x_j \to x \Rightarrow \delta_{x_j} \xrightarrow{(so)} \delta_x \Rightarrow \bar{\varphi}(\delta_{x_j}) \to \bar{\varphi}(\delta_x) \Rightarrow \pi \circ s(x_j) \to \pi \circ s(x)$. If $\mu \in \Gamma_0$ and $g \in s(\mu)$, then we can write $\mu = \rho_\mu \mid \mu \mid = \rho_\mu(\delta_g * m_K)$ and we can take ρ_μ to be a unique continuous function on the coset $s(\mu) \subset H_0$. Assigning ρ_μ in this manner for each $\mu \in \Gamma_0$ we have $\rho_{\mu*\lambda}(st) = \rho_\mu(s)\rho_\lambda(t)$ for $s \in s(\mu)$, $t \in s(\lambda)$, as indicated in the proof of 3.1.2. Define ρ on all of H_0 such that $\rho(x) = \rho_\mu(x)$ if $x \in s(\mu)$, $\mu \in \Gamma_0$; this is unambiguous since $s(\mu) = s(\lambda) \Rightarrow \mu = \lambda$ (we assume $\Gamma_0 \cap Si = \{i\}$, so 3.1.4 applies).

Consider the group topology γ on H_0 constructed as in 5.1.2 for the epimorphism $\zeta = \pi \circ s : F \to (H_0/K, \gamma_G/\pi)$.

PROPOSITION 5.1.3. (H_0, γ) is a locally compact Hausdorff group and $\rho \in (H_0, \gamma)^\wedge$.

Proof. Clearly ρ is a unimodular multiplicative function on H_0 which is continuous on cosets of K (see proof of 3.1.2). The topological group properties of (H_0, γ) were verified in 5.1.2. Given $\varepsilon > 0$ we can find a γ_G neighborhood V of the unit in G such that $\mid \rho(g_1) - \rho(g_2) \mid < \varepsilon$ for all $g_1, g_2 \in H_0$ with $g_1 = g_2 \bmod K$ and $g_1^{-1}g_2 \in V$. This is clear since $\rho_\mu{}^g(s) = \rho_\mu(gs) = \alpha\rho(s)$ for some $\mid \alpha \mid = 1$, whenever $s \in K$, $g \in s(\mu)$, $\mu \in \Gamma$ (see proof of 3.1.2), and we know ρ is uniformly continuous on K.

Let $g \in H_0$; then $g_0 \in s(x_0)$ for some $x_0 \in F$, and if U is a compact γ_F neighborhood of x_0, $N = s(U)$ is a neighborhood of g_0 in (H_0, γ). N is compact since continuity of $\pi \circ s : F \to (H/K, \gamma/\pi) \Rightarrow \pi \circ s(U)$ is compact, and since K is a γ compact subgroup in H_0. If ρ fails to be γ continuous at g_0 we can find a net $\{g_j\} \subset N$ such that $g_j \xrightarrow{(\gamma)} g_0$ while $\rho(g_j) \to \beta \neq \beta_0 = \rho(g_0)$. For each index j there exists an $x_j \in U$ such that $g_j \in s(x_j)$; we can assume that the net $\{x_j\}$ is γ_F convergent to some $x_1 \in U$, which will $\Rightarrow \mu_j = \bar{\varphi}(\delta_{x_j}) \xrightarrow{(\sigma)} \bar{\varphi}(\delta_{x_1}) = \mu_1$. But this $\Rightarrow s(\mu_1) = g_0K = s(\mu_0)$, since $g_j \xrightarrow{(\gamma_G)} g_0$, so $\mu_1 = \mu_0 = \bar{\varphi}(\delta_{x_0})$. Recall that $\mu_j = \rho(\delta_{g_j} * m_K)$ and $\mu_0 = \rho(\delta_{g_0} * m_K)$ from the definition of ρ.

If $\psi \in C_0(G)$ has sup norm one and $\langle \mu_0, \psi \rangle \neq 0$, then $\Psi(s) =$

$\int \psi(ts)\rho(t)dm_K(t)$ is in $C_0(G)$, $\|\Psi\|_\infty \leqq 1$, and $\Psi(s) = \alpha\overline{\rho(s)}$ for all $s \in g_0K$, where $\alpha = \langle \mu_0, \psi \rangle$ (a constant $\neq 0$). Furthermore,

$$\begin{aligned}\langle \mu_0, \Psi \rangle &= \int\left[\int \psi(ts)\rho(t)dm_K(t)\right]\rho(s)d[\delta_{g_0} * m_K](s) \\ &= \int\left[\int \psi(ts)\rho(ts)dm_K(t)\right]d[\delta_{g_0} * m_K](s) \\ &= \int \psi(x)\rho(x)d[m_K * \delta_{g_0} * m_K](x) \\ &= \int \psi(x)\rho(x)d[\delta_{g_0} * m_K](x) = \langle \mu_0, \psi \rangle .\end{aligned}$$

