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PRIMAL CLUSTERS

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In a series of recent publications [Math. Z, 66 (1957), 452–469; Math Z, 62 (1955), 171–188] Foster introduced and studied the theory of a "primal cluster", —a concept which embraces *classes* of algebras of such diverse nature as the classes of all (i) prime-fields, (ii) "n-fields", (iii) basic Post algebras. Here, a primal cluster is essentially a class $\{U_i\}$ of primal (=strictly functionally complete) algebras of the same species such that every finite subset of $\{U_i\}$ is "independent". The concept of independence is essentially a generalization to universal algebras of the Chinese residue Theorem in number theory. Each cluster, $\tilde{U}$, equationally defines —in terms of the identities jointly satisfied by the various finite subset of $\tilde{U}$-a class of "$\tilde{U}$-algebras", and a structure theory for these $\tilde{U}$-algebras was established by Foster, —a theory which contains well known results for Boolean rings, p-rings, and Post algebras. In order to expand the domain of applications of this theory, one should then look for primal clusters. In this paper a permutation, $\frown$, of ths residue class ring R_n, mod n, is constructed, such that $\{(R_n, \times, \frown)\}$ forms a primal cluster. In Theorem 9, which is the main result of this paper, it is shown that a much more comprehensive (and quite "heterogeneous") class K of algebras nevertheless forms a primal cluster. Indeed, K here is the union of all nonisomorphic algebras in the classes of all (i) residue class rings, (ii) basic Post algebras, and (iii) "n-fields". Thus, the primal cluster K furnishes an extension of the primal clusters which were previously given by Foster (loc. cit.).

In a series of recent publications ([1]–[3]) Foster introduced and studied the theory of a "primal cluster", —a concept which embraces *classes* of algebras of such diverse nature as (i) the class of all prime-fields, (ii) the class of all "n-fields", (iii) the class of all basic Post algebras, and (iv) the union of the primal clusters (ii) and (iii) above. Here, a primal cluster is essentially a class $\{U_i\}$ of universal algebras U_i (all of the same species), each is primal (=strictly functionally complete), and such that every finite subset of $\{U_i\}$ is "independent". The concept of independence is essentially a generalization to universal algebras of the Chinese residue Theorem in number theory. Each cluster, $\tilde{U}$, equationally defines—in terms of the identites jointly satisfied by the various finite subsets of $\tilde{U}$-a class of "$\tilde{U}$-algebras", and a structure theory for these $\tilde{U}$-algebras was established in [1],

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—a theory which contains well known results for Boolean rings, p-rings, and Post algebras. In order to expand the domain of applications of this theory, we should then always look for primal clusters. Thus, our present object, in part, will be to first establish that the class $\{R_n\}$ of all residue class rings, mod n, can be converted to a primal cluster (with respect to suitably chosen operations). Indeed, we shall *construct* a permutation, $\frown$, of R_n such that $(R_n, \times, \frown)$ is a primal algebra (Theorem 7). We will thus obtain a confirmation to an *existence theorem* proved in [2, pp. 70–81], —a theorem which implies the *existence* (but does not give the constructability) of such a permutation, $\frown$. In Theorem 9, which is the main result of this paper, we show that a much more comprehensive (and quite "heterogeneous") class K of algebras nevertheless forms a primal cluster. Indeed, K here is the union of all nonisomorphic algebras in the classes of all (i) residue class rings, (ii) basic Post algebras, and (iii) "n-fields". Thus, the primal cluster K furnishes an extension of the primal clusters which were previously given by Foster [3; p. 179].

