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INTEGRAL SOLUTIONS TO THE INCIDENCE EQUATION FOR FINITE PROJECTIVE PLANE CASES OF ORDERS $n \equiv 2 \pmod 4$

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A finite projective plane of order $n \geqq 2$ can be considered as a $\langle v, k, \lambda \rangle$ design where $v = n^2 + n + 1$, $k = n + 1$, and $\lambda = 1$. As such, it can be characterized by its point-line 0, 1 incidence matrix A of order v satisfying the incidence equation

$$(*) \qquad AA^T = nI + J,$$

where J is the matrix of order v consisting entirely of l's. Thus, if a plane of order n exists then (*) has an integral solution A. Ryser has shown that if A is a normal integral solution to (*) or if A is merely an integral solution to (*) where n is odd, then A can be made into an incidence matrix for a plane of order n by suitably multiplying its columns by -1. Such an integral solution to (*) we shall call a type I solution. When A is merely an integral solution to (*) where n is even, then A may be a type I solution but may also be not of this type. These latter integral solutions to (*) we shall call type II solutions. Ryser has constructed type II solutions for $n = 2$ and for all $n \equiv 0 \pmod 4$ for which there exists a Hadamard matrix of order n, and Hall and Ryser have constructed a type II solution for $n = 10$. In this paper we construct type II solutions for some infinite classes of values of $n \equiv 2 \pmod 4$. Basic to these constructions is a special class of $\langle v, k, \lambda \rangle$ designs called skew-Hadamard designs whose incidence matrices form a part of the substructure of our type II solutions. We exhibit examples for $n = 26$ and 50 and also derive examples for $n = 10$ and 18.

A $\langle v, k, \lambda \rangle$ *design* is an arrangement of v elements $x_1, x_2, \cdots, x_v$ into v sets $S_1, S_2, \cdots, S_v$ such that every set contains exactly k elements, every pair of sets has exactly λ elements in common, and to avoid certain degenerate situations, $0 \leqq \lambda < k \leqq v - 1$. A $\langle v, k, \lambda \rangle$ design can be characterized by its *incidence matrix* $A = [a_{ij}]$ by writing the elements $x_1, x_2, \cdots, x_v$ in a row and the sets $S_1, S_2, \cdots, S_v$ in a column and setting $a_{ij} = 1$ if $x_j \in S_i$ and $a_{ij} = 0$ if $x_j \notin S_i$. This matrix A, of order v, consists entirely of 0's and 1's and, by the conditions given above, is easily seen to satisfy the *incidence equation*:

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$$AA^T = (k - \lambda)I + \lambda J \equiv B \, , \tag{1.1}$$

where A^T is the transpose of A, I is the identity matrix of order v, and J is the matrix of order v consisting entirely of 1's. Conversely, if $0 \leqq \lambda < k \leqq v - 1$, a matrix A of order v consisting entirely of 0's and 1's and satisfying equation (1.1) is an incidence matrix for some $\langle v, k, \lambda \rangle$ design. Ryser [13] showed for a $\langle v, k, \lambda \rangle$ design with incidence matrix A that $\lambda(v - 1) = k(k - 1)$ and that A is *normal*, i.e., $A^T A = AA^T = B$, which means that every element is contained in exactly k of the sets and every pair of elements are together in exactly λ of the sets. When $\lambda = 0$ or $k = v - 1$ we have the $\langle v, 1, 0 \rangle$ or $\langle v, v - 1, v - 2 \rangle$ designs, respectively. These designs exist for every integer $v \geqq 2$ and are quite trivial. Two classes of $\langle v, k, \lambda \rangle$ designs will be of particular interest to us here. These are the *finite projective planes* of orders $n \geqq 2$ where $v = n^2 + n + 1$, $k = n + 1$, $\lambda = 1$, and the *Hadamard designs* where $v = 4m - 1$, $k = 2m - 1$, $\lambda = m - 1$, $m \geqq 1$ on integer.

We now let A be an integral solution to the incidence equation. Although an integral solution to the incidence equation is more general than a 0, 1 solution, Ryser [14] has shown that if A is normal or if $gcd(k, \lambda)$ is squarefree and $k - \lambda$ is odd, then by suitable multiplication of the columns of A by -1 we can obtain a 0, 1 incidence matrix for a $\langle v, k, \lambda \rangle$ design. Hence, for odd n the existence of a finite projective plane of order n is equivalent to the existence of an integral solution to the corresponding incidence equation. For even n, however, we do not have this equivalence. When n is even, more exotic integral solutions may and do occur. We may, of course, have design type integral solutions like those for odd n, which we shall call type *I* solutions, or we may have integral solutions which are not of that type, which we shall call type *II* solutions. Ryser [14] showed that a type *II* solution exists for $n = 2$ and for $n \equiv 0 \pmod 4$ whenever n is the order of a Hadamard matrix, and Hall and Ryser [11] exhibit a type *II* solution for $n = 10$. Here we shall construct type *II* solutions for some infinite classes of values of $n \equiv 2 \pmod 4$ which satisfy the Bruck-Ryser criterion [4]. This criterion is equivalent to saying that $n = a^2 + b^2$ where a and b are odd integers. It rules out the existence of integral solutions for all orders $n \equiv 6 \pmod 8$ along with some orders $n \equiv 2 \pmod 8$. Basic to these constructions is a special class of Hadamard designs called skew-Hadamard designs, whose incidence matrices form part of the substructure of our integral solutions.

2. Skew-Hadamard matrices and designs. Let $H = [h_{ij}]$ be a matrix of order n where $h_{ij} = 1, -1$; $j = 1, \cdots, n$. We call H a

Hadamard matrix if $HH^T = nI$. By an inequality of Hadamard [10], H is a Hadamard matrix if and only if $|\det(H)| = n^{n/2}$. We immediately see that a Hadamard matrix is normal. It is easy to show that a Hadamard matrix can only exist when $n = 1, 2$ or $n = 4m$, $m \geqq 1$ an integer, and that a direct product of two Hadamard matrices is a Hadamard matrix, which means that from Hadamard matrices of orders m and n we can construct one of order mn. In [19] J. A. Todd showed that from a Hadamard matrix of order $4m$ we can obtain a related Hadamard design incidence matrix of order $4m - 1$, and conversely, $m \geqq 1$ an integer. Hadamard matrices and their related Hadamard designs have been studied extensively [1], [2], [3], [5], [7], [8], [9], [10], [12], [16], [17], [18], [19], [20], [21]. Hadamard matrices exist for infinitely many orders $4m, m \geqq 1$ an integer, and are conjectured to exist for all such orders. We call a Hadamard matrix H *skew-Hadamard* if $H + H^T = 2I$. These also exist for infinitely many orders, as will be shown later. We also call a Hadamard design and its corresponding incidence matrix A *skew-Hadamard* if $A + A^T = J - I$. This agreement in terminology will be justified by the next theorem. Skew-Hadamard design incidence matrices are a special type of round robin tournament matrix [15]. As such, they occur in the statistical method of paired comparisons [6]. Corresponding to Todd's result for Hadamard matrices and designs, we have the following result for skew-Hadamard matrices and designs.

THEOREM 2.1. *From a skew-Hadamard matrix of order* $4m$ *we can obtain a skew-Hadamard design incidence matrix of order* $4m - 1$, *and conversely,* $m \geqq 1$ *an integer.*

Proof. By multiplying the appropriate rows and the corresponding columns of a skew-Hadamard matrix by -1, we can bring this matrix to the form

$$H = \left(\begin{array}{c|c} 1 & 1 \cdots 1 \\ \hline -1 & \\ \vdots & H_1 \\ -1 & \end{array}\right).$$

Without loss of generality, assume that our original skew-Hadamard matrix is H. Here H_1 consists of 1's and -1's and satisfies

$$H_1H_1^T = 4mI - J$$

and

$$H_1 + H_1^T = 2I .$$

Now let $A = (J - H_1)/2$. Then A consists of 0's and 1's and satisfies

$$\begin{aligned} AA^T &= \frac{1}{4}(J^2 - JH_1^T - H_1J + H_1H_1^T) \\ &= \frac{1}{4}((4m-1)J - J - J + 4mI - J) \\ &= mI + (m-1)J \end{aligned}$$

and

$$\begin{aligned} A + A^T &= J - \frac{1}{2}(H_1 + H_1^T) \\ &= J - I\,. \end{aligned}$$

Hence A is a skew-Hadamard design incidence matrix of order $4m - 1$. By reversing the above argument, we have the converse.