If $\varepsilon > 0$ we can insure that $|\Psi(g_js) - \Psi(g_0s)| < \varepsilon$ for all $s \in K$ and $j \geqq j_\varepsilon$ since $g_j \xrightarrow{(\gamma_G)} g_0$, and this means that $\alpha = \langle \mu_0, \Psi \rangle \longleftarrow \langle \mu_j, \Psi \rangle = \int \Psi\rho d[\delta_{g_j} * m_K] = \int \Psi(g_js)\rho(g_js)dm_K(s)$. The last integral is eventually within ε of

$$\begin{aligned}\int \Psi(g_0s)\rho(g_js)dm_K(s) &= \alpha \int \overline{\rho(g_0s)}\rho(g_js)dm_K(s) \\ &= \alpha \int \overline{\rho(g_0)}\rho(g_j)dm_K(s) \longrightarrow \alpha\beta_0\overline{\beta} .\end{aligned}$$

Since this is true for all $\varepsilon > 0$, and $\beta_0 \neq \beta$, we have a contradiction.

COROLLARY 5.1.4. *If F is a compact group and $\varphi : L^1(F) \to M(G)$ is a norm decreasing monomorphism, then in* 5.1.2 *$\Gamma = \overline{\varphi}(\mathscr{F})$ is a (σ) compact subgroup of $\sum_{M(G)}$, $H_0 = \operatorname{supp}(\Gamma)$ is a compact subgroup in G, and $\gamma = \gamma_G | H_0$ in H_0. Thus if ρ is defined as above, $\rho \in (H_0, \gamma_G)^\wedge$.*

Proof. Clearly Γ is compact; H_0 is then γ_G compact since $(H_0/K, \gamma_G/\pi)$ is compact (recall $\pi \circ s : F \to (H_0/K, \gamma_G/\pi)$ is a continuous epimorphism). By definition of γ the map $\pi \circ s : F \to (H_0/K, \gamma/\pi)$ is continuous and we know that $K \subset H_0$ is γ compact; thus H_0 is γ compact as well as γ_G compact. Since γ is finer than γ_G, these must be equivalent topologies on H_0.

Consider the following maps between measure algebras.

(1) Let H, G be locally compact groups and let $j : H \to G$ be a continuous monomorphism. Define $j^{**} : (M(H), (so)) \to (M(G), (\sigma))$ such that $\langle j^{**}\mu, \psi \rangle = \langle \mu, \psi \circ j \rangle$ for $\psi \in C_0(G)$.

(2) Let H be a locally compact group and let $\rho \in H^\wedge$. Define $A_\rho : (M(H), (so)) \to (M(H), (so))$ such that $A_\rho(\mu) = \rho\mu$, so $\langle A_\rho\mu, \psi \rangle = \langle \mu, \rho\psi \rangle$.

(3) Let F, H be locally compact groups, let K be a compact

normal subgroup in H, and let $\zeta: F \to (H/K, \gamma_H/\pi)$ be an open continuous epimorphism, where $\pi: H \to H/K$ is the canonical homomorphism. Then define $\Phi: (M(F),(so)) \to (M(H),(so))$ such that $\langle \Phi\mu, \psi\rangle = \langle \mu, (\pi^*\psi)\circ\zeta\rangle$, where the function $\pi^*\psi(x) = \int \psi(xt)dm_K(t)$, constant on cosets of K, is considered as a function in $C_0(H/K)$.

We assert that the maps in (1) $\cdots$ (3) are all norm decreasing homomorphisms, continuous on norm bounded sets with respect to the topologies indicated. Since $(A_\rho\mu)*f = \rho(\mu*\bar\rho f)$, this assertion is clear for (2), and follows easily from 1.1.1 for (1), because $\psi\circ j$ is uniformly continuous and bounded on H; we momentarily put off verification of (3). Once this assertion has been checked we can prove the following structure theorem.

