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THE BEHAVIOR OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $y''' + p(x)y' + q(x)y = 0$

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This paper is a study of the oscillation and other properties of solutions of the differential equation

$$(L) \qquad y''' + p(x)y' + q(x)y = 0\,.$$

Throughout, we shall assume that $p(x)$ and $q(x)$ are continuous and do not change sign on the infinite half-axis I: $a \leqq x < +\infty$. A solution of (L) will be said to be oscillatory if it change sign for arbitrarily large values of x.

Our principal results will be concerned with the existence, uniqueness, (aside from constant multiples) and asymptotic behavior of nontrivial, nonoscillatory solutions, and criteria for the existence of oscillatory solutions in terms of the behavior of nonoscillatory solutions. Other results are concerned with separation properties and the question of when the amplitudes of oscillatory solutions are increasing or decreasing.

The general properties of linear homogeneous thirdorder differential equations were first studied by Birkhoff [1]. Other investigators have been Greguš [2-11], Hanan [12], Mammana [14], Rab [15-20], Sansone [21], Švec [22, 23], Villari [24, 25], and Zlamal [26]. In this paper we shall study successively the cases

(i) $p(x) \leqq 0$, $q(x) > 0$,
(ii) $p(x) \leqq 0$, $q(x) \leqq 0$,
(iii) $p(x) \geqq 0$, $q(x) \geqq 0$,

and shall show that under certain conditions the solutions of (L) have similar qualitative properties as in the cases when $p(x)$ and $q(x)$ are nonzero constants. It is for this reason that we list the following remarks which characterize these cases when $p(x)$ and $q(x)$ are nonzero constants.

A. *If $p(x) = p < 0$ and $q(x) = q > 0$, (L) has oscillatory solutions if and only if.*

$$q - \frac{2}{3\sqrt{3}}(-q)^{3/2} > 0\,.$$

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When this condition is satisfied all solutions of (L) are oscillatory except constant multiples of one solution which does not vanish on I, and which together with all of its derivatives is monotonic on I and approaches zero as x tends to infinity.

B. *If $p(x)$ and $q(x)$ are both negative constants p and q, (L) has oscillatory solutions if and only if*

$$-q - \frac{2}{3\sqrt{3}}(-p)^{3/2} > 0 .$$

When this condition is satisfied, (L) has two independent oscillatory solutions and the zeros of any two oscillatory solutions separate on I. Moreover the absolute values of the successive maxima and minima form a decreasing sequence.

C. *When $p(x)$ and $q(x)$ are both positive constants p and q, all solutions are oscillatory except constant multiples of one solution which does not vanish on I, and which together with all of its derivatives approaches zero as x tends to infinity.*

1. We first consider the case where $p(x) \leqq 0$ and $q(x) > 0$. For this case the following lemma will be of fundamental importance.

LEMMA 1.1. *If $p(x) \leqq 0$, $q(x) \geqq 0$ and $u(x)$ is any solution of (L) satisfying the initial conditions*

$$u(c) \geqq 0 , \qquad u'(c) \leqq 0 , \qquad u''(c) > 0 , \tag{1}$$

(where c is an arbitrary number greater than a), then

$$u(x) > 0 , \qquad u'(x) < 0 , \qquad u''(x) > 0 , \tag{2}$$

for $x \in [a, c)$.

Proof. From (1) it is clear that the inequalities (2) hold in an interval (b, c), $b < c$. If the inequalities (2) failed to hold in the interval $[a, c)$ there would be a first point c', to the left of c, where the function $u(x)u'(x)u''(x)$ vanished. On the other hand

$$\begin{aligned} u(x)u'(x)u''(x))' &= (u'(x))^2u''(x) + u(x)(u''(x))^2 \\ &\quad - u(x)u'(x)(-p(x)u'(x) - q(x)u(x)) > 0 , \end{aligned}$$

for $x \in (c', c)$.

On integrating the above inequality from c' to c we would have

$$0 < \int_{c'}^{c} (u(t)u'(t)u''(t))'dt = u(c)u'(c)u''(c)$$

which is a contradiction.

The following theorem can be derived from a result due to Hartman and Winter [13]. For completeness, we shall present an elementary proof based on Lemma 1.1.