1. Fundamental concepts. In this section, we recall the following basic concepts of [1]. Let $S = (n_1, n_2, \cdots)$ be a given finitary species, where the n_1 are positive integers, and let $o_1, o_2, \cdots$ denote the primitive operation symbols of S. Here, o_i is n_i-ary, $o_i = o_i(\zeta_1, \cdots, \zeta_{n_i})$. By an *expression* $\varphi(\zeta, \cdots)$ of species S we mean a primitive composition of one or more indeterminate-symbols $\zeta, \cdots$ via the primitive operations o_i. As usual, we shall use the same symbols o_i to denote the primitive operations of the algebras $U_1, U_2, \cdots$ when these algebras are of species S. We write "$\varphi(\zeta, \cdots)(U)$" to mean that the S-expression φ is interpreted in the S-algebra U. This simply means that the primitive operation symbols are identified with the corresponding primitive operations of U, and the indeterminate-symbols $\zeta, \cdots$ are now viewed as indeterminates over U. Thus for unrelated S-algebras U_1 and U_2, $\varphi(\zeta, \cdots)(U_1)$ will in general be quite unrelated to $\varphi(\zeta, \cdots)(U_2)$. "$\varphi(\zeta, \cdots)(U)$" is also called a *strict U-function*. An identity between the strict U-functions f, g-holding throughout U-is called a *strict U-identity*, and is written as $f(\zeta, \cdots) = g(\zeta, \cdots)(U)$. A finite algebra U with more than one element is called *primal* if every (set-theoretical) mapping of $U \times \cdots \times U$ into U is expressible by a *strict* U-function. Examples of primal algebras are wide spread. Thus, for example, the two-element Boolean algebra, $(F_2, \times, \frown)(\times = \text{intersection}, \frown = \text{complement})$ was shown in [1] to be primal. Other examples of primal algebras are (see [1]):

(i) The prime-field $(F_p, \times, \frown)$, $p = \text{prime}$, and where

$$F_p = \{0, 1, 2, \cdots, p-1\}, \quad \zeta^\frown = \zeta + 1 \pmod{p}.$$

(ii) The basic Post algebra $(P_n, \times, \frown)$, n arbitrary. Here, $P_n = \{0, \rho_{n-2}, \rho_{n-3}, \cdots, p_1, 1\}$, $\zeta \times \eta = \min(\zeta, \eta)$, where "min" refers to the above ordering, and where $0^\frown = 1, 1^\frown = \rho_1, \rho_1^\frown = \rho_2, \cdots, \rho_{n-2}^\frown = 0$.

(iii) The "n-field" $(F_n, \times, \frown)$, n arbitrary. Here,

$$F_n = \{0, 1, \alpha, \alpha^2, \cdots, \alpha^{n-2}\}, \quad (\alpha^{n-1} = 1),$$

and where $0 \times \zeta = \zeta \times 0 = 0 (\zeta \in F_n)$. Furthermore, $0^\frown = 1, 1^\frown = \alpha$, $\alpha^\frown = \alpha^2, \cdots, (\alpha^{n-2})^\frown = 0$. (For further examples, see [1].)

We now procede to define the concept of independence. Let $\{U_i\} = \{U_1, \cdots, U_r\}$ be a finite set of algebras of species S. We say that $\{U_i\}$ satisfies the *Chinese residue condition*, or that $\{U_i\}$ is *independent*, if, corresponding to each set of expressions $\varphi_1, \cdots, \varphi_r$ of species S, there exists a single expression Ψ such that $\Psi = \varphi_i(U_i) (i = 1, \cdots, r)$.

A *primal cluster* of species S is simply a class $\widetilde{U} = \{\cdots, U_i, \cdots\}$ of primal algebras, of species S, any finite subset of which is independent.

It was shown in [1] that numerous classes of algebras (of rather diverse nature), including those in (i), (ii), (iii) above, form primal clusters.

2. Independence.

Let

$$n = p_1^{k_1} \cdots p_t^{k_t}; \quad p_1 > \cdots > p_t,$$

where $p_1, \cdots, p_t$ are primes and $k_1, \cdots, k_t$ are positive integers. We now have the following.

DEFINITION. With $n, p_1, \cdots, p_t, k_1, \cdots, k_t$, as above, the residue class ring R_n is of *type* 1 if $k_1 = 1$, and is of *type* 2 if $k_1 > 1$.

We procede to define a permutation, $\frown$, of R_n. This we do in two stages.

