PACIFIC JOURNAL OF MATHEMATICS
Vol. 17, No. 3, 1966

WEAK-STAR GENERATORS OF H^∞

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Let H^∞ denote the algebra of bounded analytic functions in the unit disk $D = \{z: |z| < 1\}$. A function φ in H^∞ is called a generator if the polynomials in φ are weak-star dense in H^∞. The problem to be considered here is that of characterizing the generators of H^∞.

The weak-star topology of H^∞ can be thought of as arising in the following way. By Fatou's theorem, each function ψ in H^∞ has radial limits at almost every point of the unit circle $C = \{z: |z| = 1\}$ and thus gives rise to a bounded measurable function ψ_C on C. The map $\psi \rightarrow \psi_C$ sends H^∞ isomorphically and isometrically onto a certain subspace of $L^\infty(C)$; we denote this subspace by $H^\infty(C)$. (We regard C as endowed with normalized Lebesgue measure.) The space $H^\infty(C)$ is the dual of a quotient space of $L^1(C)$ and as such has a weak-star topology (which is simply the topology induced on $H^\infty(C)$ by the weak-star topology of $L^\infty(C)$). Because of the natural correspondence between H^∞ and $H^\infty(C)$, the weak-star topology on the latter induces a topology on the former, and this is what we mean by the weak-star topology of H^∞. The convergent sequences of this topology are easily characterized.

LEMMA 1. *A sequence $\{\psi_n\}_1^\infty$ in H^∞ converges weak-star to the function ψ if and only if it is uniformly bounded and converges to ψ at every point of D.*

Proof. This is of course well-known; however we include a proof for the sake of completeness. To simplify the notation we shall write $\varphi(e^{it})$ in place of $\varphi_C(e^{it})$ for any φ in H^∞ For each point a in D let P_a denote the corresponding Poisson kernel, i.e.,

$$P_a(z) = \frac{1 - |a|^2}{|1 - \bar{a}z|^2}, \quad |z| = 1 .$$

We then have

$$\varphi(a) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{it})P_a(e^{it})dt \tag{1}$$

for all φ in H^∞ and all a in D.

Now suppose the sequence $\{\psi_n\}$ in H^∞ is uniformly bounded and converges to the function ψ at each point of D. Then it follows from

Received January 7, 1965.

(1) that

$$\int_0^{2\pi} \psi(e^{it})h(e^{it})dt = \lim_{n\to\infty} \int_0^{2\pi} \psi_n(e^{it})h(e^{it})dt \tag{2}$$

for every function h in the linear hull of the functions P_a ($a \in D$). But the linear hull of the functions P_a is dense in $L^1(C)$ (since, by Fatou's theorem, no nonidentically zero function in $L^\infty(C)$ is orthogonal to every P_a). This together with the uniform boundedness of $\{\psi_n\}$ implies that (2) holds for all h in $L^1(C)$, and thus $\psi_n \to \psi$ weak-star.

Conversely, if $\psi_n \to \psi$ weak-star, then the sequence $\{\psi_n\}$ is uniformly bounded by the principle of uniform boundedness, and $\psi_n(a) \to \psi(a)$ for all a in D because the functions P_a belong to $L^1(C)$.

The problem of characterizing the weak-star generators of H^∞ was suggested in the preceding paper [9]. It is proved there that every generator is univalent. From now on we let G denote a fixed but arbitrary bounded simply connected domain and we let φ be a conformal map of D onto G. We seek necessary and sufficient conditions on G in order that φ be a generator. Eventually we shall obtain such conditions. Although they are not particularly simple, this seems, at least to the author, to be an unavoidable concomitant of the complexities of the weak-star topology. Perhaps it is worth mentioning at this point that there are domains G for which φ is not a generator. In fact, we know from Proposition 2 of the preceding paper that if φ is a generator then φ_C is univalent almost everywhere, and it is a triviality to construct domains G for which this condition is violated.