THEOREM 1.1. *If* $p(x) \leqq 0$ *and* $q(x) > 0$, *then* (L) *has a solution* $w(x)$ *with the following properties*:

$$\begin{gathered} w'''(x)w''(x)w'(x)w(x) \neq 0 \qquad x \in [a, \infty), \\ \operatorname{sgn} w(x) = \operatorname{sgn} w''(x) \neq \operatorname{sgn} w'(x) = \operatorname{sgn} w'''(x), \\ \lim_{x \to +\infty} w''(x) = \lim_{x \to +\infty} w'(x) = 0 , \end{gathered}$$

and $w(x)$ *is asymptotic to a finite constant.*

Proof. For every positive integer n greater than a, let $y_n(x)$ be a solution of (L) satisfying the initial conditions

$$y_n(n) = 0 , \qquad y'_n(n) = 0 , \qquad y''_n(n) > 0 .$$

By Lemma 1.1, we have

$$(3) \qquad y_n(x) > 0 , \qquad y'_n(x) < 0 , \qquad y''_n(x) > 0$$

for $x \in [a, n)$. Let $z_1(x)$, $z_2(x)$, $z_3(x)$ be a set of three linearly independent solutions of (L). By multiplying each $y_n(x)$ by a suitable constant we may assume that

$$y_n(x) = c_{1n}z_1(x) + c_{2n}z_2(x) + c_{3n}z_3(x)$$

with

$$(4) \qquad c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1 .$$

Since the three sequences $\{c_{in}\}$, $i = 1, 2, 3$, are bounded, there exists a sequence of integers $\{n_j\}$ such that the subsequences $\{c_{in_j}\}$ converge to numbers c_i, $i = 1, 2, 3$. From (4) we see that

$$(5) \qquad c_1^2 + c_2^2 + c_3^2 = 1 .$$

We now consider the solution

$$(6) \qquad w(x) = c_1z_1(x) + c_2z_2(x) + c_3z_3(x) .$$

Since the sequences $\{y_{n_j}(x)\}$, $\{y'_{n_j}(x)\}$, $\{y''_{n_j}(x)\}$ converge uniformly to the functions $w(x)$, $w'(x)$, $w''(x)$ on any finite subinterval of $[a, \infty)$, it follows from (3) that

$$(7) \qquad w(x) \geqq 0 , \qquad w'(x) \leqq 0 , \qquad w''(x) \leqq 0 ,$$

and

$$w'''(x) = -p(x)w'(x) - q(x)w(x) \leqq 0$$

for $x \in [a, \infty)$. If equality held at a point $\bar{x}$ in the first inequality (7), then

$$w(x) \equiv 0 \quad \text{for} \quad x \in [\bar{x}, \infty) ,$$

which contradicts (5) and (6).
Thus

$$w(x) > 0 , \quad x \in [a, \infty) . \tag{8}$$

By a similar argument

$$w'(x) < 0 , \quad w''(x) > 0 , \quad w''(x) < 0 , \qquad \text{for all } x \in [a, \infty) . \tag{9}$$

From (8) and (9), it follows at once that

$$\lim_{x\to+\infty} w'(x) = \lim_{x\to+\infty} w''(x) = 0$$

and $w(x)$ is asymptotic to a finite constant.

LEMMA 1.2. *If $p(x) \leqq 0$, $q(x) > 0$ and $y_1(x)$, $y_2(x)$ are two independent solution of (L) such that either*

$$y_1(x_0) = y_2(x_0) = 0 , \tag{10}$$

or

$$y_1'(x_0) = y_2'(x_0) = 0 \tag{11}$$

($x_0 \in [a, \infty)$, arbitrary) then

$$W(y_1, y_2) = y_1(x)y_2'(x) - y_2(x)y_1'(x) \neq 0 \qquad \textit{for} \quad x > x_0 .$$

Proof. If

$$y_1(\bar{x})y_2'(\bar{x}) - y_2(\bar{x})y_1'(\bar{x}) = 0$$

held for a point $\bar{x} > x_0$, there would exist constants c_1 and c_2 such that

$$c_1 y_1(\bar{x}) + c_2 y_2(\bar{x}) = 0$$
$$c_1 y_1'(\bar{x}) + c_2 y_2'(\bar{x}) = 0$$

with $c_1^2 + c_2^2 \neq 0$.
By Lemma 1.1, if $w(x)$ is the nontrivial solution

$$w(x) = c_1 y_1(x) + c_2 y_2(x)$$

then $w(x)\ w'(x) \neq 0$ for $x < x_0$. But this contradicts (10) or (11).

REMARK. It follows immediately from the above that if $p(x) \leqq 0$,

$q(x) > 0$ and $y_1(x)$ and $y_2(x)$ are two independent solutions satisfying either (10) or (11), then the zeros of $y_1(x)$ and $y_2(x)$ *separate* to the right of x_0; i.e., between any two zeros of $y_1(x)$ to the right of x_0, there is precisely one zero of $y_2(x)$.

LEMMA 1.2′. *Suppose $p(x) \leqq 0$ and $q(x) > 0$. If $u(x)$ and $v(x)$ are two nontrivial solutions of (L) such that*

$$u(x_0) = v(x_0) = 0$$

and $u(x)$ is oscillatory, then $v(x)$ is also oscillatory.