Case 1. If R_n is of *type* 1. In this case, let $\{1, \alpha_1, \cdots, \alpha_\sigma\}$ be the set of all positive integers which are relatively prime to p_1, and which do not exceed n, and choose the notation so that $\alpha_1 \cdots \alpha_\sigma = \alpha_1 (\bmod n)$. Define, $\frown$, by the ordering

$$(2.1) \qquad \frown : = \mathrm{def} = 0, 1, p_1, 2p_1, \cdots, \left(\frac{n}{p_1} - 1\right)p_1, \alpha_\sigma, \alpha_{\sigma-1}, \cdots, \alpha_1;$$
$$\text{i.e., } 0^\frown = 1, 1^\frown = p_1, \cdots, \alpha_1^\frown = 0.$$

REMARK. If n is *prime*, the $[p_1, \cdots, \{(n/p_1) - 1\}p_1]$ is empty. If $n = 2$, then both $[p_1, \cdots, \{(n/p_1) - 1\}p_1]$ and $\{\alpha_\sigma, \cdots, \alpha_1\}$ are empty.

Case 2. If R_n is of *type* 2. Then $n = p_1^{k_1} \cdots p_t^{k_t}$, and where $k_1 \geqq 2$. Let $\varphi(m)$ denote the familiar Euler φ-function (=number of positive integers $\leqq m$ and relatively prime to m). Then

(2.1)′ $\varphi(p_1^{k_1-1}) = p_1^{k_1-1} - p_1^{k_1-2} \geqq p_1^{k_1-2} \geqq 2^{k_1-2} \geqq k_1 - 1$.

Let $\{\mu_1(=1), \mu_2, \cdots, \mu_{k_1-1}\}$ be an arbitrary but fixed set of distinct positive integers, each relatively prime to p_1 and each less than $p_1^{k_1-1}$. This is possible by (2.1)′. Moreover, let $\{1, \eta_2, \cdots, \eta_s\}$ be the set of all positive integers which are relatively prime to p_1 and which are less than n. Now, define, $\frown$, by the following ordering:

(2.2) If $n = 4$, $\frown: = \text{def} = 0, 1, 3, 2$.

(2.3) If $n \neq 4$, $\frown: = \text{def} = 0, 1, \eta_2, \cdots, \eta_s, p_1, \mu_2 p_1, \cdots, \mu_{k_1-1}p_1, \beta_1 p_1, \cdots,$ $\beta_v p_1$ where $\beta_v p_1 = \text{def} = (1 \cdot \eta_2 \cdots \eta_s)\{(p_1)(\mu_2 p_1) \cdots (\mu_{k_1-1}p_1)\} \pmod n$.

Here, the (possibly empty) set $\{\beta_1 p_1, \cdots, \beta_{v-1}p_1\}$ gives the remaining elements of R_n, arranged in an arbitrary but fixed way.

LEMMA 1. *Suppose that R_n is of type 2, $n \neq 4$, and suppose that, $\frown$, satisfies* (2.3). *Then $\beta_v p_1$ is distinct from $0, 1, \eta_2, \cdots, \eta_s$, $p_1, \cdots, \mu_{k_1-1}p_1$, and, $\frown$, is a permutation of R_n.*

Proof. Clearly, $\beta_v p_1 \neq 0$. Furthermore, $\beta_v p_1 \neq 1, \eta_2, \cdots, \eta_s$. If $k_1 > 2$, then $\beta_v p_1 = \lambda p_1^2$, for some λ, and the result follows since all of the η_i and μ_j are relatively prime to p_1. Hence assume $k_1 = 2$ (recall that $k_1 > 1$ since R_n is of type 2.) Now, assume that $n = p_1^{k_1} \cdots p_t^{k_t}$, and $t > 1$. Then $p_2 \in \{\eta_2, \cdots, \eta_s\}$, and hence $\beta_v p_1 = \lambda p_1 p_2$, for some λ. Again, the result follows since $\{p_1, \cdots, \mu_{k_1-1}p_1\} = \{p_1\}$. An easy application of Wilson's Theorem shows that, in this case,

$$1 \cdot \eta_2 \cdots \eta_s \equiv (-1)^{p_1} \pmod{p_1}\,, \quad (s = \varphi(p_1^2) = p_1^2 - p_1)\,,$$

and hence $\beta_v p_1 \equiv (-1)^{p_1} p_1 \pmod{p_1^2} \not\equiv p_1 \pmod{p_1^2}$ unless $p_1 = 2$ (and hence $n = p_1^2 = 4$). Hence, if $n \neq 4$, $\beta_v p_1$ is distinct from p_1, and the lemma is proved.