We note that the matrices [1] of order 1 and

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

of order 2 are skew-Hadamard. Among the matrices of order $4m$ with entries 1 and -1, $m \geqq 1$ an integer, we can characterize those that are skew-Hadamard by the following theorem.

THEOREM 2.2. *Let* $H = [h_{ij}]$, $h_{ij} = 1, -1$ *be a matrix of order* $n = 4m$, $m \geqq 1$ *an integer, and let* $G = H + H^T - 2I$. *Then the following statements are equivalent*:

(a) H *is a skew-Hadamard matrix.*

(b) $H^2 - 2H + nI = 0$.

(c) *The eigenvalues of* H *are* $1 + i\sqrt{n-1}$ *and* $1 - i\sqrt{n-1}$, *each with multiplicity* $2m$.

(d) H *is a Hadamard matrix and* $tr(G^2) = 0$.

Proof. We shall show that (a) implies (b) implies (c) implies (d) implies (a). Let H be a skew-Hadamard matrix. Then $HH^T = nI$ and $H + H^T = 2I$ imply(b). Now suppose that (b) holds. Since H cannot satisfy a first degree polynomial, $\lambda^2 - 2\lambda + n$ must be its minimal polynomial, whence only $1 + i\sqrt{n-1}$ and $1 - i\sqrt{n-1}$ are its eigenvalues. Now the trace of H is real; hence these two complex eigenvalues must occur with the same multiplicity, namely, $2m$. Now assume that (c) holds. Then

$$\det(H) = (1 + i\sqrt{n-1})^{2m}(1 - i\sqrt{n-1})^{2m} = n^{n/2}$$

whence H is a Hadamard matrix. Since the eigenvalues of H^2 are $2 - n + 2i\sqrt{n-1}$ and $2 - n - 2i\sqrt{n-1}$, each with multiplicity $2m$, we have, moreover, that

$$\begin{aligned} tr(G^2) &= tr[H^2 + (H^T)^2 + 4I + HH^T + H^TH - 4H - 4H^T] \\ &= 2tr(H^2) + 4\,tr(I) + 2\,tr(nI) - 8\,tr(H) \\ &= 2[2m(4 - 2n)] + 4n + 2n^2 - 8[2m \cdot 2] \\ &= 16m - 8mn + 4n + 2n^2 - 32m \\ &= 0\,, \end{aligned}$$

hence (d) is satisfied. Now suppose (d) holds. Since G is symmetric, $tr(G^2) = 0$ implies that the sum of the squares of the elements of G is 0. Hence $G = 0$ and H is a skew-Hadamard matrix.

We now inquire as to whether there is a direct product type of construction for skew-Hadamard matrices as there is for Hadamard matrices. Such a result can be obtained as a corollary to the following lemma of Williamson [20] in which I_r denotes the identity matrix of order r and $\dot{x}$ denotes the direct product.

LEMMA 2.3. *Let C be a matrix of order n such that $C^T = \varepsilon C$, $\varepsilon = 1, -1$, and $CC^T = (n-1)I_n$, and let D and E be two matrices of order m satisfying $DD^T = EE^T = mI_m$ and $DE^T = -\varepsilon ED^T$. Then the matrix $K = D\dot{x}I_n + E\dot{x}C$ satisfies $KK^T = mnI_{mn}$.*

The result of interest to us here for skew-Hadamard matrices is the following corollary.

COROLLARY 2.4. *Let $C + I$ be a skew-Hadamard matrix of order n, and let D be a skew-Hadamard and E a symmetric Hadamard matrix of order m such that $DE^T = ED^T$. Then the matrix $K = D\dot{x}I_n + E\dot{x}C$ is a skew-Hadamard matrix of order mn.*

Proof. Clearly K consists entirely of 1's and -1's. Since $C + I$ is a skew-Hadamard matrix, $C^T = -C$ and $CC^T = (n-1)I_n$, and since D and E are both Hadamard matrices, $DD^T = EE^T = mI_m$. Now $\varepsilon = -1$ and we have $DE^T = ED^T$. Thus, by Lemma 2.3, we have $KK^T = mnI_{mn}$. Now since D is skew-Hadamard and E is symmetric,

$$\begin{aligned} K + K^T &= D\dot{x}I_n + E\dot{x}C + (D\dot{x}I_n + E\dot{x}C)^T \\ &= D\dot{x}I_n + E\dot{x}C + D^T\dot{x}I_n + E^T\dot{x}C^T \\ &= (D + D^T)\dot{x}I_n + E\dot{x}C - E\dot{x}C \\ &= 2I_m\dot{x}I_n \\ &= 2I_{mn}\,. \end{aligned}$$

Hence K is a skew-Hadamard matrix of order mn.

Williamson [20] obtained special cases of this corollary for $m = 2$ and $m = p^\alpha + 1 \equiv 0 \pmod 4$, p a prime, $\alpha \geqq 1$ an integer, by obtaining the desired pair of matrices of order m. In a different vein, Goldberg [8] constructed a skew-Hadamard design incidence matrix of order $(m-1)^3$ from one of order $m-1$, in effect obtaining a skew-Hadamard matrix of order $(m-1)^3 + 1$ from one of order m. We summarize these results in the following theorem.

THEOREM 2.5. *If there exists a skew-Hadamard matrix of order n then there exists one of order*

(i) $2n$.
(ii) $n(p^\alpha + 1)$; $p^\alpha + 1 \equiv 0 \pmod 4$, *$p$ a prime, $\alpha \geqq 1$ an integer.*
(iii) $(n-1)^3 + 1$.

TABLE 1.
The Existence of Skew-Hadamard Matrices for Orders $4 \leqq n \leqq 200$

n	Form	Exists	n	Form	Exists
4	2^2	SH	104	$103+1$	SH
8	2^3	SH	108	$107+1$	SH
12	$11+1$	SH	112	$2^2(3^3+1)$	SH
16	2^4	SH	116		
20	$19+1$	SH	120	$2(59+1)$	SH
24	$2(11+1)$	SH	124		h
28	3^3+1	SH	128	2^7	SH
32	2^5	SH	132	$131+1$	SH
36		h	136	$2(67+1)$	SH
40	$2(19+1)$	SH	140	$139+1$	SH
44	$43+1$	SH	144	$2(71+1)$	SH
48	$2^2(11+1)$	SH	148		h
52		h	152	$151+1$	SH
56	$2(3^3+1)$	SH	156		h
60	$59+1$	SH	160	$2^3(19+1)$	SH
64	2^6	SH	164	$163+1$	SH
68	$67+1$	SH	168	$2(83+1)$	SH
72	$71+1$	SH	172		h
76		h	176	$2^2(43+1)$	SH
80	$2^2(19+1)$	SH	180	$179+1$	SH
84	$83+1$	SH	184		h
88	$2(43+1)$	SH	188		
92		h	192	$2^4(11+1)$	SH
96	$2^3(11+1)$	SH	196		h
100		h	200	$199+1$	SH

Since there exist skew-Hadamard matrices of orders 2 and $p^\alpha + 1 \equiv 0 \pmod 4$, p a prime, $\alpha \geqq 1$ an integer [12] [20], we can apply Theorem 2.5 to obtain the following existence theorem.

THEOREM 2.6. *There exists a skew-Hadamard matrix of order n where n is of the form*

(i) $2^c \prod_{i=1}^{r} (p_i^{\alpha_i} + 1)$; $c \geqq 0$, $r \geqq 0$ *are integers*, $p_i^{\alpha_i} + 1 \equiv 0 \pmod 4$, p_i *a prime*, $\alpha_i \geqq 1$ *an integer*, $i = 1, \cdots, r$, *where* $\prod_{i=1}^{r} (p_i^{\alpha_i} + 1) = 1$ *for* $r = 0$.