Proof. If $u(x)$ and $v(x)$ are dependent there is nothing to prove. If $u(x)$ and $v(x)$ are independent the result follows easily from the above remark.

LEMMA 1.2″. *Suppose $p(x) \leqq 0$ and $q(x) > 0$. If (L) has one oscillatory solution and $u(x)$ is any nontrivial solution with either*

$$u(x_0) = 0 \ , \quad or \quad u'(x_0) = 0 \ ,$$

(x_0 arbitrary) then $u(x)$ is also oscillatory.

Proof. Let $v(x)$ be an oscillatory solution of (L) which vanishes at x_1 and suppose $u(x_0) = 0$. Construct a solution $z(x)$ of (L) such that $z(x_0) = z(x_1) = 0$, $z(x) \not\equiv 0$. Applying Lemma 1.2′ first to the solutions $v(x)$ and $z(x)$ at the point x_1, we see that $z(x)$ is oscillatory. Next, applying Lemma 1.2′ to the solutions $z(x)$ and $u(x)$ at the point x_0, we see that $u(x)$ is oscillatory. If $u'(x_0) = 0$ and $u(x_0) \neq 0$, consider the solution $y(x)$ such that $y(x_0) = y'(x_0) = 0$ and $y''(x_0) = 1$. By the above argument $y(x)$ is oscillatory and from Lemma 1.2,

$$W(u(x)y(x)) = u(x)y'(x) - y(x)u'(x) \neq 0 \qquad \text{for} \quad x > x_0 \ ;$$

consequently $u(x)$ is oscillatory.

The above result shows that whenever $p(x) \leqq 0$, $q(x) > 0$ and (L) has one oscillatory solution, then for any nontrivial nonoscillatory solution $u(x)$, $u(x)u'(x) \neq 0$, $x \in [a, \infty)$. The following theorem will place even stronger restrictions on the nonoscillatory solutions in the event that (L) has oscillatory solutions.

THEOREM 1.2. *Suppose $p(x) \leqq 0$ and $q(x) > 0$. A necessary and sufficient condition for (L) to have oscillatory solutions is that for any nontrivial nonoscillatory solution $u(x)$,*

(12) $$u(x)u'(x)u''(x) \neq 0\ , \quad \operatorname{sgn} u(x) = \operatorname{sgn} u''(x) \neq \operatorname{sgn} u'(x)$$

for all $x \in [a, \infty)$, *and*

(13) $$\lim_{x\to+\infty} u'(x) = \lim_{x\to+\infty} u''(x) = 0\ , \quad \lim_{x\to+\infty} u(x) = c \neq \pm\infty\ .$$

Proof. The sufficiency is immediate; indeed if any nontrivial nonoscillatory solution $u(x)$ satisfies (12), any nontrivial solution which vanishes once is oscillatory. To prove the necessity, let us assume that (L) has oscillatory solutions and that $u(x) \neq 0$ is a nonoscillatory solution. By the above Lemma 1.2″ $u(x)u'(x) \neq 0$ for all $x \in [a, \infty)$. Let us assume without loss of generality that $u(x)$ is positive. Suppose $u'(x)$ were positive. Then for a suitable positive constant $\underline{b}$,

$$u(a) - \underline{b}w(a) = 0\ .$$

Here $w(x)$ is the nonvanishing solution whose existence was shown in Theorem 1.1 which we also take to be positive. We now consider the solution

$$v(x) = u(x) - \underline{b}w(x)\ ;$$

as $\operatorname{sgn} w(x) \neq \operatorname{sgn} w'(x) = -1$, $v'(x) = u'(x) - bw'(x) > 0$ for all $x \in [a, \infty)$. On the other hand, $v(a) = 0$ and thus by Lemma 1.2″ $v(x)$ would be oscillatory. This contradiction shows that $u'(x)$ is always negative. Since $u(x)$ satisfies (L) and $p(x) \leqq 0$, $q(x) > 0$, $u'''(x) = -p(x)u'(x) - q(x)u(x) < 0$ for all $x \in [a, \infty)$. Hence, $u''(x)$ is eventually of one sign. It is impossible that $u''(x) < 0$ from a certain point on, for if $u'(x) < 0$ and $u''(x) < 0$ from a certain point on, $u(x)$ would eventually be negative. Thus for a certain number $\bar{x} \in [a, \infty)$.

$$u(x) > 0\ , \quad u'(x) < 0\ , \quad u''(x) > 0$$

for $x \geqq \bar{x}$. By Lemma 1.1

$$u(x) > 0\ , \quad u'(x) < 0\ , \quad u''(x) > 0$$

for $x \in [a, \bar{x})$. Hence for all $x \in [a, \infty)$, $u(x)u'(x)u''(x) \neq 0$, $\operatorname{sgn} u(x) = \operatorname{sgn} u''(x) \neq \operatorname{sgn} u'(x) = \operatorname{sgn} u'''(x)$. The relations (13) follows at once from the above.