Before treating our problem in its full generality we consider a specialization. We shall say φ is a *sequential generator* if every function in H^∞ is the weak-star limit of a sequence of polynomials in φ. In view of Lemma 1 the following assertion is immediate.

PROPOSITION 1. For φ to be a sequential generator it is necessary and sufficient that G have the following property: for every bounded analytic function f in G, there is a sequence of polynomials which is uniformly bounded on G and converges to f at every point of G.

The domains with this property have a simple topological characterization which was discovered by O. J. Farrell [5], [6]. Before stating Farrell's result we need a few definitions.

If B is a bounded domain in the plane, then the *Carathéodory hull* (or $\mathscr{C}$-hull) of B is the complement of the closure of the unbounded component of the complement of the closure of B. We denote the $\mathscr{C}$-hull of B by B^*. Loosely speaking, B^* can be described as the interior of the outer boundary of B, and in analytic terms it can be defined as the interior of the set of all points z_0 in the plane such that

$$|p(z_0)| \leqq \sup_{z\in B} |p(z)|$$

for all polynomials p. The components of B^* are simply connected; in fact, it is a simple matter to show that each of these components has a connected complement. We denote by B^1 the component of B^* that contains B. We can now state:

FARRELL'S THEOREM. *Let B be a bounded domain in the plane and let f be a bounded analytic function in B. Then in order for there to exist a sequence of polynomials which is uniformly bounded on B and converges to f at each point of B, it is necessary and sufficient that f be the restriction of a function bounded and analytic in B^1.*

This result was recently rediscovered and extended from domains to arbitrary bounded open sets by Rubel and Shields [8]. An interesting proof of Farrell's theorem based on the theory of Dirichlet algebras has been given by Hoffman and Wermer; see [10, p. 27].

Farrell's theorem tells us immediately that our domain G satisfies the condition of Proposition 1 if and only if $G = G^1$. The sequential generators of H^∞ can thus be characterized in the following terms.

PROPOSITION 2. The function φ is a sequential generator if and only if G is a component of its $\mathscr{C}$-hull.

But Farrell's theorem tells us even more; it enables us to identify the functions in H^∞ that are weak-star limits of sequences of polynomials in φ.

PROPOSITION 3. A function ψ in H^∞ is the weak-star limit of a sequence of polynomials in φ if and only if $\psi \circ \varphi^{-1}$ is the restriction of a function bounded and analytic in G^1.

We now take up in its full generality the problem of characterizing the generators of H^∞. Let M^0 be the set of polynomials in φ, and for each countable ordinal number α define M^α inductively to be the linear manifold in H^∞ consisting of all functions that are weak-star limits of sequences of functions in $\bigcup_{\beta<\alpha} M^\beta$. It is a well-known property of weak-star topologies [1, p. 213] that the manifolds M^α eventually become constant, i.e., there is a least countable ordinal α' such that $M^{\alpha'} = M^{\alpha'+1}$. Moreover $M^{\alpha'}$ is the weak-star closure of M^0, and so is the weak-star closed subalgebra of H^∞ generated by φ and the identity. Thus φ is a generator if and only if $M^{\alpha'} = H^\infty$, in which case we call φ a *generator of order* α'. Above we used Farrell's theorem to identify the functions in the manifold M^1. A more refined application of Farrell's theorem will enable us to identify the functions in M^α for every α. First a number of preliminaries are necessary.

From now on let B denote a bounded domain in the plane. (In our applications B will be simply connected.) For any simply connected domain E containing B we define the *relative hull of B in E*, or the E-hull of B, to be the interior of the set of all points z_0 in E such that

$$|f(z_0)| \leqq \sup_{z \in B} |f(z)|$$

for every function f bounded and analytic in E. The crucial step in our reasoning will be to show that if B is contained in the open unit disk D, then the D-hull of B coincides with B^*, the $\mathscr{C}$-hull of B. For this we need:

