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ON THE CONTINUOUS IMAGE OF A SINGULAR CHAIN COMPLEX

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A continuous surjection $\pi: X \to Y$ between topological spaces is called "ductile" if, for each $y \in Y$ and neighborhood U of y there is a neighborhood V of y which contracts to y through U in such a way that this contraction can be covered by a homotopy of $\pi^{-1}(V)$. It is shown, in this note, that if $\pi: X \to Y$ is ductile and Y is paracompact then the inclusion of the image $\pi_* C_*(X)$ of the singular chain complex of X in the singular chain complex $C_*(Y)$ of Y induces an isomorphism in homology. Thus $H_*(Y)$ can be computed from those singular simplices of Y which are images of singular simplices of X.

This result does not hold, in general, when π is not ductile. This question was brought to our attention (for a specific case) by Klingenberg who plans to use our result in a study of geodesics on a Riemannian manifold. We shall now rephrase the condition that a map be ductile in a more convenient language.

Let $\mathscr{M}$ be the category whose objects are surjective maps $\pi: X \to Y$ between topological spaces and whose morphisms are commutative diagrams

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ Y & \longrightarrow & Y' \end{array}$$

of continuous maps (where $\pi, \pi' \in \mathscr{M}$). This contains an analogue of homotopy, that is a commutative diagram

$$\begin{array}{ccc} X \times I & \longrightarrow & X' \\ \downarrow{\scriptstyle\pi \times 1} & & \downarrow{\scriptstyle\pi'} \\ Y \times I & \longrightarrow & Y' \end{array}$$

For $\pi: X \to Y$ and $A \subset Y$ we let π_A denote the restriction $\pi^{-1}(A) \to A$ of π.

We will say that $\pi: X \to Y$ (in $\mathscr{M}$) is *ductile* if, for each point $y \in Y$ and neighborhood U of y, there is a neighborhood V of y with $V \subset U$ such that the inclusion $\pi_V \to \pi_U$ is homotopic (in $\mathscr{M}$) to a map into $\pi_{\{y\}}$. (Thus V contracts, through U, to $\{y\}$ and this contraction is covered by a homotopy of $\pi^{-1}(V)$.)

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Most nice mappings are ductile. The following are all examples of ductile maps:

(a) Simplicial maps.

(b) Let $A \subset X$ both be ANR's (compact metric) and π the map of identifying A to a point.

(c) $Y = Y_1 \cup Y_2$ where Y_1, Y_2 $Y_1 \cap Y_2$ are $ANR's$, $X = Y_1 + Y_2$ (disjoint union) and where π is the natural map.

(d) π is the map of a differentiable manifold X onto its orbit space under some compact Lie group acting differentiably on X (see [1, Chapter VIII, 3.8]). According to Smale this also holds when X is an infinite dimensional manifold.

(e) Let M be a compact Riemannian manifold and X the space of mappings $S^1 \to M$ in the uniform metric. Regarding S^1 as the unit circle in the complex plane, S^1 acts on X by $(zf)(z') = f(zz')$. Let Y be the orbit space of this action. According to Švarc [4], this is ductile. According to Smale it falls under the infinite dimensional case of example (d). It is this example that Klingenberg uses in studying geodesics on M.

THEOREM. *Let π be a ductile map of the space X onto the paracompact space Y. Then the inclusion $\pi_* C_*(X) \subset C_*(Y)$ of chain complexes induces an isomorphism in homology.*

For the proof, it is convenient to introduce some notation. For $\pi : X \to Y$ in $\mathscr{M}$, let $C_*(\pi) = \pi^* C_*(X)$, $H_p(\pi) = H_p(C_*(\pi))$, $C^*(\pi) = \operatorname{Hom}(C_*(\pi), Z)$, and $H^p(\pi) = H^p(C^*(\pi))$. These are functors on $\mathscr{M}$. It is clear that homotopies in $\mathscr{M}$ induce chain homotopies, and therefore that homotopic maps $\pi \to \pi'$ induce identical homomorphisms

$$H_p(\pi) \to H_p(\pi') \quad \text{and} \quad H^p(\pi') \to H^p(\pi) .$$

Note that, as a subcomplex of $C_*(Y)$, $C_*(\pi)$ admits the operation of subdivision, and that standard methods show that this operation induces an isomorphism in homology.

Also note that if π is ductile, then, with $y \in V \subset U$ as in the definition of ductile, the restriction $H^p(\pi_U) \to H^p(\pi_V)$ factors through $H^p(\pi_{\{y\}}) = H^p(y)$ and hence is trivial for $p \neq 0$ and has image Z for $p = 0$. ($H^p(\pi) \to H^p(\pi_{\{y\}})$ is clearly surjective). Thus, when π is ductile, the natural map

$$\varinjlim H^p(\pi_U) \to H^p(\pi_{\{y\}}) = H^p(y) = \begin{cases} Z, & p = 0 \\ 0, & p \neq 0 \end{cases}$$

is an isomorphism, where U ranges over the neighborhoods of y.