LEMMA 1.3. *If* $p(x) \leqq 0$, $q(x) > 0$ *and* $u(x) \not\equiv 0$ *is a nonoscillatory of* (L), *there exists a number* $c \in [a, \infty)$ *such that either*

(i) $u(x)u'(x) \leqq 0$ *for* $x \geqq c$ *or*

(ii) $u(x)u'(x) \geqq 0$ *for* $x \geqq c$, *and* $u(x) \neq 0$ *for* $x \geqq c$. *If* (i) *holds then*

$$u(x)u'(x)u''(x) \neq 0\ ,$$

$$\operatorname{sgn} u(x) = \operatorname{sgn} u''(x) \neq \operatorname{sgn} u'(x) ,$$

for all $x \in [a, \infty)$,

$$\lim_{x\to+\infty} u''(x) = \lim_{x\to+\infty} u'(x) = 0 ,$$

and $u(x)$ *is asymptotic to a finite constant.*

Proof. If $u(x) \neq 0$ is a nonoscillatory solution of (L), it follows from Lemma 1.1 that $u(x)$ cannot have more than one double zero; thus there exists a number b such that $u(x) \neq 0$ for $x \geqq b$. Without loss of generality let us assume $u(x) > 0$ for $x \geqq b$. We assert that $u'(x)$ cannot change signs more than twice in $[b, \infty)$. In fact, if we assume that x_1 and x_2 are two consecutive points in $[b, \infty)$ where $u'(x)$ changes sign, then by multiplying (L) by $u'(x)$ and integrating by parts between x_1 and x_2, we have

$$\begin{aligned} 0 &= \int_{x_1}^{x_2} u''(x)u'(x)dx + \int_{x_1}^{x_2} p(x)u'(x)^2dx + \int_{x_1}^{x_2} q(x)u(x)u'(x)dx \\ &= u''(x)u'(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} u''(x)^2dx + \int_{x_1}^{x_2} p(x)u'(x)^2dx \\ &+ \int_{x_1}^{x_2} q(x)u'(x)u(x)dx \\ &= -\int_{x_1}^{x_2} u''(x)^2dx + \int_{x_1}^{x_2} p(x)u'(x)^2dx + \int_{x_1}^{x_2} q(x)u(x)u'(x)dx . \end{aligned}$$

Since $p(x) \leqq 0$ and $q(x) > 0$, it follows from the above that $u(x)u'(x)$ is positive in (x_1, x_2), and from this condition the assertion follows easily. Thus there exists a c such that either $u(x)u'(x) \leqq 0$ for $x > c$, or $u(x)u'(x)u'(x) \geqq 0$, for $x > c$. If the first alternative holds then

$$u'''(x) = -p(x)u'(x) - q(x)u(x) < 0 ,$$

for $x \geqq c$ and by essentially repeating part of the argument given in the above Theorem 1.2, one can show that

$$u(x)u'(x)u''(x) \neq 0 , \quad \operatorname{sgn} u''(x) \neq \operatorname{sgn} u'(x) \neq \operatorname{sgn} u(x)$$

$x \in [a, \infty)$, and

$$\lim_{x\to+\infty} u'(x) = \lim_{x\to+\infty} u''(x) = 0 .$$

We now derive an oscillation condition for (L) under the conditions $p(x) \leqq 0$ and $q(x) > 0$.

THEOREM 1.3. *If* $p(x) \leqq 0$, $q(x) > 0$ *and*

$$\int_a^\infty \left[q(x) - \frac{2}{2\sqrt{3}}(-p(x))^{3/2} \right] dx = +\infty ,$$

then (L) *has oscillatory solutions.*

We note (see Synopsis) that this condition is necessary as well as sufficient if p and q are nonzero constants.

Proof. Suppose $u(x)$ is any nonoscillatory solution of (L). By the above Lemma 1.3, there exists a number $\underline{c}$ such that either

$$t(x) = u'(x)/u(x) \geqq 0 , \quad x \geqq \underline{c} \tag{41a}$$

or

$$t(x) = u'(x)/u(x) \leqq 0 , \quad x \geqq \underline{c} . \tag{41b}$$

We assert that (41a) is impossible. To prove this we assume the contrary and observe that $t(x)$ satisfies the second-order nonlinear Riccati equation

$$t''(x) + 3t'(x)t(x) = -(t(x)^3 + p(x)t(x) + q(x)) . \tag{15}$$

If $t(x) \geqq 0$, for $x \geqq \underline{c}$, then by considering the minimum of the function

$$F(y, x) = y^3 + p(x)y + q(x) \quad \text{for} \quad y \geqq 0 ,$$

and substituting this minimum in (15), we would find that

$$\frac{d}{dx}\left(t'(x) + \frac{3}{2}t^2(x)\right) \leqq -q(x) + \frac{2}{3\sqrt{3}}(-p(x))^{3/2} . \tag{16}$$