Following [1], we define a *frame* to be an algebra $(U, \times, \frown; 0, 1)$ of species (2, 1) possessing distinguished elements $0, 1$ $(0 \neq 1)$ such that

$$0 \times \zeta = \zeta \times 0 = 0, 1 \times \zeta = \zeta \times 1 = \zeta \qquad (\zeta \in U)\,,$$

and where $\zeta^\frown$ is a *cyclic* permutation of the elements of U such that $0^\frown = 1$.

Now, let, $\smile$, denote the inverse of, $\frown$, and as in [1], define

$$a \times_\smile b = (a^\frown \times b^\frown)^\smile\,.$$

It is readily verified that $a \times_\smile 0 = 0 \times_\smile a = a$.

We shall now state the following result of [1, p. 456] which is very useful in proving the independence of algebras.

LEMMA 2. *Let $U_1, U_2, \cdots, U_r$ be a set of frames,*

$$U_i = (U_i, \times, \frown; 0_i, 1_i)\ .$$

If there exist $\binom{r}{2}$ expressions $|_{ij}$ such that $|_{ij} = \begin{cases} 1_i(U_i) \\ 0_j(U_j) \end{cases} (1 \leqq i < j \leqq r)$, then the algebras $U_1, \cdots, U_r$ are independent.

Proof. The proof is essentially a combination of the proofs of Lemma 5.1 and Theorem 5.2 of [1]. Indeed, if $|_{ij}$ is as above, then by defining

$$|'_{ij} = \text{def} = [|_{ij}\{((|_{ij})^\frown)^\frown\}]^\frown\ ,$$

one readily verifies that

$$|'_{ij} = \begin{cases} 0_i(U_i) \\ 1_j(U_j) \end{cases}.$$

Hence, there exist expressions $|_{ij}$ such that $|_{ij} = \begin{cases} 1_i(U_i) \\ 0_j(U_j) \end{cases} (i \neq j;\ i, j = 1, \cdots, r)$. Now, for each $i = 1, \cdots, r$, define

$$|_i(\zeta) = \text{def} = |_{i1}(\zeta) \cdot |_{i2}(\zeta) \cdots |_{ir}(\zeta)\ (\text{no } |_{ii} \text{ term})\ .$$

It is easily verified that $|_i(\zeta) = \begin{cases} 1_i(U_i) \\ 0_j(U_j) \end{cases}$, for all $j \neq i$. Now, to prove the independence of $U_1, \cdots, U_r$, let $\varphi_1(\zeta, \cdots), \cdots, \varphi_r(\zeta, \cdots)$ be any set of expressions of species $\times, \frown$, and define

$$\Psi(\zeta, \cdots) = \{\varphi_1(\zeta, \cdots) \times |_1(\zeta)\} \times_\frown \cdots \times_\frown \{\varphi_r(\zeta, \cdots) \times |_r(\zeta)\}\ .$$

Then it is easily seen that $\Psi(\zeta, \cdots) = \varphi_i(\zeta, \cdots)(U_i)$, $i = 1, \cdots, r$, and the lemma is proved.

Suppose $n = p_1^{k_1} \cdots p_t^{k_t}$, $p_1 > \cdots > p_t$, where each p_i is prime and where each k_j is a positive integer. Suppose that

$$r_n = \text{def} = n - \frac{n}{p_1} + k_1\ . \tag{2.4}$$

It is readily verified that $2 \leqq r_n \leqq n$, for all $n \geqq 2$. We now have the following.