(ii) N, *where N is derivable from* (i) *by Theorem* 2.5.

Table 1 gives the existence of skew-Hadamard matrices for orders $4 \leqq n \leqq 200$ according to Theorem 2.6. For comparison, this table also gives the currently known existence of Hadamard matrices for the same range of n, based on constructions in the references mentioned earlier. The symbols SH indicate that a skew-Hadamard matrix exists, while the symbol h indicates that only non-skew-Hadamard matrices are known to exist.

3. Constructions. By § 4 of [11] we know that we can put any type *II* solution $A = [a_{ij}]$ of order $v = n^2 + n + 1$ for the finite projective plane case of order n into a form where $a_{11} = 0$, $a_{i1} = 1$ for $2 \leqq i \leqq v$, $a_{1j} = 1$ for $j \equiv 2 \pmod n$ and $a_{1j} = 0$ for $j \not\equiv 2 \pmod n$ where $2 \leqq j \leqq v$, and where the remaining entries form a submatrix C of order $v - 1 = n(n+1)$ which has n 1's and n^2 0's in each of the $n + 1$ columns under a 1 in row 1 of A and which satisfies the matrix equation $CC^T = C^TC = nI$. The constructions given in [11] and [14] have C in the form $C = A_n \dotplus A_n \dotplus \cdots \dotplus A_n$, where this direct sum contains A_n, of order n, $n + 1$ times and where A_n has all entries in column 1 equal to 1 and satisfies the matrix equation $A_nA_n^T = nI$. These conditions on A_n are sufficient for the construction of a type *II* solution for order n. We shall confine ourselves here to this form of type *II* solution. This restriction reduces the construction of a type *II* solution A of order $n^2 + n + 1$ to that of an integral matrix A_n of order n satisfying the above conditions. Type *II* solutions need not, however, be of this direct sum form to within permutations of rows and columns of A. This can be seen from the following example for $n = 4$. Here the entries in the blank parts of A are 0's.

$$A=\left(\begin{array}{c|cccc|cccc|cccc|cccc|cccc}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline
1 & 1 & 1 & 1 & -1 & & & & & & & & & & & & & & & & \\
1 & 1 & 1 & -1 & 1 & & & & & & & & & & & & & & & & \\
1 & 1 & -1 & & & & & 1 & -1 & & & & & & & & & & & & \\
1 & 1 & -1 & & & & & -1 & 1 & & & & & & & & & & & & \\
1 & & & 1 & 1 & 1 & 1 & & & & & & & & & & & & & & \\
1 & & & -1 & -1 & 1 & 1 & & & & & & & & & & & & & & \\
1 & & & & & 1 & -1 & & & & & 1 & -1 & & & & & & & & \\
1 & & & & & 1 & -1 & & & & & -1 & 1 & & & & & & & & \\
1 & & & & & & & 1 & 1 & 1 & 1 & & & & & & & & & & \\
1 & & & & & & & -1 & -1 & 1 & 1 & & & & & & & & & & \\
1 & & & & & & & & & 1 & -1 & & & & & 1 & -1 & & & & \\
1 & & & & & & & & & 1 & -1 & & & & & -1 & 1 & & & & \\
1 & & & & & & & & & & & 1 & 1 & 1 & 1 & & & & & & \\
1 & & & & & & & & & & & -1 & -1 & 1 & 1 & & & & & & \\
1 & & & & & & & & & & & & & 1 & -1 & & & & & 1 & -1 \\
1 & & & & & & & & & & & & & 1 & -1 & & & & & -1 & 1 \\
1 & & & & & & & & & & & & & & & 1 & 1 & 1 & 1 & & \\
1 & & & & & & & & & & & & & & & -1 & -1 & 1 & 1 & & \\
1 & & & & & & & & & & & & & & & & & 1 & -1 & 1 & 1 \\
1 & & & & & & & & & & & & & & & & & 1 & -1 & -1 & -1
\end{array}\right)$$

Let K be a skew-Hadamard design incidence matrix of order $q \equiv 3 \pmod 4$. Here $v = q = 4m - 1$, $k = 2m - 1$, $\lambda = m - 1$, where $m \geqq 1$ is an integer,

$$KK^T = K^TK = mI + (m-1)J\,, \tag{3.1}$$

and

$$K + K^T = J - I\,. \tag{3.2}$$

We obtain from K a matrix $K(t, u, x)$ by substituting t for each of the main diagonal 0's, u for each of the remaining 0's and x for each of the 1's. From (3.1) and (3.2), any two rows of $K(t, u, x)$ can be schematically represented as

$$\begin{array}{l}
t,\, u,\, u,\, \cdots,\, u,\, u,\, \cdots,\, u,\, x,\, \cdots,\, x,\, x,\, \cdots,\, x \\
x,\, t,\, \underbrace{x,\, \cdots,\, x}_{m-1},\, \underbrace{u,\, \cdots,\, u}_{m-1},\, \underbrace{x,\, \cdots,\, x}_{m-1},\, \underbrace{u,\, \cdots,\, u}_{m}
\end{array}$$

where there are $4m-1$ entries in each row, $2m-1$ each of u's and x's. The inner product of a row of $K(t, u, x)$ with itself is thus

$$t^2 + (2m-1)(x^2+u^2) = t^2 + \frac{1}{2}(q-1)(x^2+u^2) . \tag{3.3}$$

Also, the inner product of two distinct rows of $K(t, u, x)$ is

$$\begin{aligned} &t(x+u) + (m-1)(x^2+u^2) + (2m-1)xu \\ &\quad = t(x+u) + \frac{1}{4}(q-1)(x+u)^2 - \frac{1}{2}(x^2+u^2) . \end{aligned} \tag{3.4}$$

We now form $Y=[y_{ij}]=K(t_1, u_1, x_1)$ and $Z=[z_{ij}]=K(t_2, u_2, x_2)$ of order q and then form

$$N = \begin{bmatrix} Y & Z \\ -Z^T & Y^T \end{bmatrix} . \tag{3.5}$$

We then set

$$w \equiv t_1^2 + t_2^2 + \frac{1}{2}(q-1)(x_1^2 + u_1^2 + x_2^2 + u_2^2) . \tag{3.6}$$

LEMMA 3.1. *The matrix equation*

$$NN^T = wI \tag{3.7}$$

is satisfied if and only if

$$w = \left[t_1 + \frac{1}{2}(q-1)(x_1+u_1)\right]^2 + \left[t_2 + \frac{1}{2}(q-1)(x_2+u_2)\right]^2 . \tag{3.8}$$

Proof. By (3.5) we have

$$NN^T = \begin{bmatrix} YY^T + ZZ^T, & ZY - YZ \\ (ZY-YZ)^T, & Y^TY + Z^TZ \end{bmatrix} . \tag{3.9}$$

Since, by (3.1), K is a normal matrix, the statements about inner product values of $K(t, u, x)$ are true when the word row(s) is replaced by column(s); hence $K(t, u, x)$ is normal whence Y and Z are normal or

$$Y^TY = YY^T \quad \text{and} \quad Z^TZ = ZZ^T . \tag{3.10}$$

Now

$$\begin{aligned} Y &= t_1 I + x_1 K + u_1(J-K) - u_1 I \\ &= (t_1 - u_1)I + (x_1 - u_1)K + u_1 J , \end{aligned}$$

and similarly

$$Z = (t_2 - u_2)I + (x_2 - u_2)K + u_2 J .$$

Since I commutes with both K and J and

$$KJ = JK = (2m - 1)J ,$$

i.e., K commutes with J, Y commutes with Z so that

$$ZY - YZ = 0 . \tag{3.11}$$

Then by (3.10) and (3.11), (3.9) becomes

$$NN^T = (YY^T + ZZ^T) \dotplus (YY^T + ZZ^T) . \tag{3.12}$$

The diagonal entries of NN^T are, by (3.3) and (3.12),

$$t_1^2 + t_2^2 + \frac{1}{2}(q - 1)(x_1^2 + u_1^2 + x_2^2 + u_2^2) = w , \tag{3.13}$$

and the nondiagonal entries of the direct summands in (3.12) are, by (3.4),

$$t_1(x_1 + u_1) + t_2(x_2 + u_2) + \frac{1}{4}(q - 1)[(x_1 + u_1)^2 + (x_2 + u_2)^2] - \frac{1}{2}(x_1^2 + u_1^2 + x_2^2 + u_2^2) = y . \tag{3.14}$$

We note that (3.7) is satisfied if and only if $y = 0$. Now solving (3.14) for $(x_1^2 + u_1^2 + x_2^2 + u_2^2)/2$ and substituting the result into (3.13) we obtain

$$\left[t_1 + \frac{1}{2}(q - 1)(x_1 + u_1)\right]^2 + \left[t_2 + \frac{1}{2}(q - 1)(x_2 + u_2)\right]^2 - (q - 1)y = w . \tag{3.15}$$

Hence by (3.13), (3.14), and (3.15), we see that (3.7) is true if and only if (3.8) is.