THEOREM 5.1.5. *If we are given groups and maps as in* (1) $\cdots$ (3) *then the map* $\bar\varphi = j^{**}\circ A_\rho\circ\Phi: (M(F),(so)) \to (M(G),(\sigma))$ *is a norm decreasing homomorphism, continuous on norm bounded sets, with the special property that* $\Gamma_0 \cap Si = \{i\}$, *where* $\Gamma_0 = \bar\varphi(\mathscr{F}_0)$ *and* $i \in \Gamma$ *is its unit. Conversely, let* $\varphi: L^1(F) \to M(G)$ *be a norm decreasing homomorphism whose extension* $\bar\varphi$ *(as described in* 4.1.1*) has the special property* $\Gamma_0 \cap Si = \{i\}$, *where* $\Gamma_0 = \bar\varphi(\mathscr{F}_0)$ *and* $i = \rho m_K$ *is its unit. If* $H_0 = \operatorname{supp}(\Gamma)$, *then we get* $\bar\varphi = j^{**}\circ A_\rho\circ\Phi$ *by taking groups* $H = (H_0, \gamma) \supset K = (K, \gamma)$ *and maps* $\zeta = \pi\circ s: F \to (H_0/K, \gamma/\pi)$, $j = id: (H_0, \gamma) \to (G, \gamma_G)$, *where* $\rho \in (H_0, \gamma)^\wedge$ *is the unique function on* H_0, *continuous on cosets of* K, *with the property* $\mu = \rho|\mu|$ *for all* $\mu \in \Gamma_0$.

REMARK. In the first part, $\bar\varphi$ is clearly the extension of $\varphi = \bar\varphi | L^1(F)$. Furthermore, the unit of Γ will be $i = \rho m_K$ and $\operatorname{supp}(\Gamma) = H$, when H and K are regarded as subgroups in G. In the second part the γ topology in H_0 is defined as in 5.1.2.

Proof. In the first part consider H and K as subgroups of G (with new group topologies) and j as the identity injecting H into G; H has a topology finer than $\gamma_G | H$, but since j is continuous, it is a homeomorphism on compacta and on cosets of K in particular. If $x \in F$ it is easy to verify that $\bar\varphi(\delta_x) = \rho(\delta_g * m_K)$ for any $g \in \bar\pi^{-1}\circ\zeta(x)$. From this it is clear that Γ has unit $i = \rho m_K$, and that $\Gamma_0 \cap Si = \{i\}$.

Conversely let $\varphi: L^1(F) \to M(G)$ be given. If we take $H = (H_0, \gamma)$, $K = (K, \gamma)$ and let $\zeta = \pi\circ s$, $j = id: (H_0, \gamma) \to (G, \gamma_G)$, we see that H is a locally compact group and that $\zeta: F \to (H_0/K, \gamma/\pi)$ is an open, continuous homomorphism (5.1.2); thus, the maps j^{**}, A_ρ, Φ are well defined. We know $\rho \in (H_0, \gamma)^\wedge$ from 5.1.3.

If $x \in F$ then $|\bar\varphi(\delta_x)| = \delta_g * m_K$ for any $g \in s(x)$ and $\bar\varphi(\delta_x) = \rho(\delta_g * m_K)$

by definition of ρ. It is a simple matter to verify that

$$id^{**} \circ A_\rho \circ \Phi(\delta_x) = \rho(\delta_g * m_K)$$

for any $g \in s(x)$, so that $\bar{\varphi} = id^{**} \circ A_\rho \circ \Phi$ on $\mathscr{F}$. Since the maps on each side of this identity are continuous on norm bounded sets, as maps of $(M(F), (so))$ into $(M(G), (\sigma))$, we get $\bar{\varphi} = id^{**} \circ A_\rho \circ \Phi$ on all of $M(F)$ from 1.1.3.

COROLLARY 5.1.6. *A norm decreasing homomorphism $\varphi : L^1(F) \to M(G)$ is a monomorphism $\Leftrightarrow$ its extension has the structure $\bar{\varphi} = id^{**} \circ A_\rho \circ \Phi$, as in* 5.1.5, *where the map $\zeta = \pi \circ s$ which induces Φ is an isomorphism of F onto H_0/K.*

Proof. If φ is a monomorphism, so is $\bar{\varphi} \mid F$ (see 5.1.1); now 5.1.5 applies and it is clear that $\zeta = \pi \circ s$ is an isomorphism, as required for $(\Rightarrow)$. Notice that the maps A_ρ and id^{**} are always monomorphisms in (2) and (3) above. Conversely, in (3) we have $\Phi = \pi^{**} \circ \zeta^{**}$, where $\langle \zeta^{**}\mu, \psi \rangle = \langle \mu, \psi \circ \zeta \rangle$ and $\langle \pi^{**}\mu, \psi \rangle = \langle \mu, \pi^*\psi \rangle$ define maps

$$M(F) \xrightarrow{\zeta^{**}} M(H_0/K, \gamma/\pi) \xrightarrow{\pi^{**}} M(H_0, \gamma) .$$

Since $\zeta = \pi \circ s : F \to (H_0/K, \gamma/\pi)$ is a topological isomorphism if ζ is $1:1$, ζ^{**} is a monomorphism. It is easy to verify that $\pi^*(C_0(H_0, \gamma))$ is sup norm dense in $C_0(H_0/K, \gamma/\pi)$; hence π^{**} is always a monomorphism.