LEMMA 3. *Let $n, p_1, \cdots, p_t, k_1, \cdots, k_t$ be as above. Then*

$$n - \frac{n}{p_1} + k_1 > n - \frac{n}{p_i} + k_i$$

for each $i = 2, \cdots, t$.

Proof. Let

$$d = \left(n - \frac{n}{p_1} + k_1\right) - \left(n - \frac{n}{p_i} + k_i\right), \qquad (i \neq 1).$$

Then,

$$d = \frac{n(p_1 - p_i)}{p_1 p_i} + k_1 - k_i \geqq \frac{n}{p_1 p_i} + (k_1 - k_i).$$

If $k_1 - k_i \geqq 0$, then, clearly, we are through. Assume $k_1 < k_i$. Then, $k_i \geqq 2$. Hence,

$$\begin{aligned} d &\geqq \frac{n}{p_1 p_i} - (k_i - k_1) \geqq p_1^{k_1 - 1} p_i^{k_i - 1} - (k_i - 1) \\ &\geqq p_i^{k_i - 1} - (k_i - 1) \geqq 2^{k_i - 1} - (k_i - 1) > 0, \end{aligned}$$

and the lemma is proved.

COROLLARY 4. *Any subset of* $\{n - (n/p_1) + k_1 - 1\}$ *distinct elements taken from* $\{0, 1, 2, \cdots, n-1\}$ *contains at least* k_i *multiples of the prime* $p_i (i = 2, \cdots, t)$.

Proof. This follows readily from Lemma 3.

Again, we shall denote the elements comprising the residue class ring, mod n, by R_n. We now have the following.

THEOREM 5. *Let* $n_1, \cdots, n_t$ *be any set of* t *distinct positive integers, each* $n_i > 1$, *and let,* $^\frown$, *be defined as in* (2.1) *if* R_{n_i} *is of type* 1 *and as in* (2.2), (2.3) *if* R_{n_i} *is of type* $2 (i = 1, \cdots, t)$. *Then the algebras* $(R_{n_1}, \times, ^\frown), \cdots, (R_{n_t}, \times, ^\frown)$ *are independent.*

Proof. In view of Lemma 2, and its proof, we will be through if we can prove the existence of the expressions $|_{ij}$ of Lemma 2. We shall construct these expressions in several stages. To simplify the notation, denote by $(R_n, \times, ^\frown)$ any $(R_{n'}, \times, ^\frown)$ any two *distinct* algebras in the above set, and define

$$(2.5) \qquad E = \zeta \zeta^\frown \zeta^{\frown 2} \cdots \zeta^{\frown nn'-1}, \quad \text{where } \zeta^{\frown k} = (\cdots((\xi)^\frown)^\frown \cdots)^\frown,$$

k-iterations.

Case 1. R_n and $R_{n'}$ are both of type 1. Let r_n and $r_{n'}$ be defined as in (2.4). We now distinguish the following subcases.

Case 1(a). $r_{n'} < r_n$. By (2.4), (2.5), (2.1), and Corollary 4, it is easily seen that

$$\{(E^{\smile}E^{\smile 2}\cdots E^{\smile r_n-2})(E^{\smile n-1})\}^{\frown}=\begin{cases}0(R_n)\\1(R_{n'})\,.\end{cases}$$

Case 1(b). $r_n < r_{n'}$. By symmetry, this is essentially the same as Case 1(a).

Case 1(c). $r_n = r_{n'}$. Since $n \neq n'$, assume, without any loss in generality, that $n' < n$. We distinguish two subcases.