We now define the matrices $E_r = (r + 2)I/2 - J$ of even order r, F_r of size $r \times 2$ consisting entirely of 1's, and G_r of size $r \times 2$ whose first column consists entirely of 1's and whose second column consists entirely of -1's. In the constructions which follow we shall be taking $t_1 = (r + 2)/2$ and $x_1 + u_1 = 2$. We then note that

$$F_r F_r^T + E_r E_r^T = G_r G_r^T + E_r E_r^T = \left[\frac{1}{2}(r + 2)\right]^2 I = t_1^2 I , \tag{3.16}$$

$$F_r F_r^T + 2E_r = G_r G_r^T + 2E_r = (r + 2)I = (x_1 + u_1)t_1 I , \tag{3.17}$$

and

$$F_r G_r^T = G_r F_r^T = 0 . \tag{3.18}$$

We substitute for the entries y_{ii} in Y and Y^T the matrix E_r and for all other entries y_{ij}, $i \neq j$, the matrix $y_{ij}I$ of order r to obtain the matrices Y_* and Y_*^T, respectively, of order rq, and substitute for the entries z_{ij} in Z and Z^T the matrix $z_{ij}\ I$ of order r to obtain the matrices Z_* and Z_*^T, respectively, also of order rq. These matrices will appear in the constructions which follow, bordered by the matrices F_{rq} and G_{rq}.

We can now obtain two existence theorems for type *II* solutions to the incidence equation for finite projective plane cases of orders $n \equiv 2 \pmod 4$. After each one are theorems which cover the various cases of the theorem.

THEOREM 3.2. *Let* (3.8) *be satisfied in integers* $t_1, t_2, u_1, u_2, x_1,$ *and* x_2 *where* $q \equiv 3 \pmod 4$ *is the order of a skew-Hadamard design incidence matrix and* w *is defined in* (3.6), *and where* $x_1 + u_1 = 2$ *and* $t_1 = (r+2)/2$ *and* $w = 2rq + 2$ *for the positive even integer* r. *Then we can construct a type II solution to the incidence equation for the finite projective plane case of order* $n = 2rq + 2$.

Proof. We have

$$N = \begin{bmatrix} Y & Z \\ -Z^T & Y^T \end{bmatrix}; \quad Y = [y_{ij}] , \quad Z = [z_{ij}] ,$$

where

$$y_{ii} = t_1 = \frac{1}{2}(r+2) , \tag{3.19}$$

$$y_{ij} + y_{ji} = x_1 + u_1 = 2 ; \quad 1 \leqq i \leqq q , \quad 1 \leqq j \leqq q , \quad i \neq j ,$$

and

$$NN^T = (2rq + 2)I . \tag{3.20}$$

Since (3.8) is satisfied we have

$$\left[\frac{1}{2}(r+2) + (q-1)\right]^2 + \left[t_2 + \frac{1}{2}(q-1)(x_2+u_2)\right]^2 = 2rq + 2 , \tag{3.21}$$

or

$$\left[q - \frac{1}{2}r\right]^2 + \left[t_2 + \frac{1}{2}(q-1)(x_2+u_2)\right]^2 = 2 .$$

Since q, $r/2$, t_2, $(q-1)/2$, x_2, and u_2 are integers this means that

$$(3.22)\quad q - \frac{1}{2}r = \varepsilon_1,\quad t_2 + \frac{1}{2}(q-1)(x_2+u_2) = \varepsilon_2;\ \varepsilon_1,\ \varepsilon_2 = 1,\ -1\,.$$

We form two matrices U and V of size $2 \times rq$ according to the values of ε_1 and ε_2 as follows:

$$(3.23)\quad \begin{array}{ccl} U & V & \\ \begin{bmatrix} -1 \cdots -1 \\ 1 \cdots \quad 1 \end{bmatrix} & \begin{bmatrix} -1 \cdots -1 \\ -1 \cdots -1 \end{bmatrix} & \text{if } \varepsilon_1 = \varepsilon_2 = 1\,. \\ \begin{bmatrix} 1 \cdots \quad 1 \\ -1 \cdots -1 \end{bmatrix} & \begin{bmatrix} 1 \cdots \quad 1 \\ 1 \cdots \quad 1 \end{bmatrix} & \text{if } \varepsilon_1 = \varepsilon_2 = -1\,. \\ \begin{bmatrix} -1 \cdots -1 \\ -1 \cdots -1 \end{bmatrix} & \begin{bmatrix} 1 \cdots \quad 1 \\ -1 \cdots -1 \end{bmatrix} & \text{if } \varepsilon_1 = -\varepsilon_2 = 1\,. \\ \begin{bmatrix} 1 \cdots \quad 1 \\ 1 \cdots \quad 1 \end{bmatrix} & \begin{bmatrix} -1 \cdots -1 \\ 1 \cdots \quad 1 \end{bmatrix} & \text{if } \varepsilon_1 = -\varepsilon_2 = -1\,. \end{array}$$

Finally, we construct A_n of order $n = 2rq + 2$:

$$(3.24)\qquad A_n = \left(\begin{array}{cc|cc} 1 & 1 & U & V \\ 1 & -1 & & \\ \hline \multicolumn{2}{c|}{F_{rq}} & Y_* & Z_* \\ \multicolumn{2}{c|}{G_{rq}} & -Z_*^T & Y_*^T \end{array}\right).$$

By (3.23) the first two rows of A_n are orthogonal and have self inner products equal to $2rq + 2 = n$. Since the row and column sums of Y_* are $q - r/2$ and those of Z_* are $t_2 + (q-1)(x_2+u_2)/2$, we have by (3.22) and (3.23) that rows one and two are orthogonal to all the other rows of A_n. We now look upon the submatrix of A_n below row 2 and to the right of F_{rq} and G_{rq} as a matrix with the matrix entries E_r, u_1I, x_1I, t_2I, u_2I, and x_2I, all of order r. These matrices naturally divide the entire submatrix of A_n below 2 into r-row blocks. Since these matrices commute with one another they behave multiplicatively among themselves as scalars. Thus (3.16), (3.19) and (3.20) imply that the inner product of an r-row block with itself is $(2rq+2)I = nI$ of order r, (3.17), (3.19) and (3.20) imply that any two r-row blocks intersecting either F_{rq} or G_{rq} are orthogonal, and (3.18) and (3.20) imply that any r-row block intersecting F_{rq} is orthogonal to any r-row block intersecting G_{rq}. Hence $A_nA_n^T = nI$, and since the first column of A_n consists entirely of 1's we see that we have a type *II* solution to the incidence equation for the finite projective plane case of order $n = 2rq + 2$.