From the condition of the theorem it would then follow that

$$\begin{aligned} t'(x) \leqq t'(c) &+ \frac{3}{2}t^2(c) - \frac{3}{2}t^2(x) \\ &- \int_c^x \left[q(t) - \frac{2}{3\sqrt{3}}(-p(t))^{3/2} \right] dt \\ &\longrightarrow -\infty \quad \text{as} \quad x \longrightarrow +\infty ; \end{aligned}$$

consequently $t(x)$ would eventually become negative. Hence (41a) is impossible and $u(x)u'(x) \geqq 0$ for $x \geqq c$.

By Lemma 1.3,

$$u(x)u'(x)u''(x) \neq 0 , \quad \text{and} \quad \operatorname{sgn} u(x) \neq \operatorname{sgn} u'(x) \neq \operatorname{sgn} u''(x)$$

for all $x \in [a, \infty)$. Since $u(x)$ was taken to be any nonoscillatory solution it now follows from Theorem 1.2 that (L) has oscillatory solutions.

Greguš [11] has shown that if $p(x) \leqq 0$, $q(x) \geqq 0$, $2q(x) - p'(x) > 0$

except at isolated points, and (L) has one oscillatory solution, then all solutions oscillate except constant multiples of one nonvanishing solution. In Theorem 1.4 below we shall establish another condition which will insure this type of behavior. Although Greguš' method of obtaining the nonvanishing solution is based on the inequality $2q - p' \geqq 0$, his construction is similar to that used in Theorem 1.1. As the following example shows, the condition $2q - p' \geqq 0$ is not necessary for oscillation when $p(x) \leqq 0$ and $q(x) > 0$.

EXAMPLE 1.5. Consider the differential equation

$$y''' - (2 - \sin x^2)^{2/3} y' + \left(\frac{4}{3\sqrt{3}} + b\right) y = 0\,, \quad b > 0\,. \tag{17}$$

Here $p(x) = -(2 - \sin x^2)^{2/3} < 0$, $q(x) = 4/3\sqrt{3} + b > 0$ and

$$\int_a^\infty \left[q(t) - \frac{3}{3\sqrt{3}}(-p(t))^{3/2}\right] dt = \int_a^\infty \left(b - \frac{2}{3\sqrt{3}} \sin t^2\right) dt = +\infty\,.$$

and thus by Theorem 1.3, the equation has oscillatory solutions.

On the other hand

$$2q(x) - p'(x) = \left(2b + \frac{8}{3\sqrt{3}}\right) + \frac{4}{3} x \cos x^2 (2 - \sin x^2)^{-1/3}\,,$$

which is negative for arbitrarily large values of x. Therefore we cannot use Greguš' condition to show that all nonoscillatory solutions of equation (17) are constant multiples of one nonvanishing solution. However Theorem 1.4 below will show that this is still true for this example.

LEMMA 1.4. *If* $q(x) > 0$ (< 0) *and*

$$2\frac{p(x)}{q(x)} + \frac{d^2}{dx^2}(q(x)^{-1}) \leqq 0 \ (\geqq 0)$$

the absolute values of a solution at its successive maxima and minima form a nondecreasing (nonincreasing) sequence.

Proof. If $u(x)$ is any solution of (L), then as can be verified through differentiation, we have the identity

$$\begin{aligned} H[u(x)] &= u^2(x) + \frac{2u'(x)u''(x)}{q(x)} + \frac{q'(x)u'(x)^2}{q(x)^2} \\ &= H[u(a)] + 2\int_a^x \frac{u''(t)^2}{q(t)} dt - \int_a^x \left[\frac{2p(t)}{q(t)} + \frac{d^2 q(t)^{-1}}{dt^2}\right] u'^2(t) dt\,. \end{aligned}$$

By the conditions of the theorem $H[u(x)]$ is a nondecreasing (nonincreasing) function of x. At a maximum or minimum point of $u(x)$ where $u'(x) = 0$, $H[u(x)] = u^2(x)$; hence the squares of the maxima and minima of $u(x)$, and hence the corresponding values of $|u(x)|$ form a nondecreasing (nonincreasing) sequence.

THEOREM 1.4. *If* $q(x) > 0$, $p(x) \leqq 0$

$$\frac{2p(x)}{q(x)} + \frac{d^2}{dx^2}(q(x)^{-1}) \leqq 0 ,$$

and (L) *has one oscillatory solution then all solutions oscillate except constant multiples of the nonvanishing solution whose existence was proven in Theorem* 1.1.