In the following paragraphs we digress to study the map defined in (3) and prove the assertions about it which were used to prove 5.1.5. Then in 5.2, we will use these observations to study the structure of special norm decreasing homomorphisms.

THEOREM 5.1.7. *Let F and H be locally compact groups, let $K \subset H$ be a compact normal subgroup, and let $\zeta : F \to H/K$ be an open, continuous epimorphism. Then the map $\Phi : (M(F), (so)) \to (M(H), (so))$, defined such that $\langle \Phi\mu, \psi \rangle = \langle \mu, (\pi^*\psi) \circ \zeta \rangle$, is a norm decreasing homomorphism, continuous on norm bounded sets, if we identify $\pi^*\psi(x) = \int \psi(xt) dm_K(t)$ (constant on cosets of K) with a function in $C_0(H/K)$ for each $\psi \in C_0(H)$.*

Proof. Consider the maps shown in Figure 2,

$$\begin{array}{ccc} (M(F), (so)) & \xrightarrow{\Phi} & (M(H), (so)) \\ \Big\downarrow{\scriptstyle \Phi_1} & & \Big\downarrow{\scriptstyle \pi^{**}} \\ (M(F/F_0), (so)) & \xrightarrow{\Phi_2} & (M(H/K), (so)) \end{array}$$

Figure 2

where $F_0 = \mathrm{Ker}(\zeta)$, $\langle \Phi_1\mu, \psi\rangle = \langle \mu, \psi\circ\pi_0\rangle$ ($\pi_0: F \to F/F_0$ is the canonical homomorphism), $\langle \Phi_2\mu, \psi\rangle = \langle \mu, \psi\circ\zeta\circ\pi_0^{-1}\rangle$, and where $\langle \pi^{**}\mu, \psi\rangle = \langle \mu, \pi^*\psi\rangle$. Clearly Φ_2 is bicontinuous with respect to the topologies in Figure 2. Continuity of Φ follows from the lemmas below, since we can verify by direct computation that $\Phi = \pi^{**}\circ\Phi_2\circ\Phi_1$ on $M(F)$.

LEMMA 5.1.8. *Let Q be a locally compact group with $Q_0 \subset Q$ a closed, normal subgroup, and let $\pi_0: Q \to Q/Q_0$ be the canonical homomorphism. Define $\Phi: (M(Q), (so)) \to (M(Q/Q_0), (so))$ such that $\langle \Phi\mu, \psi\rangle = \langle \mu, \psi\circ\pi_0\rangle$ for $\psi \in C_0(Q/Q_0)$. Then Φ is a norm decreasing homomorphism, continuous on norm bounded sets.*

Proof. It is easy to verify that Φ is a norm decreasing homomorphism. We assert that $\Phi(M(Q)) = M(Q/Q_0)$, and in fact $\Phi(\Sigma_{M(Q)}) = \Sigma_{M(Q/Q_0)}$; from this it will follow that $\Phi(L^1(Q))$ is a two sided ideal in $M(Q/Q_0)$ since $L^1(Q)$ is a two sided ideal in $M(Q)$. If $\Sigma_X = \{\mu : \|\mu\| \leqq 1,\ s(\mu) \subset X\}$ for $X \subset Q$, we will show that $\Phi(\Sigma_K) = \Sigma_{\pi_0 K}$ for all compacta $K \subset Q$; since π_0 is open, this means that every μ with compact support in $M(Q/Q_0)$ is the Φ-image of some $\mu \in M(Q)$ with $\|\mu\| = \|\lambda\|$. Clearly $\Phi(\Sigma_K) \subset \Sigma_{\pi_0 K}$, and K compact $\Rightarrow$ the map

$$\Phi: (\Sigma_K, (\sigma)) \longrightarrow (M(Q/Q_0), (\sigma))$$

is continuous, in fact if $\{\mu_j\} \subset \Sigma_K$ with $\mu_j \xrightarrow{(\sigma)} \mu$ and if $f \in C_0(Q)$ has $f = 1$ on K, then for any $\psi \in C_0(Q/Q_0)$ we get $\langle \Phi\mu_j, \psi\rangle = \langle \mu_j, \psi\circ\pi_0\rangle = \langle \mu_j, f\cdot(\psi\circ\pi_0)\rangle \to \langle \mu, f\cdot(\psi\circ\pi_0)\rangle = \langle \Phi\mu, \psi\rangle$. Now Σ_K is precisely the (σ)-closed convex span of $\{\alpha\delta_x : |\alpha| = 1,\ x \in K\}$, so $\Phi(\Sigma_K)$ is (σ)-compact; since $\Phi(\delta_x) = \delta_{\pi_0 x}$ we have $\Phi(\Sigma_K) \supset co\{\alpha\delta_{\pi_0 x} : |\alpha| = 1,\ x \in K\}$, which gives the converse inclusion.