Letting $c = x_2 + u_2$ and combining (3.22) with (3.6), noting that

$t_1 = (r + 2)/2 = q - \varepsilon_1 + 1$, we have

$$(3.25)\quad [q - \varepsilon_1 + 1]^2 + \left[\varepsilon_2 - \frac{1}{2}(q-1)c\right]^2 + \frac{1}{2}(q-1)[x_1^2 + (2 - x_1)^2 + x_2^2 + (c - x_2)^2] = 2q \cdot 2(q - \varepsilon_1) + 2 ,$$

or

$$- \varepsilon_2 c(q-1) + \frac{1}{4}c^2(q-1)^2 + \frac{1}{2}(q-1)\left[2(x_1-1)^2 + 2\left(x_2 - \frac{1}{2}c\right)^2 + \frac{1}{2}c^2 + 2\right] = 3q^2 - 2\varepsilon_1 q + 2\varepsilon_1 - 2q - 1 = [3q - (2\varepsilon_1 - 1)](q-1) ,$$

or

$$- \varepsilon_2 c + \frac{1}{2}c^2(q-1) + (x_1 - 1)^2 + \left(x_2 - \frac{1}{2}c\right)^2 + \frac{1}{4}c^2 + 1 = 3q - 2\varepsilon_1 + 1 ,$$

whence

$$(3.26)\qquad (12 - c^2)q + 4\varepsilon_2 c - 8\varepsilon_1 = (2x_1 - 2)^2 + (2x_2 - c)^2 .$$

By (3.26)

$$(12 - c^2)q + 4\varepsilon_2 c - 8\varepsilon_1 \geqq 0 ,$$

and since $q \geqq 3$,

$$(3.27)\qquad c^2 - \frac{4\varepsilon_2}{q}c + \frac{4}{q^2} \leqq 12 - \frac{8\varepsilon_1}{q} + \frac{4}{q^2} \leqq \frac{136}{9} .$$

Since c is an integer we can readily conclude that

$$(3.28)\qquad |c| \leqq 4 .$$

We let $a = 2x_1 - 2$ and $b = 2x_2 - c$. Since $q = 4m - 1$, where $m > 0$ is an integer, we have from (3.26) that

$$(3.29)\qquad (12 - c^2)(4m - 1) + 4\varepsilon_2 c - 8\varepsilon_1 = a^2 + b^2 .$$

Now suppose for given values of $\varepsilon_1 = 1, -1$, $\varepsilon_2 = 1, -1$, and c that (3.29) has a solution in integers a and b. If c is even the left side of (3.29) is divisible by 4 whence a and b must both be even, while if c is odd the left side of (3.29) is odd whence one of these integers, say a, is even while the other, b, is odd. So in either case we can solve the equations $a = 2x_1 - 2$ and $b = 2x_2 - c$ for integral values of x_1 and x_2. Thus we have a solution to (3.26) in integers x_1, x_2, and c. These values then determine the values $u_1 = 2 - x_1$ and $u_2 = c - x_2$. Then taking $t_1 = q - \varepsilon_1 + 1$, $t_2 = \varepsilon_2 - (q-1)c/2$, and $r = 2(q - \varepsilon_1)$ and noting that (3.25) is equivalent to (3.26) we have by (3.25) that

$$t_1^2 + t_2^2 + (q-1)[x_1^2 + u_1^2 + x_2^2 + u_2^2]/2 = 2rq + 2 = w\ .$$

Then since (3.21) is equivalent to (3.22) and (3.22) holds we have by (3.21) that

$$[t_1 + (q-1)(x_1 + u_1)/2]^2 + [t_2 + (q-1)(x_2 + u_2)/2]^2 = 2rq + 2 = w$$

where $t_1 = (r+2)/2$. So if $q = 4m - 1$ is the order of a skew-Hadamard design incidence matrix, the conditions of Theorem 3.2 are satisfied and we can construct a type *II* solution according to this theorem. Now in deciding whether or not (3.29) has a solution in integers a and b we have, by (3.28), nine values of $\varepsilon_2 c$ to consider for each of the values $\varepsilon_1 = 1, -1$. We take the nine cases for $\varepsilon_1 = 1$.

Case 1. $\varepsilon_2 c = 4$: $-16m + 12 = a^2 + b^2$, impossible since $-16m + 12 < 0$ for $m > 0$.

Case 2. $\varepsilon_2 c = 3$: $12m + 1 = a^2 + b^2$, possible since, e.g., $12(1) + 1 = 13 = 3^2 + 2^2$. Here $3q + 4 = a^2 + b^2$.

Case 3. $\varepsilon_2 c = 2$: $8(4m - 1) = a^2 + b^2$ or $4m - 1 = a_1^2 + b_1^2$, a_1, b_1 integers, impossible since $4m - 1 \equiv 3 \pmod 4$.

Case 4. $\varepsilon_2 c = 1$: $44m - 15 = a^2 + b^2$, possible since, e.g., $44(1) - 15 = 29 = 5^2 + 2^2$. Here $11q - 4 = a^2 + b^2$.

Case 5. $\varepsilon_2 c = 0$: $48m - 20 = a^2 + b^2$ or $12m - 5 = a_1^2 + b_1^2$, a_1, b_1 integers, impossible since $12m - 5 \equiv 3 \pmod 4$.

Case 6. $\varepsilon_2 c = -1$: $44m - 23 = a^2 + b^2$, possible since, e.g., $44(2) - 23 = 65 = 8^2 + 1^2$. Here $11q - 12 = a^2 + b^2$.

Case 7. $\varepsilon_2 c = -2$: $32m - 24 = a^2 + b^2$ or $4m - 3 = a_1^2 + b_1^2$, a_1, b_1 integers, possible since, e.g., $4(2) - 3 = 5 = 2^2 + 1^2$. Here $8q - 16 = a^2 + b^2$ or $q - 2 = a_1^2 + b_1^2$.

Case 8. $\varepsilon_2 c = -3$: $12m - 23 = a^2 + b^2$, possible since, e.g., $12(3) - 23 = 13 = 3^2 + 2^2$. Here $3q - 20 = a^2 + b^2$.

Case 9. $\varepsilon_2 c = -4$: $-16m - 20 = a^2 + b^2$, impossible since $-16m - 20 < 0$ for $m > 0$.

Now when $\varepsilon_1 = 1$ we have $r = 2(q-1)$, hence $n = 4q^2 - 4q + 2 = (2q-1)^2 + 1$. So by Theorem 3.2 we have the following result.

THEOREM 3.3. *There exists a type II solution to the incidence equation for the finite projective plane case of order* $n = (2q-1)^2 + 1$

whenever q is the order of a skew-Hadamard design incidence matrix and any of the following expressions is the sum of two integral squares: $3q + 4$, $11q - 4$, $11q - 12$, $q - 2$, $3q - 20$.

When $\varepsilon_1 = -1$ we have $r = 2(q + 1)$ hence $n = 4q^2 + 4q + 2 = (2q + 1)^2 + 1$. Analyzing this case as was done above for $\varepsilon_1 = 1$, we have by Theorem 3.2 the corresponding result:

THEOREM 3.4. *There exists a type II solution to the incidence equation for the finite projective plane case of order $n = (2q + 1)^2 + 1$ whenever q is the order of a skew-Hadamard design incidence matrix and any of the following expressions is the sum of two integral squares*: $3q - 4$, $11q + 4$, $11q + 12$, $q + 2$, $3q + 20$.