Proof. Suppose that (L) had an oscillatory solution and that in addition to the nonvanishing solution $w(x)$ of Theorem 1.1 there was a second independent nonoscillatory solution $v(x)$. By Theorem 1.1 and Theorem 1.2 there would exist constants c_1 and c_2 such that

$$\lim_{x\to+\infty} w(x) = c_1 \quad \text{and} \quad \lim_{x\to+\infty} v(x) = c_2 .$$

Let b be a number such that $v(a) - bw(a) = 0$ and consider the nonzero solution

$$u(x) = v(x) - bw(x) .$$

Since $u(a) = 0$, we see by Theorem 1.2 that $u(x)$ would be oscillatory. On the other hand

$$\lim_{x\to+\infty} u(x) = c_2 - bc_1 .$$

This $c_2 - bc_1 = 0$ and $\lim_{x\to+\infty} u(x) = 0$, otherwise $u(x)$ could not be oscillatory. But, by Lemma 1.4 and the hypothesis on $p(x)$ and $q(x)$, the absolute values of the successive maxima and minima of $u(x)$ form a nondecreasing sequence and consequently

$$\limsup_{x\to+\infty} u(x) > 0 \quad \text{and} \quad \liminf_{x\to+\infty} u(x) < 0 .$$

Thus contradiction proves the theorem.

By considering the case of constant coefficients, one might be led to conjecture that whenever $p(x) \leqq 0$, $q(x) > 0$, and (L) has one oscillatory solution, then every nonoscillatory solution tends to zero as x tends to infinity. Whether or not this conjecture is true still remains an open question, although Švec [22] Villari [25] have proved it for the case when $p(x)$ is identically zero. In the following theorem we will prove it with the added restriction

$$\int_a^\infty x^2 q(x)dx = +\infty .$$

THEOREM 1.5. *If $p(x) \leqq 0$, $q(x) > 0$, $\int_a^\infty x^2 q(x)dx = +\infty$, and (L) has one oscillatory solution, then any nonoscillatory solution tends to zero as x tends to infinity.*

Proof. Let $u(x)$ and $v(x)$ be the solutions of (L) defined by the initial conditions

$$u(a) = u''(a) = 0 , \quad u'(a) = 1$$
$$v(a) = v'(a) = 0 , \quad v''(a) = 1 .$$

By Lemma 1.2
$W(u(x), v(x)) = u(x)v'(x) - v(x)u'(x) \neq 0$ for $x > a$, and since

$$W(u(a), v(a)) = 0 , \quad W'(u(a), v(a)) = 0 , \quad W''(u(a), v(a)) = 1 , \tag{18}$$

we see that

$$W(u(x), v(x)) > 0, \quad \text{for} \quad x > a . \tag{19}$$

Furthermore, as can be shown through differentiation,

$$u'(x)v''(x) - v'(x)u''(x) = 1 + \int_a^x q(t) W(u(t), v(t))dt \tag{20}$$

and

$$W''(u(x), v(x)) = 1 + \int_a^x q(t) W(u(t), v(t))dt - p(x) W(u(x), v(x)) . \tag{21}$$

From (19), (21) and the fact that $p(x) \leqq 0$, we see that

$$W''(u(x), v(x)) \geqq 1 \quad \text{for} \quad x \geqq a ;$$

therefore by (18) and (20),

$$\begin{aligned} W'(u(x), v(x)) &= u(x)v''(x) - v(x)u''(x) > x - a , \\ W(u(x), v(x)) &= u(x)v'(x) - v(x)u'(x) > \frac{(x-a)^2}{2} , \\ u'(x)v''(x) - v'(x)u''(x) &> 1 + \int_a^x q(t)\frac{(t-a)^2}{2}dt \end{aligned} \tag{22}$$

for $x > a$.

Hence by the conditions of the theorem

$$u(x)v'(x) - v(x)u'(x) > 0 , \qquad uv''(x) - v(x)u''(x) > 0 , \tag{23}$$

$$u'(x)v''(x) - v(x)u''(x) > 0$$

for $x > a$, and

$$\lim_{x \to +\infty} u(x)v'(x) - v(x)u'(x) = \lim_{x \to +\infty} u(x)v''(x) - v(x)u''(x) \tag{24}$$
$$= \lim_{x \to +\infty} u'(x)v''(x) - v'(x)u''(x) = +\infty .$$

Suppose now that $z(x)$ is any nontrivial nonoscillatory solution of (L). By Theorem 1.2 and the assumption that (L) has oscillatory solutions it follows that

$$z(x)z'(x)z''(x) \neq 0 , \quad \operatorname{sgn} z(x) = \operatorname{sgn} z''(x) \neq \operatorname{sgn} z'(x)$$

for all $x \in [a, \infty)$, and we may assume without loss of generality that

$$z(x) > 0 , \quad z'(x) < 0 , \quad z''(x) > 0 \tag{25}$$

for all $x \in [a, \infty)$.