LEMMA 2. *Let f be a bounded analytic function in a bounded simply connected domain A. For each point a on ∂A define*

$$m(f, A, a) = \lim_{n \to \infty} \sup \{|f(z)| : z \in A, |z - a| < 1/n\}$$

(in other words $m(f, A, a)$ is the maximum of the moduli of all cluster values of f at a). Let $ac(A)$ denote the set of points on ∂A that are accessible from A. Then

$$\sup_{z \in A} |f(z)| = \sup_{a \in ac(A)} m(f, A, a) . \tag{3}$$

Proof. Although this is well-known we include a proof for the sake of completeness. Let w be a conformal map of the unit disk D onto A. Let S be the set of points on the unit circle C at which both w and $f \circ w$ have radial limits. Fatou's theorem implies that $C - S$ has measure zero. Since $f \circ w$ is the Poisson integral of its boundary values, it follows that:

$$\begin{aligned} \sup_{z \in A} |f(z)| &= \sup_{b \in S} \lim_{r \to 1} |f(w(rb))| \\ &\leqq \sup_{b \in S} m(f, A, w(b)) . \end{aligned}$$

But if b is in S then $w(b)$ is in $ac(A)$, and thus the right side of the preceding inequality is no greater than the right side of (3). This proves the lemma.

REMARK 1. With the notations of the preceding lemma, let a_0 be any point in $ac(A)$. Then the supremum on the right side of (3) is equal to

$$\sup_{\substack{a \in ac(A) \\ a \neq a_0}} m(f, A, a) .$$

This follows from the fact that the set of points on C at which the radial limit of w equals a_0 is a null set [7, p. 52].

REMARK 2. The conclusion of Lemma 2 remains true if one drops the assumption that A is simply connected. To show this, take a uniformizer w of A and repeat *verbatim* the above proof, using the easily proved fact that all radial limits of w are boundary points of A. In our application the domain A will be simply connected.

LEMMA 3. *Let the domain B be contained in the unit disk D. Then the D-hull of B is equal to B^*.*

Proof. The D-hull of B is obviously contained in B^*. To prove the reverse inclusion we must show that if z_0 is any point of B^* and if f is any bounded analytic function in D, then

$$|f(z_0)| \leqq \sup_{z\in B} |f(z)| \,. \tag{4}$$

This is obvious if z_0 is in $\bar{B}$, and so we may suppose that z_0 is in $B^* - \bar{B}$. Let A be the component of $B^* - \bar{B}$ containing z_0. We assert that at most one point of $ac(A)$ lies on the unit circle C. In fact, if $ac(A) \cap C$ contained two distinct points b_1 and b_2, then we could join b_1 and b_2 by a Jordan arc J lying except for its end points in A. The arc J would then separate D into two disjoint nonempty domains D_1 and D_2, and since B is connected it would have to lie either entirely in D_1 or entirely in D_2. But this is absurd because A meets both D_1 and D_2 and $\bar{B}$ separates A from ∞. This contradiction proves our assertion that $ac(A) \cap C$ contains at most one point.

Now it is easy to verify that A has a connected complement, i.e., A is simply connected. Therefore by Lemma 2 and Remark 1 following it, for any function f bounded and analytic in D we have

$$\begin{aligned} |f(z_0)| &\leqq \sup_{a\in\partial A\cap D} m(f, A, a) \\ &= \sup_{a\in\partial A\cap D} |f(a)| \,. \end{aligned}$$

This implies (4) because $\partial A \subset \partial B$. The proof of the lemma is complete.

COROLLARY. *Let the domain B be contained in the bounded simply connected domain E. Then the components of the E-hull of B are simply connected.*

Proof. Let w be a conformal map of E onto D. By Lemma 3 w sends the E-hull of B onto the $\mathscr{C}$-hull of $w(B)$. Hence the corollary follows from the already observed fact that the components of a $\mathscr{C}$-hull are simply connected.

Lemma 3 yields the following extension of Farrell's theorem.