Both of these theorems yield infinitely many type *II* solutions. There exist skew-Hadamard design incidence matrices of orders

$$q_1 = 2^{2d-2}(11 + 1) - 1 = 3 \cdot 2^{2d} - 1$$

and

$$q_2 = 2^{2d-2}(43 + 1) - 1 = 11 \cdot 2^{2d} - 1$$

for each integer $d \geqq 1$. Then $3q_1 + 4 = (3 \cdot 2^d)^2 + 1^2$, and $11q_2 + 12 = (11 \cdot 2^d)^2 + 1^2$. The first five orders for which each of these theorems yields a type *II* solution correspond to $q = 3, 7, 11, 15$, and 19 and are $n = 26$, 170, 442, 842, and 1370, respectively, by Theorem 3.3, and $n = 50$, 226, 530, 962, and 1522, respectively, by Theorem 3.4. As an example we construct A_{26}. For $n = 26$ we have $q = 3$ and $\varepsilon_1 = 1$ hence $r = 4$ whence $t_1 = 3$. Now by case 2 above, $\varepsilon_2 c = 3$ and

$$3q + 4 = 13 = 2^2 + 3^2 = (2x_1 - 2)^2 + (2x_2 - c)^2 .$$

We take $2x_1 - 2 = 2$ or $x_1 = 2$ and $2x_2 - c = 3$. Letting $\varepsilon_2 = 1$, we have $c = 3$ whence $x_2 = 3$ and $t_2 = -2$. Then $u_1 = u_2 = 0$. Now $E_4 = 3I - J$ of order 4 and since $\varepsilon_1 = \varepsilon_2 = 1$,

$$U = \begin{bmatrix} -1 \cdots -1 \\ 1 \cdots 1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} -1 \cdots -1 \\ -1 \cdots -1 \end{bmatrix},$$

of size 2×12. The matrices F_4 and G_4 are of size 4×2 and a skew-Hadamard design incidence matrix of order 3 is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Hence we have

$$A_{26} = \left(\begin{array}{cc|ccc|ccc} 1 & 1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\ 1 & -1 & 1 & \cdots & 1 & -1 & \cdots & -1 \\ \hline 1 & 1 & 3I-J & 2I & 0 & -2I & 3I & 0 \\ \vdots & & 0 & 3I-J & 2I & 0 & -2I & 3I \\ 1 & 1 & 2I & 0 & 3I-J & 3I & 0 & -2I \\ \hline 1 & -1 & 2I & 0 & -3I & 3I-J & 0 & 2I \\ \vdots & & -3I & 2I & 0 & 2I & 3I-J & 0 \\ 1 & -1 & 0 & -3I & 2I & 0 & 2I & 3I-J \end{array}\right) .$$

The second existence theorem for type *II* solutions is the following one.

Theorem 3.5. *Let* (3.8) *be satisfied in integers* t_1, t_2, u_1, u_2, x_1, *and* x_2 *where* $q \equiv 3 \pmod 4$ *is the order of a skew-Hadamard design incidence matrix and* w *is defined in* (3.6), *and where* $x_1 + u_1 = 2$ *and* $t_1 = (r+2)/2$ *and* $w = 2rq + 1$ *for the positive even integer* r. *Then we can construct a type II solution to the incidence equation for the finite projective plane case of order* $n = 4rq + 2$.

Proof. We have

$$N = \begin{bmatrix} Y & Z \\ -Z^T & Y^T \end{bmatrix}; \qquad Y = [y_{ij}] , \qquad Z = [z_{ij}] ,$$

where

$$y_{ii} = t_1 = \frac{1}{2}(r+2) , \tag{3.30}$$

$$y_{ij} + y_{ji} = x_1 + u_1 = 2 ; \quad 1 \leqq i \leqq q , \quad 1 \leqq j \leqq q , \quad i \neq j ,$$

and

$$NN^T = (2rq + 1)I . \tag{3.31}$$

Since (3.8) is satisfied we have

$$\left[\frac{1}{2}(r+2) + (q-1)\right]^2 + \left[t_2 + \frac{1}{2}(q-1)(x_2 + u_2)\right]^2 = 2rq + 1 , \tag{3.32}$$

or

$$\left[q - \frac{1}{2}r\right]^2 + \left[t_2 + \frac{1}{2}(q-1)(x_2+u_2)\right]^2 = 1 .$$

Since q, $r/2$, t_2, $(q-1)/2$, x_2, and u_2 are integers this means that

$$(3.33)\qquad q-\frac{1}{2}r=\varepsilon_1\,,\quad t_2+\frac{1}{2}(q-1)(x_2+u_2)=\varepsilon_2\,;$$
$$\varepsilon_1^2+\varepsilon_2^2=1\,;\qquad \varepsilon_1,\ \varepsilon_2=1,\ 0,\ -1\,.$$

We form two matrices U and V of size $2\times rq$ according to the values of ε_1 and ε_2 as follows:

$$(3.34)\qquad
\begin{array}{ccl}
U & V & \\
\begin{bmatrix}-2\cdots-2\\ 0\cdots\ \ 0\end{bmatrix} & \begin{bmatrix}0\cdots\ \ 0\\ -2\cdots-2\end{bmatrix} & \text{if}\quad \varepsilon_1=1,\varepsilon_2=0\,.\\
\begin{bmatrix}2\cdots\ \ 2\\ 0\cdots\ \ 0\end{bmatrix} & \begin{bmatrix}0\cdots\ \ 0\\ 2\cdots\ \ 2\end{bmatrix} & \text{if}\quad \varepsilon_1=-1,\ \varepsilon_2=0\,.\\
\begin{bmatrix}0\cdots\ \ 0\\ 2\cdots\ \ 2\end{bmatrix} & \begin{bmatrix}-2\cdots-2\\ 0\cdots\ \ 0\end{bmatrix} & \text{if}\quad \varepsilon_1=0,\ \varepsilon_2=1\,.\\
\begin{bmatrix}0\cdots\ \ 0\\ -2\cdots-2\end{bmatrix} & \begin{bmatrix}2\cdots\ \ 2\\ 0\cdots\ \ 0\end{bmatrix} & \text{if}\quad \varepsilon_1=0,\ \varepsilon_2=-1\,.
\end{array}$$

We set

$$f=t_1+\frac{1}{2}(q-1)(x_1+u_1)=\frac{1}{2}r+q=\varepsilon_1+r$$

and

$$g=t_2+\frac{1}{2}(q-1)(x_2+u_2)=\varepsilon_2\,.$$

Then f and g are integers and by (3.8)

$$(3.35)\qquad f^2+g^2=w=2rq+1\,.$$

Finally, we construct A_n of order $n=4rq+2$:

$$(3.36)\qquad A_n=\left(\begin{array}{cc|cc|cc}
1 & 1 & & & & \\
1 & -1 & U & V & 0 & 0\\
\hline
F_{rq} & & Y_* & Z_* & fI_{rq} & gI_{rq}\\
F_{rq} & & Y_* & Z_* & -fI_{rq} & -gI_{rq}\\
\hline
G_{rq} & & -Z_*^T & Y_*^T & gI_{rq} & -fI_{rq}\\
G_{rq} & & -Z_*^T & Y_*^T & -gI_{rq} & fI_{rq}
\end{array}\right).$$

By (3.34) the first two rows of A_n are orthogonal and have self inner products equal to $4rq+2=n$. Since the row and column sums of Y_* are $q-r/2$ and those of Z_* are $t_2+(q-1)(x_2+u_2)/2$, we have by (3.33) and (3.34) that rows one and two are orthogonal to all the other rows of A_n. We now look upon the submatrix of A_n below row 2 and to the right of the F_{rq}'s and G_{rq}'s as a matrix with the matrix entries E_r, u_1I, x_1I, t_2I, u_2I, and x_2I, all of order r. These matrices naturally divide the entire submatrix of A_n below row 2 into r-row blocks. Since these matrices commute with one another they behave multiplicatively among themselves as scalars. Thus (3.16), (3.17), (3.30), (3.31), and (3.35) imply that the inner product of an r-row block with itself is $(4rq + 2)I = nI$ of order r and that any two r-row blocks both intersecting F_{rq}'s or both intersecting G_{rq}'s are orthogonal, and (3.18) and (3.31) imply that any r-row block intersecting an F_{rq} is orthogonal to any r-row block intersecting a G_{rq}. Hence $A_nA_n^T = nI$, and since the first column of A_n consists entirely of 1's we see that we have a type *II* solution to the incidence equation for the finite projective plane case of order $n = 4rq + 2$.