(i) If $n' \nmid n$. It is readily verified that

$$\{(E^{\smile n+1}E^{\smile n+2}\cdots E^{\smile n+r_n-2})(E^{\smile n-1})\}^{\frown}=\begin{cases}0(R_n)\\1(R_{n'})\,.\end{cases}$$

(ii) If $n' \mid n$. Then, one easily verifies that

$$\{(E^{\smile n'+1}E^{\smile n'+2}\cdots E^{\smile n'+r_{n'}-2})(E^{\smile n'-1})\}^{\frown}=\begin{cases}1(R_n)\\0(R_{n'})\,.\end{cases}$$

Case 2. R_n and $R_{n'}$ are both of type 2. The argument here is quite similar to the one given in Case 1. One need only replace in the above proof $(E^{\smile}E^{\smile 2}\cdots E^{\smile r_n-2})(E^{\smile n-1})$ by $E^{\frown}E^{\frown 2}\cdots E^{\frown r_n-1}$ (see (2.4), (2.3), (2.2), and Corollary 4); $(E^{\smile n+1}\cdots E^{\smile n+r_n-2})(E^{\smile n-1})$ by $(E^{\frown n+1}\cdots E^{\frown n+r_n-1})$, etc.

Case 3. R_n and $R_{n'}$ are of opposite types. Assume, without loss of generality, that R_n is of type 1 and $R_{n'}$ is of type 2. We distinguish three subcases.

Case 3(a). n is *prime*. Since $R_{n'}$ is of type 2, therefore, by (2.1), (2.2), (2.3), (2.5), and Corollary 4, $(E^{\smile})^2=\begin{cases}0(R_{n'})\\1(R_n).\end{cases}$

Case 3(b). n *not* prime, $n' = 4$. Then by (2.1), (2.2), and Corollary 4, $(E^{\smile}E^{\frown 2})^{\frown}=\begin{cases}0(R_{n'})\\1(R_n).\end{cases}$

Case 3(c). n *not* prime, $n' \neq 4$. Then by (2.1), (2.3), and Corollary 4,

$$\{(E^{\smile}E^{\smile 2}\cdots E^{\smile n-2})(E^{\smile r_n-2})\}^{\frown}=\begin{cases}0(R_n)\\1(R_n')\,.\end{cases}$$

(Observe that, since, in addition, R_n is of type 1, therefore, using (2.4), $r_n \geqq 4$). The proof of Theorem 5 is now completed upon using Lemma 2.

3. Primal clusters; principal theorem. In this section, we shall prove, among other things, that the residue class frames $(R_n, \times, \frown)$ are primal algebras, where, $\frown$, satisfies (2.1) if R_n is of type 1, and satisfies (2.2)–(2.3) if R_n is of type 2. First, we recall the following result which is an immediate consequence of [2; Theorem 3].

LEMMA 6. *Let* $(U, \times, \frown)$ *be a finite frame, and let* $\Delta(\zeta) = 0$ *if* $\zeta = 0$ *and* $\Delta(\zeta) = 1$ *if* $\zeta \neq 0\ (\zeta \in U)$. *Suppose that* $\Delta(\zeta)$ *is expressible as a strict U-function. Then* $(U, \times, \frown)$ *is primal.*

We now have the following

THEOREM 7. *The residue class algebra* $(R_n, \times, \frown)$, *where,* $\frown$, *is as in* (2.1)–(2.3), *is primal.*

Proof. Case 1. R_n is of type 1. In this case, it is readily verified, by (2.1), (2.4) and Corollary 4, that

$$\Delta(\zeta) = \{(\zeta^{\frown}\zeta^{\frown 2} \cdots \zeta^{\frown r_n - 2})(\zeta^{\frown n-1})\}^{\frown} .$$

The result now follows readily from Lemma 6.

Case 2. R_n is of type 2. In this case, it is easily seen, by (2.2), (2.3), (2.4), and Corollary 4, that

$$\Delta(\zeta) = (\zeta^{\frown}\zeta^{\frown 2}\zeta^{\frown 3} \cdots \zeta^{\frown r_n - 1})^{\frown} .$$

Again, the result follows from Lemma 6, and the theorem is proved.