We now consider the Wronskian

$$\begin{vmatrix} z(x) & u(x) & v(x) \\ z'(x) & u'(x) & v'(x) \\ z''(x) & u''(x) & v''(x) \end{vmatrix} ,$$

where $u(x)$ and $v(x)$ are the solutions studied in the above. By Liouville's identity,

$$\begin{vmatrix} z(x) & u(x) & v(x) \\ z'(x) & u'(x) & v'(x) \\ z''(x) & u''(x) & v''(x) \end{vmatrix} \equiv \begin{vmatrix} z(a) & u(a) & v(a) \\ z'(a) & u'(a) & v'(a) \\ z''(a) & u''(a) & v''(a) \end{vmatrix}$$
$$= \begin{vmatrix} z(a) & 0 & 0 \\ z'(a) & 1 & 0 \\ z''(a) & 0 & 1 \end{vmatrix} = z(a) .$$

Thus, on expanding the determinant, we have

$$z(x)(u'(x)v''(x) - v'(x)u''(x)) - z'(x)(u(x)v''(x) - v(x)u''(x))$$
$$+ z''(x)(u(x)v'(x) - v(x)u'(x)) = z(a) .$$

According to (23) and (25) all the terms in the left hand side of the above equation are positive and consequently,

$$0 < z(x)(u'(x)v''(x) - v'(x)u''(x)) < z(a) .$$

From (24) and the above inequality, it follows immediately that

$$\lim_{x \to +\infty} z(x) = 0 .$$

2. In this section we shall first investigate some rather general properties of the solutions of (L) for the case $p(x) \leqq 0$ and $q(x) \leqq 0$. By placing stronger conditions on $p(x)$ and $q(x)$ we shall then give two conditions under which the zeros of two linearly independent solutions of L separate. Finally we shall give an oscillation condition for the case $p(x) \leqq 0$ and $q(x) - p'(x) < 0$.

LEMMA 2.1. *If $p(x) \leqq 0$, $q(x) \leqq 0$ and $y(x)$ is any solution of (L) satisfying the initial conditions*

$$y(x_0) \geqq 0\ , \quad y'(x_0) \geqq 0\ , \quad \textit{and} \quad y''(x_0) > 0$$

($x_0 \in [a, \infty)$ arbitrary), then

$$y(x) > 0\ , \quad y'(x) > 0\ , \quad y''(x) > 0\ , \quad y'''(x) \geqq 0$$

for $x > x_0$ and

$$\lim_{x \to +\infty} y(x) = \lim_{x \to +\infty} y'(x) = +\infty\ .$$

Proof. We assert that $y''(x) > 0$ for $x \geqq x_0$. To prove this we consider the function

$$w(x) = y(x)y'(x)y''(x)\ .$$

If $y''(x)$ vanished for some value of x greater than x_0 there would be a smallest number $x_1 > x_0$ such that $y''(x_1) = 0$. Since $y(x_0) \geqq 0$, $y'(x_0)\ 0$, $y''(x_0) > 0$ we would have $y(x) > 0$, $y'(x) > 0$ for $x \varepsilon (x_0, x_1)$, $w(x_0) \geqq 0$, and $w(x_1) = 0$.

Moreover, since $p(x) \leqq 0$ and $q(x) \leqq 0$ it would follow that

$$\begin{aligned}\frac{dw(x)}{dx} &= (y''(x))^2 y(x) + y''(x)(y'(x))^2 - p(x)y'(x)^2 y(x)\\ &\quad - q(x)y(x)^2 y'(x) > 0 \qquad \text{for} \quad x \in (x_0, x_0)\ .\end{aligned}$$

But, by integrating the above inequality between x_0 and x_1, we would obtain the impossible inequality

$$0 = w(x_0) + \int_{x_0}^{x_1} w'(t)dt > 0\ .$$

Thus $y''(x) > 0$ for $x \geqq x_0$ and since $y(x_0) \geqq 0$ and $y'(x_0) \geqq 0$, we see that $y(x) > 0$, $y'(x) > 0$, and $y''(x) > 0$ for $x > x_0$. Finally $y'''(x) = -p(x)y'(x) - q(x)y(x) \geqq 0$ for $x > x_0$, and from the above inequalities it follows easily that

$$\lim_{x \to +\infty} y(x) = \lim_{x \to +\infty} y'(x) = +\infty\ .$$

LEMMA 2.2. *If $p(x) \leqq 0$, $q(x) \geqq 0$, and $u(x) \not\equiv 0$ is any nonoscillatory solution of (L) then there exists a number $c \in [a, \infty)$ such that either*