Now, for $\pi : X \to Y$ fixed, let S^* be the (differential) presheaf on

Y defined by $S^*(U) = C^*(\pi_U) = \mathrm{Hom}(\pi_* C_*(\pi^{-1}(U)), Z)$. S^* clearly satisfies the axiom $(F2)$ of Godement [2]. Let $\mathscr{S}^*$ be the sheaf generated by S^*. The kernel $S_0^*(Y)$ of the natural map $C^*(\pi) = S^*(Y) \to \mathscr{S}^*(Y)$ consists of those cochains with empty support (that is, which vanish on "small" simplices of $C_*(\pi)$).

LEMMA. $H^*(S_0^*(Y)) = 0$.

Proof. For an open covering $\mathfrak{U}$ of Y let $C_*^{\mathfrak{U}}(\pi)$ be the subcomplex of $C_*(\pi)$ generated by those singular simplices which are contained in some member of $\mathfrak{U}$. A standard argument using subdivision shows that $H_*(C_*^{\mathfrak{U}}(\pi)) \to H_*(C_*(\pi))$ is an isomorphism. If $C_{\mathfrak{U}}^*(\pi) = \mathrm{Hom}(C_*^{\mathfrak{U}}(\pi), Z)$ it follows that $H^*(C^*(\pi)) \to H^*(C_{\mathfrak{U}}^*(\pi))$ is an isomorphism. Thus if $K_{\mathfrak{U}}^* = \ker\{C^*(\pi) \to C_{\mathfrak{U}}^*(\pi)\}$ then $H^*(K_{\mathfrak{U}}^*) = 0$. But clearly $S_0^*(Y) = \bigcup_{\mathfrak{U}} K_{\mathfrak{U}}^* = \varinjlim K_{\mathfrak{U}}^*$. Thus $H^*(S_0^*(Y)) = H^*(\varinjlim K_{\mathfrak{U}}^*) = \varinjlim H^*(K_{\mathfrak{U}}^*) = 0$.

Now suppose that Y is paracompact. Then by [2; 3,9.1, p. 159], the sequence

$$0 \to S_0^*(Y) \to S^*(Y) \to \mathscr{S}^*(Y) \to 0$$

is exact, so that $H^*(\pi) = H^*(S^*(Y)) \approx H^*(\mathscr{S}^*(Y))$.

Since each S^p is an S^0-module, it follows that each $\mathscr{S}^p$ is an $\mathscr{S}^0$-module. $\mathscr{S}^0$ is just the ordinary singular cochain sheaf of Y in degree zero and hence it is flabby. Since Y is paracompact it follows that each $\mathscr{S}^p$ is soft.

Let $\mathscr{H}^*(\mathscr{S}^*)$ be the derived sheaf of $\mathscr{S}^*$. By standard facts, the stalk of this sheaf at $y \in Y$ is $\mathscr{H}^*(\mathscr{S}^*)_y = \varinjlim_{y \in U} H^*(S^*(U)) = \varinjlim_{y \in U} H^*(\pi_U)$ (U ranging over the neighborhoods of y). We have seen that, when π is ductile, this is identified with $H^*(y)$. Thus, when π is ductile, $\mathscr{S}^*$ is a resolution of the constant sheaf Z.

If $\mathscr{C}^*$ is the ordinary singular sheaf of Y, the diagram

$$\begin{array}{ccc} H^*(C^*(Y)) \to & H^*(S^*(Y)) = H^*(C^*(\pi)) \\ \downarrow & \downarrow \\ H^*(\mathscr{C}^*(Y) \to & H^*(\mathscr{S}^*(Y)) \end{array}$$

commutes (note that $\mathscr{C}^* = \mathscr{S}^*$ when π is the identity). If Y is paracompact, the vertical maps are isomorphisms and so is the lower map when π is ductile (see [2, 4.6.2, p. 178]).

We wish to obtain this isomorphism on the homology level. Note that $C_*(Y)/C_*(\pi)$ is a free chain complex (generated by those singular simplices not in the image of π). We wish to show that $H_*(C_*(Y)/C_*(\pi)) = 0$, under the hypotheses of the theorem. We know that the cohomology of this chain complex is trivial. Thus, by the universal coefficient theorem, it suffices to show that, for any abelian group A, $\operatorname{Hom}(A, Z) = 0 = \operatorname{Ext}(A, Z)$ implies that $A = 0$. This is proved in [3, Theorem 8.5] and completes the proof of our theorem.

In conclusion we give an example of a map $\pi ; X \to Y$ which is not ductile even though each point $y \in Y$ has a neighborhood U such that $\pi^{-1}(U)$ can be deformed into $\pi^{-1}(y)$. Indeed the conclusion of the theorem does not hold for this example.

Let Y_1 be the interval $[0, 1]$ on the x-axis of the $x - y$ plane and for $n > 1$ let Y_n be the upper semicircle ($y \geqq 0$) with radius $1/n$ and center at $(1/n, 0)$. Let $Y = \bigcup_{n=1}^{\infty} Y_n$ and let X be the disjoint union of the Y_n with $\pi : X \to Y$ the natural map.

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