Now if $\lambda \in M(Q/Q_0)$ there are measures λ_n with compact support such that $\|\lambda_n - \lambda\| \to 0$ and $\|\lambda\| = \|\lambda_1\| + \sum_{j=1}^{\infty} \|\lambda_{n+1} - \lambda_n\|$ (restrict λ to increasingly large compacta). Then there exist $\mu_n \in M(Q)$ with $\Phi(\mu_1) = \lambda_1$, $\|\mu_1\| = \|\lambda_1\|$ and $\Phi(\mu_{n+1}) = (\lambda_{n+1} - \lambda_n)$, $\|\mu_{n+1}\| = \|\lambda_{n+1} - \lambda_n\|$ for $n \geqq 1$; hence $\mu = \sum_{n=1}^{\infty} \mu_n$ converges in $M(Q)$, $\|\mu\| = \|\lambda\|$, and $\Phi(\mu) = \lambda$ as required.

Next we show $\Phi(L^1(Q)) \subset L^1(Q/Q_0)$; in fact, if $\mu = \Phi f$ and $x \in Q/Q_0$, then given $\varepsilon > 0$ and $g \in \pi_0^{-1}(x)$ we can find a compact neighborhood V of g with $\|\delta_h * f - f\| < \varepsilon$ for all $h \in V$. Thus $\|\Phi(\delta_h) * \Phi f - \Phi f\| = \|\delta_y * \mu - \mu\| < \varepsilon$ for $y \in \pi_0 V$. Since π_0 is open and continuous, this means $\mu = \Phi f \in L^1(Q/Q_0)$ (see Rudin [9], p. 230; the abelian hypothesis used there is superfluous). To prove 5.1.8 it is now sufficient to show that $\Phi(L^1(Q))$ is norm dense in $L^1(Q/Q_0)$. To prove density, let $\{e_j\} \subset L^1(Q)$ be a left approximate identity such that $e_j > 0$, $\|e_j\| = 1$, and $s(e_j)$ are compacta which are eventually within any fixed neighborhood of the unit in Q. Then $s(\Phi e_j)$ are compacta shrinking to the unit in Q/Q_0. But $\langle \Phi e_j, \psi \rangle \rightarrow \psi(e)$ for all $\psi \in C_0(Q/Q_0)$, hence $\Phi e_j \xrightarrow{(\sigma)} \delta_e$ and $\| \Phi e_j \| \leqq 1$; these facts together imply that $\lim \{\| \Phi e_j \|\} = 1$. Since $\Phi e_j \geqq 0$ (clear), $\{\Phi e_j\} \subset L^1(Q/Q_0)$ is an approximate identity for $L^1(Q/Q_0)$. Since $\Phi(L^1(Q))$ is an ideal in $M(Q/Q_0)$, norm density of $\Phi(L^1(Q))$ in $L^1(Q/Q_0)$ follows.

LEMMA 5.1.9. *If Q is a locally compact group, $K \subset Q$ a compact normal subgroup with canonical homomorphism $\pi: Q \rightarrow Q/K$, define $\pi^{**}: (M(Q/K), (so)) \rightarrow (M(Q), (so))$ such that $\langle \pi^{**}\mu, \psi \rangle = \langle \mu, \pi^*\psi \rangle$ where $\pi^*\psi(x) = \int \psi(xt)dm_K(t)$ (constant on cosets of K) is regarded as a function in $C_0(Q/K)$. Then π^{**} is a norm decreasing homomorphism, continuous on norm bounded sets.*

