PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 4, 1965

FIXED POINTS IN A TRANSFORMATION GROUP

HSIN CHU

In this paper, the following result is proved: "Let (X, T, π) be a transformation group, where X is a Peano continuum with an end point fixed under T. If the group T is one of the following two types: (1) It contains a subgroup R^n such that G/R^n is compact or (2) It contains a s subgroup $Z \cdot R^n$ such that $G/(Z \cdot R^n)$ is compact, where Z is isomorphic to the discrete additive group of all integers, then T has another fixed point."

Professor A. D. Wallace, in [4], proved the following: "Let (X, Z, π) be a transformation group, where $Z=$ the discrete additive group of all integers. If X is a Peano continuum with a fixed end point under Z, then Z has another fixed point." An interesting question, (See [5]) has been raised by Wallace: "Can one reach the same conclusion about either compact groups or abelian groups"? In the case of compact groups, Professor H. C. Wang answered the question in the affirmative (See [6]). We also give an affirmative answer to the question in the case of abelian groups when the abelian group is of the type either $R^n \cdot K$ or $Z \cdot R^n \cdot K$ where R^n is a vector group of dimension n and k is a compact abelian group. Actually, we also cover the case of non-abelian groups. The same conclusion can be reached if the group, G, is one of the following two types:

(1) It contains a subgroup R^n such that G/R^n is compact or

(2) It contains a subgroup $Z \cdot R^n$ such that $G/(Z \cdot R^n)$ is compact.

2. We divide that proof of our main result into several steps.

LEMMA 1. *Let (X, T, π) be a transformation group, where X is an arcwise connected Hausdorff space with an end point e fixed under T. If X has a closed invariant set A under T which does not contain e then T has another fixed point. Let $1(t)$, $0 \leqq t \leqq 1$, be an arc connecting e and some point x in A such that $1(0) = e$ and $1(1) = x$. All the points which separate e and A lie on $1(t)$. Let S be the set of all those points. S is not empty. Introduce a linear ordering in $1(t)$, $0 \leqq t \leqq 1$, by the natural linear ordering of t. Then the upper limit point of S is a fixed point, other than e, under T.*

Proof. The first part of the lemma is an equivalent statement of a theorem, in [6], of Professor H. C. Wang. Under the same assumption

Received June 8, 1964. This work was supported by Contract NAS8-1646, with the George C. Marshall Space Flight Center, NASA, Huntsville, Alabama.

as our lemma, Wang's conclusion is that T has no other fixed point if and only if, given any neighborhood U of e, the orbit UT under T coincides with the whole space X. We notice that if S is a closed invariant set under T which does not contain e, then $U = X - S$ is a neighborhood of e and $UT = U$ which does not coincide with the whole space X and vice versa.

The proof of the second part of this lemma can be obtained from the proof of Wang's theorem. (See [6]).

LEMMA 2. *Let (X, Z, π) be a transformation group. If X is a compact, connected, Hausdorff space which is more than a point and has a fixed end point e, then there is a closed set $H \subset X - e$, which is invariant under Z.*

Proof. This is a theorem by Wallace, See [4].

By Lemma 1 and Lemma 2, we obtain Wallace's result.

LEMMA 3. *Let (X, Z, π) be a transformation group. If X is a Peano continuum with a fixed end point e under Z, then Z has another fixed point.*

LEMMA 4. *Let (X, T, π) be a transformation group. If X is a Peano continuum with a fixed end point e under T and T contains a syndetic subgroup Z (i.e. T contains a integer group Z such that T/Z is a compact set), then T has another fixed point. If, furthermore, T is connected, then the assumption on the given end point being fixed under T is not necessary.*

Proof. Consider the transformation group (X, Z, π) induced by (X, T, π). From Lemma 3, we know that there is another fixed point p under Z. Since Z is syndetic, there is a compact subset K in T such that $T = Z \cdot K$. Consequently, $pT = (pZ)K = pK$ which is compact and therefore, is closed. It is clear that $e \notin pK$. We know pK is closed and invariant under T. By Lemma 1, X has another fixed point q under T.