THEOREM 1. *Let the domain B be contained in the bounded simply* *connected domain E, and let $\tilde{B}$ be the component of the E-hull of B that contains B. Let f be a bounded analytic function in B. Then in order for there to exist a sequence of functions bounded and analytic in E which is uniformly bounded on B and converges to f at every point of B, it is necessary and sufficient that f be the restriction of a function bounded and analytic in $\tilde{B}$.*

Proof. If a sequence of functions bounded and analytic in E converges to f in the manner described, then by Vitali's theorem [3, p. 186] this sequence converges uniformly on compact subsets of $\tilde{B}$ to a bounded analytic function $\tilde{f}$, and we have $f = \tilde{f} \mid B$. To prove the converse we may by a conformal map reduce the general case to the case where $E = D$. But when $E = D$ the desired conclusion follows immediately from Lemma 3 and Farrell's theorem.

It might be worth while to try to find a more direct proof of Theorem 1, one that does not use Farrell's theorem and conformal mapping. Such a proof could conceivably be illuminating.

Before applying Theorem 1 to the problem at hand we obtain a topological description of relative hulls. This description will be in terms of the notion of a crosscut. If E is a domain then a *crosscut* of E is a Jordan arc contained in E except for its end points. If E is simply connected and J is a crosscut of E, then $E - J$ consists of two disjoint nonempty domains E_1 and E_2, and we say that J *separates* the points of E_1 from the points of E_2 [2, pp. 328–329].

PROPOSITION 4. *Let the domain B be contained in the bounded simply connected domain E and let F denote the relative closure in E of the E-hull of B. Then $E - F$ consists of those points of E that can be separated from B by a crosscut of E.*

REMARK. This proposition really does give a description of the E-hull of B, because the E-hull of B is the interior of F.

Proof. We first show that it suffices to consider the case where $E = D$. For this let w be a conformal map of E onto D and let z_0 be a point of E. We must show that z_0 can be separated from B by a crosscut of E if and only if $w(z_0)$ can be separated from $w(B)$ by a crosscut of D. The implication in one direction follows from the fact that w maps crosscuts of E onto crosscuts of D; see [2, p. 353, Satz XVII]. To obtain the reverse implication, suppose J is a crosscut of D separating $w(z_0)$ from $w(B)$. It may not be true that $w^{-1}(J)$ is a crosscut of E. However by modifying J slightly we can replace it by a crosscut J' of D which still separates $w(z_0)$ from $w(B)$ and has the additional properties:

(i) the radial limits of w^{-1} exist at the endpoints a and b of J';
(ii) $w^{-1}(a) \neq w^{-1}(b)$;
(iii) J' approaches a and b radially.

The map w^{-1} then sends J' onto a crosscut of E, as desired.

We may thus suppose that $E = D$. By Lemma 3 the D-hull of B is B^*. If the point z_0 of D can be separated from B by a crosscut of D, then z_0 obviously belongs to the unbounded component of the complement of $\bar{B}$ and therefore is in $D - F$. Suppose on the other hand that z_0 is in $D - F$. Then we can join z_0 to ∞ by a polygonal arc that does not meet $\bar{B}$. Let a be the first point at which this arc meets the unit circle C, and let J_1 denote that portion of the arc between z_0 and a (inclusive). The point a, and therefore some circular neighborhood of a, is contained in the unbounded component of the complement of $\bar{B}$. Therefore, from some point b on C, near but distinct from a, we can draw a segment into D which is contained except for b in the same component of $D - F$ as is $J_1 - \{a\}$, and which moreover does not meet J_1. We can now continue this segment so as to obtain a polygonal arc J_2 joining b to z_0, not meeting $\bar{B}$, and not meeting J_1 except at z_0. Then $J = J_1 \cup J_2$ is a crosscut of D which does not meet $\bar{B}$. This crosscut separates D into two disjoint nonempty domains and the set $\bar{B} \cap D$, being connected, lies entirely in one of them. Thus by modifying J slightly in the vicinity of z_0 we can produce a crosscut of D which separates z_0 from B.