Proof. Normality of K in $Q \Rightarrow h * m_K = m_K * h = m_K * h * m_K$ for all $h \in L^1(Q)$. Define $\xi: M(Q) \rightarrow M(Q/K)$ such that $\langle \xi\mu, \psi \rangle = \langle \mu, \psi \circ \pi \rangle$. It is a simple matter to verify that (1) $\pi^{**}\xi(\mu) = \mu * m_K$ for all $\mu \in M(Q)$, and (2) $\xi\pi^{**}(\mu) = \mu$ for all $\mu \in M(Q/K)$. One can also verify by direct computation that $\pi^{**}\mu = (\pi^{**}\mu) * m_K$ for $\mu \in M(Q/K)$. From (1) we see that $\pi^{**}(M(Q/K)) = M(Q) * m_K$, so that $\pi^{**}(L^1(Q/K))$ is closed under right or left multiplication by elements of $M(Q) * m_K$. Finally, $\pi^{**}(L^1(Q/K)) \subset L^1(Q)$; for if $\varepsilon > 0$ and $x \in Q$, and if $f \in L^1(Q/K)$, we can find a neighborhood V of $\pi(x)$ with $\| \delta_u f - f \| < \varepsilon$ whenever $u \in V$. Thus, if W is a neighborhood of x such that $\pi(W) \subset V$, we have $\| \delta_y * (\pi^{**}f) - \pi^{**}f \| \leqq \| \xi(\delta_y) * \xi(\pi^{**}f) - \xi(\pi^{**}f) \| = \| \delta_{\pi y} * f - f \| < \varepsilon$ for all $y \in W$. Thus $\pi^{**}f \in L^1(Q)$ (again see Rudin [9], p. 230). If $\{e_j\}$ is a norm one approximate identity in $L^1(Q/K)$, then $e_j \xrightarrow{(so)} \delta_e$ and it is easy to show that $\pi^{**}e_j \xrightarrow{(\sigma)} m_K = \pi^{**}(\delta_e)$ from 1.1.1. We can arrange that the supports $s(\pi^{**}e_j)$ shrink to $s(m_K) = K$, a compact set; thus we get $\pi^{**}e_j \xrightarrow{(so)} m_K$ by applying 1.1.2. Since $\pi^{**}(L^1(Q/K))$ is closed under right multiplication by elements of $m_K * M(Q) = M(Q) * m_K$, we get (for any $h \in L^1(Q)$) $\| (\pi^{**}e_j) * m_K * h - m_K * m_K * h \| \rightarrow 0$, which $\Rightarrow \pi^{**}(L^1(Q/K))$ is norm dense in $m_K * L^1(Q)$.

Consider $\mu_j \xrightarrow{(so)} \mu$ in $M(Q/K)$ with $\| \mu_j \| \leqq 1$; if $h \in L^1(Q)$ then $(\pi^{**}\mu_j) * h = (\pi^{**}\mu_j) * m_K * h$. But we can approximate $m_K * h$ in norm by some $\pi^{**}f (f \in L^1(Q/K))$ and we know that $(\pi^{**}\mu_j) * (\pi^{**}f) = \pi^{**}(\mu_j * f) \xrightarrow{\text{norm}} (\pi^{**}\mu) * (\pi^{**}f)$.

5.2. Norm decreasing homomorphisms which map $L^1(F)$ to $L^1(G)$. Suppose φ actually maps $L^1(F)$ onto $L^1(G)$, then the structure of φ is exceedingly simple. First recall that if φ is a norm decreasing isomorphism of $L^1(F)$ onto $L^1(G)$ it is actually an isometry; furthermore, an isometric isomorphism has the special structure

$$\langle \varphi f, \psi \rangle = \int_F \rho \circ s(x) \psi \circ s(x) df(x)$$

where $s: F \to G$ is any topological isomorphism and $\rho \in G^\wedge$, as was first proved by Wendel [10], [11]. Although the structure theorem 5.1.4 could be used as the basis for a direct proof of these results, it only gives conditions on the structure of φ which are necessary (but not sufficient) if we are to have $\varphi(L^1(F)) = L^1(G)$. To identify these norm decreasing isomorphisms (or isometries) precisely we would have to retrace some of Wendel's analysis rather than do this we use Wendel's analysis as a starting point.

THEOREM 5.2.1. *Let $\varphi: L^1(F) \to L^1(G)$ be a norm decreasing epimorphism. Then there exists a closed normal subgroup $F_0 \subset F$, an isometric isomorphism $\Lambda: L^1(F/F_0) \to L^1(G)$, and $\beta \in F^\wedge$ with* $\operatorname{Ker} \beta \subset F_0$ *such that $\varphi = \Lambda \circ (\pi^* A_\beta)$, where $A_\beta(\mu) = \beta\mu$, and the canonical homomorphism $\pi: F \to F/F_0$ gives $\langle \pi^*(\mu), \psi \rangle = \langle \mu, \psi \circ \pi \rangle$ for $\psi \in C_0(F/F_0)$.*

Proof. First notice that, if $s(x) = s(\bar{\varphi}(\delta_x))$, then $s: F \to G$ is a continuous homomorphism; in fact, $\bar{\varphi}(\delta_e) = \rho m_K$ for compact subgroup $K \subset G$ and $\rho \in K^\wedge$, and if $h \in L^1(G)$ we can write $h = \varphi f$ for some $f \in L^1(F)$. Thus $h * \rho m_K = \varphi(f) * \bar{\varphi}(\delta_e) = \varphi f = h$, which is impossible for all h unless $K = \{e\}$, so $\bar{\varphi}$ maps $\mathscr{E}_{M(F)}$ into $\mathscr{E}_{M(G)}$. For continuity of s see remarks preceding 5.1.3. Hence $F_0 = \{x \in F: s(x) = e \text{ in } G\}$ is a closed normal subgroup in F. If we define $\beta(x) = \alpha \in S \Leftrightarrow \bar{\varphi}(\delta_x) = \alpha\delta_{s(x)}$, the continuity properties of $\bar{\varphi}$ (see 4.1.1) insure that $\beta \in F^\wedge$; thus $A_\beta: \mu \to \beta\mu$ is an isometric automorphism of $M(F)$. The map $\pi^*: M(F) \to M(F/F_0)$ has been discussed in 5.1.8; we assert that π^* has the following properties (which will be verified at the end of this proof):