If T is connected, it is easy to see that every end point is fixed under T (See [5]). Suppose e is an end point and $e \neq et$ for some $t \in T$. Then, because e is an end point and eT is connected, there is $s \in eT$ such that s separates e and et. Consequently, there exists some $t' \in T$ such that $s = et'$. It follows that as t' is a homeomorphism of X, et' is also an end point as well as a cut point. A contradiction!

As a direct consequence of Lemma 4, we have:

LEMMA 5. *Let (X, R, π) be a transformation group. If X is a Peano continuum with an end point, then R has another fixed point.*

LEMMA 6. *Let (X, R^n, π) be a transformation group where n is a positive integer. If X is a Peano continuum with an end point e, then R^n has another fixed point.*

Proof. By Lemma 4, we know that the end point e is fixed under R^n for all n. The proof of this lemma is by induction. Suppose the statement is true for $n = k$. Consider $n = k + 1$. Let $(x_1, \cdots, x_k, x_{k+1})$ be a coordinate system of R^{k+1}. Let A and B be the closed subgroups determined by $x_1 = 0$ and $x_2 = 0$ respectively. Then $A \cong B \cong R^k$. Let the transformation groups (X, A, π) and (X, B, π) both be induced by (X, R^{k+1}, π). By the inductive assumption, we know there are two points p and q such that p is invariant under A and q is invariant under B. Both p and q are distinct from e. Let C_1 be the subgroup of R^{k+1} determined by $x_2 = 0, \cdots, x_{k+1} = 0$. Let C_2 be the subgroup of R^{k+1} determined by $x_1 = 0, x_3 = 0, \cdots, x_{k+1} = 0$. Then $C_1 \cong C_2 \cong R$ and, as direct products $R^{k+1} = C_1 \cdot A = C_2 \cdot B$. Consider the orbit, $(p)R^{k+1}$, of p under R^{k+1} and the orbit, $(q)R^{k+1}$, of q under R^{k+1}. It is clear that $(p)R^{k+1} = (p)C_1$ and $(q)R^{k+1} = (q)C_2$, where $(p)C_1$ and $(q)C_2$ both are connected.

We know both $cl((p)C_1)$ and $cl((q)C_2)$ are invariant under R^{k+1}. If e is not in either $cl((p)C_1)$ or $cl((q)C_2)$, then, by Lemma 1, R^{k+1} has another fixed point. Suppose e is in both $cl((p)C_1)$ and $cl((q)C_2)$. This implies that every neighborhood of e contains points from both $(p)C_1$ and $(q)C_2$.

Let U_e be a neighborhood of e such that $\{p, q\} \cap U_e = \phi$. Since e is a fixed end point, there exists $x \in U_e$ such that $X - x = X_1 \cup X_2$ for some sets X_1 and X_2 with the properties:

$$X_1 \cap cl(X_2) = cl(X_1) \cap X_2 = \phi \quad \text{and } e \in X_1 \subset U_e \,.$$

Consequently, $\{p, q\} \subset X_2$. Notice that X_1 is open in X. It follows that X_1 contains points from both $(p)C_1$ and $(q)C_2$. Since both $(p)C_1$ and $(q)C_2$ are connected, it follows that $x \in (p)C_1 \cap (q)C_2$. Since R^{k+1} is abelian, we have $p = q$ and p is a fixed point under R^{k+1} other than e. Complete the proof by Lemma 5.