COROLLARY. *Let B and E be as in Proposition 4. Then the E-hull of B equals E if and only if every crosscut of E meets B.*

After these preliminaries we are prepared to discuss generators of H^∞. Recall that we are letting G denote a bounded simply connected domain and φ a conformal map of D onto G. We have already defined G^1 to be the component of the $\mathscr{C}$-hull of G that contains G. We now define inductively for every countable ordinal number α a simply connected domain G^α containing G as follows. If α has an immediate predecessor we let G^α be the component of the $G^{\alpha-1}$-hull of G that contains G. (G^α is then simply connected by the corollary to Lemma 3.) If α has no immediate predecessor we define G^α to be the component of the interior of $\bigcap_{\beta<\alpha} G^\beta$ that contains G. (It is easily verified that G^α then has a connected complement, and so is simply connected.) By Proposition 4, if the inclusion $G^{\alpha+1} \subset G^\alpha$ is proper then $G^\alpha - G^{\alpha+1}$ contains interior points. Hence the inclusion is proper for at most countably many α, and so there is a least countable ordinal γ such that $G^\gamma = G^{\gamma+1}$. We call γ the *order* of G. Obviously $G^\alpha = G^\gamma$ for $\alpha > \gamma$.

THEOREM 2. *The manifold M^α consists of all functions ψ in H^∞*

such that $\psi \circ \varphi^{-1}$ is the restriction of a function bounded and analytic in G^{α}.

Proof. The case $\alpha = 1$ is given by Proposition 3. We proceed by induction, assuming that the theorem holds for all ordinals less than α. If α has an immediate predecessor the desired conclusion follows immediately from Theorem 1. We pass on to the case where α has no immediate predecessor. Suppose first that ψ is a function in M^{α}. Then by our induction hypothesis, there is a sequence of functions $\{f_n\}_1^{\infty}$ with the following properties:

(i) each f_n is a bounded analytic function in G^{β} for some $\beta < \alpha$ (perhaps a different β for each n);

(ii) the sequence $\{f_n\}$ is uniformly bounded in G;

(iii) $\lim_{n\to\infty} f_n(\varphi(z)) = \psi(z)$ for all z in D.

But then the sequence $\{f_n\}$ is uniformly bounded on G^{α}, so that by Vitali's theorem it converges on G^{α} to a bounded analytic function f, and we have $\psi \circ \varphi^{-1} = f \mid G$. This takes care of one half of the induction. For the other half we choose a strictly increasing sequence of ordinals $\{\alpha_n\}_1^{\infty}$ such that α is the least ordinal exceeding every α_n. By our induction hypothesis it will suffice to show that if f is a bounded analytic function in G^{α}, then there is a sequence of functions $\{f_n\}$ with the following properties:

(i′) each f_n is a bounded analytic function in G^{α_n};

(ii′) the sequence $\{f_n\}$ is uniformly bounded on G;