(1) $\pi^* L^1(F) = L^1(F/F_0)$, and

(2) $\|\pi^*(\mu)\| = \inf\{\|\mu + n\|: n \in \operatorname{Ker}(\pi^*)\}$,

the quotient norm in $M(F)/\operatorname{Ker}(\pi^*)$.

Clearly $\mu \in \operatorname{Ker}(\pi^* A_\beta) \Leftrightarrow \langle \pi^* A_\beta(\mu), \psi \rangle = \int_F \beta(x)\psi(\pi x) d\mu(x) = 0$ for all $\psi \in C_0(F/F_0)$; it is not hard to show that $\mu \in \operatorname{Ker} \bar{\varphi} \Leftrightarrow \langle \bar{\varphi}\mu, \psi \rangle = \int_F \langle \bar{\varphi}(\delta_x), \psi \rangle d\mu(x) = \int_F \beta(x) \langle \delta_{s(x)}, \psi \rangle d\mu(x) = 0$ for $\psi \in C_0(G)$. The nontrivial first equality here can be seen from 4.2.2, or directly by looking at the action of $\bar{\varphi}$ on finite sums of point masses and using the (*so*) continuity of $\bar{\varphi}$. We assert that $\operatorname{Ker} \bar{\varphi} \supset \operatorname{Ker}(\pi^* A_\beta)$, so

$$\Lambda = \bar{\varphi} \circ (\pi^* A_\beta)^{-1}: M(F/F_0) \longrightarrow M(G)$$

is a well defined homomorphism.

LEMMA 5.2.2. *If* $\{f_j\}$ *is a net of bounded functions in* $C(F)$ *with* $M = \sup\{\|f_j\|_\infty\} < \infty$ *and* $f_j \to f$ *uniformly on compacta, then* $\langle \mu, f_j\rangle \to \langle \mu, f\rangle$ *for all* $\mu \in M(F)$.

Proof. As usual, for bounded $f \in C(F)$ we define $\langle \mu, f\rangle = \langle \mu, \chi_E f\rangle$ where $E \in \boldsymbol{B}(F)$ and $E \supset s(\mu)$. For K compact in F we obviously have $\langle(\mu \mid K), f_j\rangle \to \langle(\mu \mid K), f\rangle$ and for suitably chosen compacta $K_n \subset s(\mu)$ we have $\|\mu - (\mu \mid K_n)\| \to 0$; hence $\langle \mu, f_j\rangle \to \langle \mu, f\rangle$.

Each function $\Psi(x) = \langle \delta_{s(x)}, \psi\rangle$ is continuous, bounded, and constant on cosets of F_0 in F, if $\psi \in C_0(G)$. But any bounded $f \in C(F)$ which is constant on cosets of F_0 can be approximated uniformly on compacta by a uniformly bounded net of functions selected from $\{h \circ \pi : h \in C_0(F/F_0)\}$; in fact, if $K \subset F$ is compact so is πK, and if U is a relatively compact open neighborhood of πK, we can find continuous h such that $h \equiv 1$ on πK, $h \equiv 0$ outside U and $0 \leqq h \leqq 1$. Then $f \cdot (h \circ \pi)$ coincides with $h_1 \circ \pi$ on F, where $h_1(x) = h(x) \cdot f(\pi^{-1}(x)) \in C_0(F/F_0)$; we have $h_1 \circ \pi \equiv f$ on K and $\|h_1 \circ \pi\|_\infty \leqq \|f\|_\infty$ as desired. Taking Ψ as the uniform on compacta limit of uniformly bounded net $\{h_j \circ \pi\}$ we get

$$\langle \bar{\varphi}\mu, \psi\rangle = \langle \beta\mu, \Psi\rangle = \lim\{\langle \beta\mu, h_j \circ \pi\rangle\} = \lim\{\langle \pi^* A_\beta(\mu), h_j\rangle\} = 0$$

if $\mu \in \operatorname{Ker}(\pi^* A_\beta)$, so $\operatorname{Ker}\bar{\varphi} \supset \operatorname{Ker}(\pi^* A_\beta)$.