Letting $c = x_2 + u_2$ and combining (3.33) with (3.6), noting that $t_1 = (r + 2)/2 = q - \varepsilon_1 + 1$, we have

$$(3.37) \qquad [q - \varepsilon_1 + 1]^2 + \left[\varepsilon_2 - \frac{1}{2}(q - 1)c\right]^2 + \frac{1}{2}(q - 1)[x_1^2 + (2 - x_1)^2 + x_2^2 + (c - x_2)^2] = 2q \cdot 2(q - \varepsilon_1) + 1 ,$$

which, because of (3.33), again yields (3.26). Since the argument from (3.26) to (3.28) depends only on $|\varepsilon_1|$, $|\varepsilon_2| \leqq 1$ and $q \geqq 3$, and since this is true here too, we obtain (3.28). Again, letting $a = 2x_1 - 2, b = 2x_2 - c$, and $q = 4m - 1$, $m > 0$ an integer, we obtain as before

$$(3.38) \qquad (12 - c^2)(4m - 1) + 4\varepsilon_2 c - 8\varepsilon_1 = a^2 + b^2,$$

where

$$(3.39) \qquad |c| \leqq 4 .$$

Now suppose for given values of $\varepsilon_1 = 1, -1, \varepsilon_2 = 0$ or $\varepsilon_1 = 0, \varepsilon_2 = 1, -1$ and c that (3.38) has a solution in integers a and b. We can then show, as we did before, that if $q = 4m - 1$ is the order of a skew-Hadamard design incidence matrix, then the conditions of Theorem 3.5 are satisfied and we can construct a type *II* solution according to that theorem.

Now in deciding whether or not (3.38) has a solution in integers a and b we have, by (3.39), five values of $|c|$ to consider for each of the two sets of values $\varepsilon_1 = 1, \varepsilon_2 = 0$ and $\varepsilon_1 = -1, \varepsilon_2 = 0$ and nine values of $\varepsilon_2 c$ to consider for the value $\varepsilon_1 = 0$. We take the five cases for $\varepsilon_1 = 1, \varepsilon_2 = 0$.

Case 1. $|c| = 4$: $-16m - 4 = a^2 + b^2$, impossible since $-16m - 4 < 0$ for $m > 0$.

Case 2. $|c| = 3$: $12m - 11 = a^2 + b^2$, possible since, e.g., $12(2) - 11 = 13 = 3^2 + 2^2$. Here $3q - 8 = a^2 + b^2$.

Case 3. $|c| = 2$: $32m - 16 = a^2 + b^2$ or $2m - 1 = a_1^2 + b_1^2$, a_1, b_1 integers, possible since, e.g., $2(3) - 1 = 5 = 2^2 + 1^2$. Here $8q - 8 = a^2 + b^2$ or $q - 1 = a_2^2 + b_2^2$, a_2, b_2 integers.

Case 4. $|c| = 1$: $44m - 19 = a^2 + b^2$, possible since, e.g., $44(1) - 19 = 25 = 5^2 + 0^2$. Here $11q - 8 = a^2 + b^2$.

Case 5. $|c| = 0$: $48m - 20 = a^2 + b^2$ or $12m - 5 = a_1^2 + b_1^2$, a_1, b_1 integers, impossible since $12m - 5 \equiv 3 \pmod 4$.

Now when $\varepsilon_1 = 1$ we have $r = 2(q - 1)$, hence $n = 8q^2 - 8q + 2 = 2(2q - 1)^2$. So by Theorem 3.5 we have the following result.

THEOREM 3.6. *There exists a type II solution to the incidence equation for the finite projective plane case of order* $n = 2(2q - 1)^2$ *whenever* q *is the order of a skew-Hadamard design incidence matrix and any of the following expressions is the sum of two integral squares*: $3q - 8$, $q - 1$, $11q - 8$.

When $\varepsilon_1 = -1$ we have $r = 2(q + 1)$, hence $n = 8q^2 + 8q + 2 = 2(2q + 1)^2$. Analyzing this case as was done above for $\varepsilon_1 = 1$, we have by Theorem 3.5 the corresponding result:

THEOREM 3.7. *There exists a type II solution to the incidence equation for the finite projective plane case of order* $n = 2(2q + 1)^2$ *whenever* q *is the order of a skew-Hadamard design incidence matrix and any of the following expressions is the sum of two integral squares*: $3q + 8$, $q + 1$, $11q + 8$.

When $\varepsilon_1 = 0$ we have $r = 2q$, hence $n = 8q^2 + 2 = (2q - 1)^2 + (2q + 1)^2$. Analyzing this case as was done for Theorem 3.3 we have by Theorem 3.5 the following result.

THEOREM 3.8. *There exists a type II solution to the incidence equation for the finite projective plane case of order* $n = (2q - 1)^2 + (2q + 1)^2$ *whenever* q *is the order of a skew-Hadamard design incidence matrix and any of the following expressions is the sum of two integral squares*: $3q + 12$, $q + 1$, $11q + 4$, $3q$, $11q - 4$, $q - 1$, $3q - 12$.

All three theorems yield infinitely many type *II* solutions. There exist skew-Hadamard design incidence matrices of orders

$q_1 = 4(3^{2d-1} + 1) - 1 = 4 \cdot 3^{2d-1} + 3$ and $q_2 = 2^{2d} - 1$ for each integer $d \geqq 1$. Then $3q_1 - 8 = (2 \cdot 3^d)^2 + 1^2$, and $q_2 + 1 = 2^{2d} + 0^2$. The first four orders for which each of these theorems yields a type *II* solution correspond to $q = 3$, 7, 11, and 15 and are $n = 50$, 338, 882, and 1682, respectively, by Theorem 3.6, $n = 98$, 450, 1058, and 1922, respectively, by Theorem 3.7, and $n = 74$, 394, 970, and 1802, respectively, by Theorem 3.8. As an example we construct A_{50}. For $n = 50$ we have $q = 3$, $\varepsilon_1 = 1$, and $\varepsilon_2 = 0$ hence $r = 4$ whence $t_1 = 3$. Now by case 4 above, $|c| = 1$ and

$$11q - 8 = 25 = 0^2 + 5^2 = (2x_1 - 2)^2 + (2x_2 - c)^2 .$$

We take $2x_1 - 2 = 0$ or $x_1 = 1$ and $2x_2 - c = 5$. Letting $c = 1$ we have $x_2 = 3$ and $t_2 = -1$. Then $u_1 = 1$ and $u_2 = -2$, $f = 5$ and $g = 0$. Now $E_4 = 3I - J$ of order 4 and since $\varepsilon_1 = 1$ and $\varepsilon_2 = 0$,

$$U = \begin{bmatrix} -2 \cdots -2 \\ 0 \cdots \quad 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 0 \cdots \quad 0 \\ -2 \cdots -2 \end{bmatrix},$$

of size 2×12. The matrices F_4 and G_4 are of size 4×2, and a skew-Hadamard design incidence matrix of order 3 is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Hence we have

$$A_{50} = \left(\begin{array}{cc|ccc|ccc|ccc|ccc}
1 & 1 & \multicolumn{3}{c|}{-2 \cdots -2} & \multicolumn{3}{c|}{0 \cdots \quad 0} & \multicolumn{3}{c|}{} & \multicolumn{3}{c}{} \\
1 & -1 & \multicolumn{3}{c|}{0 \qquad 0} & \multicolumn{3}{c|}{\times 2 \cdots -2} & \multicolumn{3}{c|}{0} & \multicolumn{3}{c}{0} \\
\hline
1 & 1 & 3I-J & I & I & -I & 3I & -2I & 5I & 0 & 0 & & & \\
\vdots & & I & 3I-J & I & -2I & -I & 3I & 0 & 5I & 0 & & 0 & \\
1 & 1 & I & I & 3I-J & 3I & -2I & -I & 0 & 0 & 5I & & & \\
\hline
1 & 1 & 3I-J & I & I & -I & 3I & -2I & -5I & 0 & 0 & & & \\
\vdots & & I & 3I-J & I & -2I & -I & 3I & 0 & -5I & 0 & & 0 & \\
1 & 1 & I & I & 3I-J & 3I & -2I & -I & 0 & 0 & -5I & & & \\
\hline
1 & -1 & I & 2I & -3I & 3I-J & I & I & & & & -5I & 0 & 0 \\
\vdots & & -3I & I & 2I & I & 3I-J & I & & 0 & & 0 & -5I & 0 \\
1 & -1 & 2I & -3I & I & I & I & 3I-J & & & & 0 & 0 & -5I \\
\hline
1 & -1 & I & 2I & -3I & 3I-J & I & I & & & & 5I & 0 & 0 \\
\vdots & & -3I & I & 2I & I & 3I-J & I & & 0 & & 0 & 5I & 0 \\
1 & -1 & 2I & -3I & I & I & I & 3I-J & & & & 0 & 0 & 5I
\end{array}\right)$$

The above constructions are all based on the existence of a skew-Hadamard design incidence matrix of a certain order $q \equiv 3 \pmod 4$.