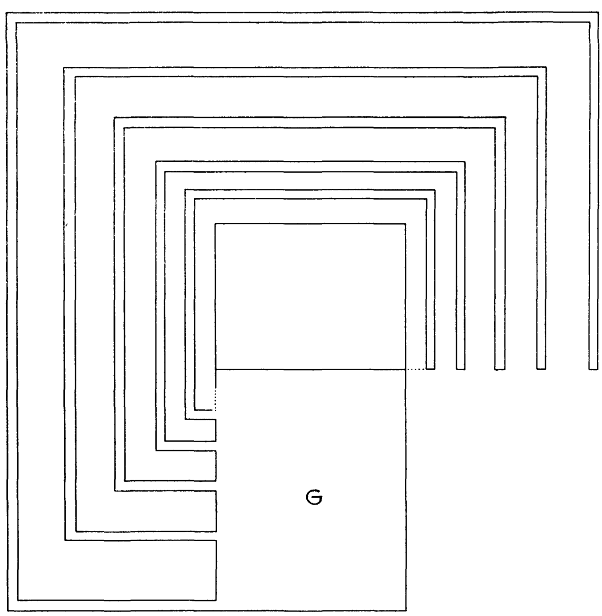

FIGURE 1

(iii′) $\lim_{n\to\infty} f_n(z) = f(z)$ for all z in G.
To do this, choose a point z_0 in G^α and for each n let w_n be the conformal map of G^α onto G^{α_n} satisfying $w_n(z_0) = z_0$ and $w_n'(z_0) > 0$. Since the sequence of domains $\{G^{\alpha_n}\}$ converges to G^α in the sense of Carathéodory it follows that $\lim_{n\to\infty} w_n(z) = z$ for all z in G^α [4, p. 76]. Hence, given a bounded analytic function f in G^α, we can achieve conditions (i′)–(iii′) by defining $f_n = f \circ w_n^{-1}$. (This reasoning is of course well-known.) The proof of the theorem is complete.

COROLLARY 1. *If the function φ is a generator of H^∞ of order γ then the domain G has order γ and $G^\gamma = G$. Conversely, if G has order γ and $G^\gamma = G$, then φ is a generator of order γ.*

COROLLARY 2. *The function φ fails to be a generator if and only if there is a domain B containing G properly such that*

$$\sup_{z\in B} |f(z)| = \sup_{z\in G} |f(z)|$$

for every function f bounded and analytic in B.

Proof. If such a domain B exists then it is contained in every G^α and so φ is not a generator. Conversely, if φ is not a generator and G has order γ, then the domain G^γ has the required property.

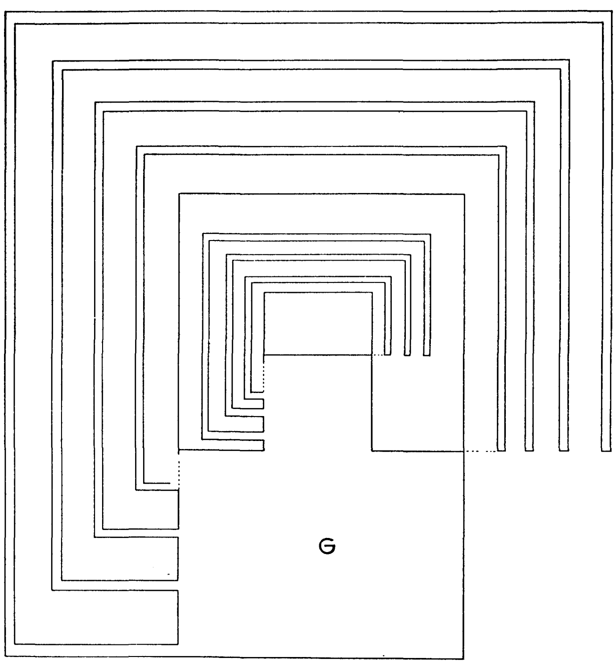

FIGURE 2

COROLLARY 3. *If φ is a generator then G is the interior of its closure.*

Proof. If $G \neq \operatorname{int}(\bar{G})$ then $\operatorname{int}(\bar{G})$ satisfies the condition of Corollary 2.

COROLLARY 4. *The weak-star closed subalgebra of H^∞ generated by φ and the identity is isometrically isomorphic to H^∞.*

Proof. Let φ_0 be a conformal map of D onto G^γ, where γ is the order of G. Then the map

$$\psi \longrightarrow \psi \circ \varphi_0^{-1} \circ \varphi$$

is an isometric isomorphism of H^∞ onto the weak-star closed subalgebra generated by φ and the identity.

In conclusion we give two examples of generators of orders greater than one. The reader can convince himself that for the domains G of Figures 1 and 2 the corresponding mapping functions φ are generators of orders two and three respectively. It is easy to see how, by compounding the method used to obtain the domains of Figures 1 and 2, one can produce a generator of infinite order, for example of order ω. However the author has been unable to construct generators of arbitrary order.

The author is indebted to Professors Allen Shields and Lee Rubel for helpful discussions.

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