Now $\|\Lambda\| \leqq 1$ since (2) insures that

$$\begin{aligned} \|\pi^* A_\beta(\mu)\| &= \inf\{\|\mu + n\| : n \in \operatorname{Ker}(\pi^* A_\beta)\} \\ &\geqq \inf\{\|\mu + n\| : n \in \operatorname{Ker}\bar{\varphi}\} \\ &\geqq \inf\{\|\bar{\varphi}\mu + \bar{\varphi}n\| = \|\bar{\varphi}\mu\|\} = \|\bar{\varphi}\mu\| \end{aligned}$$

for $\mu \in M(F)$. Since $\pi^* : (M(F), (so)) \to (M(F/F_0), (so))$ is continuous on norm bounded sets (see 5.1.8), $(\pi^*)^{-1}$ is open on $\Sigma_{M(F/F_0)}$ relative to the (so) topologies; hence $\Lambda : (M(F/F_0), (so)) \to (M(G), (\sigma))$ is continuous on norm bounded sets. From (1) we see that Λ maps $L^1(F/F_0)$ onto $L^1(G)$, so Λ on $M(F/F_0)$ coincides with the extension $\bar{\Lambda}$ from $L^1(F/F_0)$ discussed in 4.1.1. Furthermore,

$$\Lambda(\mathscr{E}_{F/F_0}) = \bar{\varphi} \circ (\pi^* A_\beta)^{-1}(\mathscr{E}_{F/F_0}) = \bar{\varphi}\{\overline{\beta(x)}\delta_x : x \in F\} = \{\delta_{s(x)} : x \in F\},$$

so $\Lambda(\mathscr{E}_{F/F_0}) \cap S\{\delta_e\} = \{\delta_e\}$ in $\mathscr{E}_{M(G)}$ and the analysis of 5.1 applies; i.e. we can write $\Lambda = j^{**} \circ A_\rho \circ \zeta^* : M(F/F_0) \to M(G)$ as in 5.1.5. In our present context some of these maps are trivial since $\Lambda(\delta_x) = \delta_{s(\pi^{-1}(x))}$ for $x \in F/F_0$; indeed, ρ and K are trivial, j is the injection of $H = \{s(x) : x \in F\}$ into G, and $\zeta : F/F_0 \to H$ is given by $\zeta(x) = s(\Lambda(\delta_x)) = s(\pi^{-1}(x))$. But $F_0 = \operatorname{Ker} s = \operatorname{Ker}\pi$ in F, so ζ is an isomorphism of F/F_0 onto H; hence, as indicated in 5.1.6, Λ must be a monomorphism on $M(F/F_0)$. Thus Λ is a norm decreasing isomorphism between $L^1(F/F_0)$ and $L^1(G)$, and Wendel's analysis applies to Λ.

In proving 5.1.8 we showed that $\pi^*(M(F)) = M(F/F_0)$ and that $\pi^*(\Sigma_{M(F)}) = \Sigma_{M(F/F_0)}$. The latter identity proves assertion (2) above. Furthermore, we showed $\pi^*(L^1(F)) \subset L^1(F/F_0)$ is norm dense, and that a right approximate identity $\{e_j\}$ of norm one in $L^1(F)$ is mapped to the same sort of approximate identity $\{\pi^* e_j\}$ in $L^1(F/F_0)$. Let $f \in L^1(F/F_0)$, say with $\|f\| = 1$, and let $\mu \in M(F)$ be chosen with $\|\mu\| = 1$, $\pi^*\mu = f$; then $\pi^*(\mu * e_j) = (\pi^*\mu)*(\pi^* e_j) = f*(\pi^* e_j) \xrightarrow{\text{norm}} f$ and $\mu * e_j \in L^1(F)$ with $\|\mu * e_j\| \leqq \|\mu\| = \|f\|$. Hence we see $\pi^*(\Sigma_{L^1(F)})$ is norm dense in $\Sigma_{L^1(F/F_0)}$. We can find $g_1 \in L^1(F)$ with $\|g_1\| \leqq 1$ and $\|\pi^* g_1 - f\| \leqq 1/2$. Since $\pi^* g_1 - f \in L^1(F/F_0)$, there exists $g_2 \in L^1(F)$ with $\|g_2\| \leqq 1/2$ and $\|\pi^* g_2 - (f - \pi^* g_1)\| < (1/2)^2$. By continuing this selection we get $g_n \in L^1(F)$ with $\|g_n\| \leqq (1/2)^{n-1}$ and $\|\pi^* g_n - (f - \sum_{j=1}^{n-1} \pi^* g_j)\| < (1/2)^n$. Thus $g = \sum_{n=1}^{\infty} g_n$ converges in $L^1(F)$ and $\pi^* g = \sum_{n=1}^{\infty} \pi^* g_n = f$, proving assertion (1) above.

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