However, let us examine these constructions to see whether other constructions like these are possible. As a very simple possibility, let us consider replacing the skew-Hadamard design incidence matrix by the matrix [0] of order 1. Here corresponding to (3.5) we have

$$N = \begin{bmatrix} t_1 & t_2 \\ -t_2 & t_1 \end{bmatrix},$$

and setting

$$w \equiv t_1^2 + t_2^2 \tag{3.40}$$

we automatically have

$$NN^T = wI \, . \tag{3.41}$$

Let us consider the form of construction in Theorem 3.2. We let (3.40) be satisfied in integers $t_1 = (r+2)/2$, t_2, and $w = 2r+2$, for the positive even integer r. Then

$$\frac{1}{4}(r+2)^2 + t_2^2 = 2r + 2 \, ,$$

or

$$\frac{1}{4}(r-2)^2 + t_2^2 = 2 \, ,$$

hence

$$1 - \frac{1}{2}r = \varepsilon_1 \, , \qquad t_2 = \varepsilon_2 \, ; \qquad \varepsilon_1, \ \varepsilon_2 = 1, \ -1 \, .$$

For $\varepsilon_1 = 1$ we have $r = 0$, hence we get no nontrivial construction. For $\varepsilon_1 = -1$ we obtain $r = 4$ whence $n = w = 10$. We have $E_4 = 3I - J$ of order 4 and F_4 and G_4, as defined previously, of size 4×2. Then corresponding to $\varepsilon_2 = 1, -1$ we obtain by the form of construction in Theorem 3.2.

$$A_{10} = \left(\begin{array}{cc|cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & & & & & & & & \\ 1 & 1 & \multicolumn{4}{c|}{3I-J} & \multicolumn{4}{c}{I} \\ 1 & 1 & & & & & & & & \\ 1 & 1 & & & & & & & & \\ \hline 1 & -1 & & & & & & & & \\ 1 & -1 & \multicolumn{4}{c|}{-I} & \multicolumn{4}{c}{3I-J} \\ 1 & -1 & & & & & & & & \\ 1 & -1 & & & & & & & & \end{array}\right) , \quad \left(\begin{array}{cc|cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & & & & & & & & \\ 1 & 1 & \multicolumn{4}{c|}{3I-J} & \multicolumn{4}{c}{-I} \\ 1 & 1 & & & & & & & & \\ 1 & 1 & & & & & & & & \\ \hline 1 & -1 & & & & & & & & \\ 1 & -1 & \multicolumn{4}{c|}{I} & \multicolumn{4}{c}{3I-J} \\ 1 & -1 & & & & & & & & \\ 1 & -1 & & & & & & & & \end{array}\right)$$

respectively, each of which satisfy $A_{10}A_{10}^T = 10I$. These are essentially the same as the A_{10} constructed by Hall and Ryser [11]. Now let us consider the form of construction in Theorem 3.5. We let (3.40) be satisfied in integers $t_1 = (r+2)/2$, t_2, and $w = 2r+1$, for the positive even integer r. Then

$$\frac{1}{4}(r+2)^2 + t_2^2 = 2r + 1 \ ,$$

or

$$\frac{1}{4}(r-2)^2 + t_2^2 = 1 \ ,$$

hence

$$1 - \frac{1}{2}r = \varepsilon_1 \ , \quad t_2 = \varepsilon_2 \ ; \quad \varepsilon_1^2 + \varepsilon_2^2 = 1 \ ; \quad \varepsilon_1, \ \varepsilon_2 = 1, \ 0, \ -1 \ .$$

For $\varepsilon_1 = 1$ we again get no nontrivial construction. For $\varepsilon_1 = 0$ we obtain $r = 2$ whence $n = 2w = 10$. We have $E_2 = 2I - J$ of order 2 and F_2 and G_2, as defined previously, of size 2×2. Then corresponding to $\varepsilon_2 = 1, -1$ we have $f = 2$ and $g = 1, -1$, respectively, and we obtain by the form of construction in Theorem 3.5

$$A_{10} = \left(\begin{array}{rr|rrrr|rrrr}
1 & 1 & 0 & 0 & -2 & -2 & \multicolumn{4}{c}{0} \\
1 & -1 & 2 & 2 & 0 & 0 & & & & \\
\hline
1 & 1 & 1 & -1 & 1 & 0 & 2 & 0 & 1 & 0 \\
1 & 1 & -1 & 1 & 0 & 1 & 0 & 2 & 0 & 1 \\
1 & 1 & 1 & -1 & 1 & 0 & -2 & 0 & -1 & 0 \\
1 & 1 & -1 & 1 & 0 & 1 & 0 & -2 & 0 & -1 \\
\hline
1 & -1 & -1 & 0 & 1 & -1 & 1 & 0 & -2 & 0 \\
1 & -1 & 0 & -1 & -1 & 1 & 0 & 1 & 0 & -2 \\
1 & -1 & -1 & 0 & 1 & -1 & -1 & 0 & 2 & 0 \\
1 & -1 & 0 & -1 & -1 & 1 & 0 & -1 & 0 & 2
\end{array}\right) ,$$

$$\left(\begin{array}{rr|rrrr|rrrr}
1 & 1 & 0 & 0 & 2 & 2 & \multicolumn{4}{c}{0} \\
1 & -1 & -2 & -2 & 0 & 0 & & & & \\
\hline
1 & 1 & 1 & -1 & -1 & 0 & 2 & 0 & -1 & 0 \\
1 & 1 & -1 & 1 & 0 & -1 & 0 & 2 & 0 & -1 \\
1 & 1 & 1 & -1 & -1 & 0 & -2 & 0 & 1 & 0 \\
1 & 1 & -1 & 1 & 0 & -1 & 0 & -2 & 0 & 1 \\
\hline
1 & -1 & 1 & 0 & 1 & -1 & -1 & 0 & -2 & 0 \\
1 & -1 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & -2 \\
1 & -1 & 1 & 0 & 1 & -1 & 1 & 0 & 2 & 0 \\
1 & -1 & 0 & 1 & -1 & 1 & 0 & 1 & 0 & 2
\end{array}\right)$$

respectively, each of which satisfy $A_{10}A_{10}^T = 10I$. These, however, are essentially different from the A_{10}'s previously exhibited. This shows that type II solutions of the direct sum type are not necessarily unique to within permutations of the rows and columns of A_n and the multiplication of the columns of A_n by -1. Finally, for $\varepsilon_1 = -1$, $\varepsilon_2 = 0$, we obtain $r = 4$ whence $n = 2w = 18$. We have $E_4 = 3I - J$ of order 4 and F_4 and G_4, as previously defined, of size 4×2. Here $f = 3$ and $g = 0$. We obtain by the form of construction in Theorem 3.5

$$A_{18} = \left(\begin{array}{rr|c|c|c|c}
1 & 1 & 2\cdots 2 & 0\cdots 0 & 0 & 0 \\
1 & -1 & 0\cdots 0 & 2\cdots 2 & & \\ \hline
1 & 1 & 3I-J & 0 & 3I & 0 \\
\vdots & & & & & \\
1 & 1 & 3I-J & 0 & -3I & 0 \\ \hline
1 & -1 & 0 & 3I-J & 0 & -3I \\
\vdots & & & & & \\
1 & -1 & 0 & 3I-J & 0 & 3I
\end{array}\right),$$

which satisfies $A_{18}A_{18} = 18I$. Hence, summarizing, we have the following result.

THEOREM 3.9. *There exists a type II solution to the incidence equation for the finite projective plane case orders* $n = 10, 18$.

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