LEMMA 7. *Let $(X, Z \cdot R^n, \pi)$ be a transformation group. If X is a Peano continuum with a fixed end point e under $Z \cdot R^n$, then $Z \cdot R^n$ has another fixed point.*

Proof. If $n = 0$, the statement of this lemma is the same as Lemma 3. Let $n > 0$. Let (X, A, π) be a transformation group induced by $(X, Z\cdot R^n, \pi)$ where $A = Z\cdot R^{n-1}$ is a subgroup of $Z\cdot R^n$. Let $B \cong R$ be a subgroup of $Z\cdot R^n$ such that $Z\cdot R^n = A\cdot B$. Prove this lemma by induction on n. Suppose (X, A, π) has a fixed point, p, other than e, under A. Consider the orbit $(p)(Z\cdot R^n)$. It is clear that $(p)(Z\cdot R^n) = (p)B$, which is connected. The orbit-closure $cl((p)(Z\cdot R^n))$ is a connected compact Hausdorff space. Obviously, $cl((p)(Z\cdot R^n))$ is invariant under $Z\cdot R^n$. If e is not in $cl((p)(Z\cdot R^n))$, then, by Lemma 1, $Z\cdot R^n$ has another fixed point. Suppose $e \in cl((p)(Z\cdot R^n))$. Let Z' be an integer group of B. Then e is a fixed end point of the transformation group $(cl((p)(Z\cdot R^n)), Z', \pi)$. By Lemma 2, there is a Z'-invariant closed subset H of $cl((p)(Z\cdot R^n))$ such that $e \notin H$. Consider the transformation group (X, Z', π), induced by $(X, Z\cdot R^n, \pi)$. Choose a point $q \in H$ and connect e and q by an arc $1(t)$, $0 \leqq t \leqq 1$ on which $1(0) = e$ and $1(1) = q$. Let S be the set of all points which separate e and H. By Lemma 1 the upper limit point, r, of S is a fixed point, other than e, under Z'. Since $cl((p)(Z\cdot R^n))$ is connected, we have $S \subset cl((p)(Z\cdot R^n))$. Consequently, $r \in cl((p)(Z\cdot R^n))$. Since the points in $(p)(Z\cdot R^n)$ are fixed under A, the points in $cl((p)(Z\cdot R^n))$ are also fixed under A. It follows that r is fixed under both A and Z'. Let $B = Z'K'$ for some compact set K'. Then $(r)(Z\cdot R^n) = (r)K'$ which is compact. It is obvious $e \notin (r)K'$. By Lemma 1, $(Z\cdot R^n)$ has another fixed point. Complete the proof by induction.

THEOREM. *Let (X, T, π) be a transformation group. If X is a Peano continuum with a fixed end point under T and T is one of the following two types:*

(1) *It contains a subgroup R^n such that G/R^n is compact or*

(2) *It contains a subgroup $Z\cdot R^n$ such that $G/Z\cdot R^n$ is compact.*

Proof. Complete the proof by Lemma 1, Lemma 6, Lemma 7 and a similar method used in the proof of Lemma 4.

COROLLARY 1. *Let (X, T, π) be a transformation group. If X is a Peano continuum with an end point and T is locally compact, connected, abelian group, then T has another fixed point.*

We have the following application in Topological Dynamics. (See [1]). The proof is similar to the one used for the theorem.

COROLLARY 2. *Let (X, T, π) be a transformation group. If X is arcwise connected, Hausdorff with a fixed end point e and a regularly almost periodic point p, other than e, then T has another fixed point.*

Proof. By the definition of regularly almost periodic point, for a closed neighborhood U of p such that $e \notin U$, there exists a syndetic subgroup A of T such $pA \subset U$. It follows that $cl(pA) \subset U$, and thereby, $e \notin cl(pA)$. It is clear that $cl(xA)$ is invariant under A. By Lemma 1, we have another fixed point q under A. Since A is syndetic, there exists a compact set K such that $T = A \cdot K$. From $qT = (qA)K = qK$, we know qT is compact and, therefore, is closed and $e \notin qT$. Since qT is invariant under T, by Lemma 1 we have another fixed point under T. The theorem is proved.

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