Now, an easy combination of Theorem 7, Theorem 5, and the definition of primal cluster gives the following

THEOREM 8. *The residue class algebras* $\{R_2, R_3, R_4, \cdots, R_n, \cdots\}$, *where* $R_i = (R_i, \times, \frown)$ *and where,* $\frown$, *is determined by* (2.1)–(2.3), *form a primal cluster.*

It was proved in [3; p. 179] that the class consisting of $F_2, P_3, F_3, P_4, F_4, \cdots$, where F_n and P_n denote the n-field and basic Post algebra, respectively, (see examples (ii), (iii), Section 1) forms a primal cluster. We shall now prove the following

THEOREM 9. PRINCIPAL THEOREM. *The class*

$$\{F_q\}_{q \geqq 4} \cup \{P_m\}_{m \geqq 3} \cup \{R_n\}_{n \geqq 2}$$

is a primal cluster with respect to the above operations.

Proof. First, we recall that the primitive operations of the algebras under consideration have been given in examples (ii), (iii), § 1, and in (2.1)–(2.3). Now, in view of the remarks immediately preceding Theorem 9, together with Theorem 8, Lemma 2, and the definition of a primal cluster, we will be through if we can show that: $(A)(R_n, \times, {}^\frown)$, and $(P_{n'}, \times, {}^\frown)$, if nonisomorphic, are independent, and $(B)(R_n, \times, {}^\frown)$ and $(F_{n'}, \times, {}^\frown)$, if nonisomorphic, are independent.

Proof of (A). First, R_n and $P_{n'}$ are isomorphic if and only if $n = n' = 2$. Assume then that $n \geqq 2$ and $n' \geqq 3$. We now distinguish the following cases.

Case A1. $n = p \neq 2$; p, prime. Then it is easily verified, by (2.5) and Fermat's little Theorem, that

$$(((E^{\smile})^{n-1})^{\frown})^{n-1} = \begin{cases} 1(R_n) \\ 0(P_{n'}) \end{cases} .$$

Case A2. $n = 2$. Then, $n' \neq 2$. Clearly

$$(E^{\smile} E^{\smile 2})^{\frown} = \begin{cases} 1(R_n) \\ 0(P_{n'}) \end{cases} .$$

Case A3. n *not* prime, R_n of type 1. Then

$$(E^{\smile} E^{\frown 2})^{\frown} = \begin{cases} 0(P_{n'}) \\ 1(R_n) \end{cases} \quad (n' \neq 2) ,$$

$$\{(E^{\smile} E^{\smile 2} \cdots E^{\smile r_n - 2})(E^{\smile n-1})\}^{\frown} = \begin{cases} 1(P_n{}') \\ 0(R_n) \end{cases} \quad (n' = 2) .$$

Case A4. n *not* prime, R_n of type 2. Then

$$((E^{\smile})^{n})^{\frown} = \begin{cases} 0(P_{n'}) \\ 1(R_n) \end{cases} .$$

Assertion (A) now readily follows from Lemma 2.

Proof of (B). It is easily seen that $(R_n, \times, {}^\frown)$ and $(F_{n'}, \times, {}^\frown)$ are isomorphic if and only if $n = n' = 2$ or $n = n' = 3$ (see example (iii), § 1, and (2.1)). Assume then that $n' \geqq 4$, $n \geqq 2$. We now distinguish the following cases.

Case B1. $n' = n = p = prime.$ Then $((E^{\smile} E^{\smile 2} \cdots E^{\smile p-2})^{\frown})^{p-1} = \begin{cases} 1(F_{n'}) \\ 0(R_n) \end{cases}$.

Case B2. n prime, $n' \neq n$. Then $(E^{\smile}E^{\smile n})^{n'-1} = \begin{cases} 0(R_n) \\ 1(F_{n'}). \end{cases}$

Case B3. n *not* prime, R_n of type 1. Then $E^{\smile}E^{\smile 2} = \begin{cases} 1(F_{n'}) \\ 0(R_n). \end{cases}$

Case B4. n *not* prime, R_n of type 2. Then $(E^{\smile})^{(n'-1)n} = \begin{cases} 1(F_{n'}) \\ 0(R_n). \end{cases}$
Assertion (B) now readily follows from Lemma 2, and the theorem is proved.

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