$$u(x)u'(x) > 0 \quad \text{for} \quad x \geqq c\ ,$$

or

$$u(x)u'(x) \leqq 0 \quad \text{for} \quad x \geqq c\ .$$

Proof. If $u(x)$ is any nontrivial, nonoscillatory solution of (L) then by Lemma 2.1, it follows that $u(x)$ can have at most one double-zero. Without loss of generality we may suppose that $u(x) > 0$ for $x \geqq b$. To prove the lemma it is sufficient to show that $u'(x)$ can change from negative to positive values at most once in the interval $[b, \infty)$. Let c be a point such that $u(c) > 0$, $u'(c) > 0$, and $u''(c) > 0$. By Lemma 2.1, $u(x) > 0$ and $u'(x) > 0$ for $x > c$ and the proof is complete.

THEOREM 2.1. *If $p(x) \leqq 0$ $q(x) \leqq 0$ and (L) has one oscillatory solution, then for any nonzero, nonoscillatory solution $u(x)$ there exist a number $c \in [a, \infty)$ such that*

$$\operatorname{sgn} u(x) = \operatorname{sgn} u'(x) = \operatorname{sgn} u''(x) \neq 0$$

for $x \geqq c$, and

$$\lim_{x\to+\infty} |\, u(x)\,| = \lim_{x\to+\infty} |\, u'(x)\,| = +\infty\ .$$

Proof. If $u(x) \not\equiv 0$ is any nonoscillatory solution then by the above lemma there exists a number $\underline{d} \in [a, \infty)$ such that either $u(x)u'(x) > 0$, or $u(x)u'(x) \leqq 0$, for $x \geqq \underline{d}$. Thus, $\lim_{x\to+\infty} u(x)$ exists finite or infinite. Let $v(x)$ be an oscillatory solution of (L) and consider the Wronskian $W(v(x),\ u(x)) = v(x)u'(x) - v'(x)u(x)$. $W(v(x),\ u(x))$ must certainly vanish for some values of x in the interval $[a, \infty)$, otherwise the zeros of $u(x)$ and $v(x)$ would separate and $u(x)$ would be oscillatory. If $\underline{b}$ is a zero of $W(v(x),\ u(x))$, there exist constants c_1 and c_2, both not zero, such that

$$c_1v(b) + c_2u(b) = 0\ ,$$
$$c_1v'(b) + c_2u'(b) = 0\ ,$$

and

$$c_1v''(b) + c_2u''(b) > 0.$$

We now consider the solution

$$z(x) = c_1 v(x) + c_2 u(x) \, .$$

Since $z(b) = z'(b) = 0$, and $z''(b) > 0$; it follows from Lemma 2.1 that

$$\lim_{x \to +\infty} z(x) = \lim_{x \to +\infty} z'(x) = +\infty \, . \tag{26}$$

As remarked above $\lim_{x \to +\infty} u(x)$ exists finite or infinite. If the limit were finite, we would have

$$\lim_{x \to +\infty} c_1 v(x) = \lim_{x \to +\infty} (z(x) - c_2 u(x)) = +\infty \, ,$$

and $v(x)$ could not be oscillatory. Thus $\lim_{x \to +\infty} u(x) = \pm\infty$ and from Lemma 2.2 we see that there must exist a number $c \in [a, \infty)$ such that $u(x)u'(x) > 0$ for $x \geqq c$. Without loss of generality let us suppose that $u(x) > 0$ and $u'(x) > 0$ for $x \geqq c$ so that

$$u'''(x) = -p(x)u'(x) - q(x)u(x) \geqq 0$$

for $x \geqq c$. From this it follows that for some $d \geqq c$ either $u''(x) > 0$, or $u''(x) \leqq 0$ for $x \geqq d$. If the second alternative held $\lim_{x \to +\infty} u'(x)$ would be finite since $u'(x) > 0$ for $x \geqq d \geqq c$. But in this case by (26)

$$\lim_{x \to +\infty} c_2 v'(x) = \lim_{x \to +\infty} (z'(x) - c_1 u'(x)) = +\infty$$

and $v(x)$ could not oscillate.
Hence

$$\operatorname{sgn} u(x) = \operatorname{sgn} u'(x) = \operatorname{sgn} u''(x) \neq 0$$

for $x \geqq d$, and

$$\lim_{x \to +\infty} | u(x) | = \lim_{x \to +\infty} | u'(x) | = +\infty \, .$$

Whether or not the converse of this theorem is true remain an open question. In the next theorem we will give a condition under which the converse holds.

THEOREM 2.2. *If* $p(x) \leqq 0$, $q(x) \leqq 0$, $p(x)' - 2q(x) \geqq 0$, *and*

$$\int_a^\infty t^4 (p'(t) - 2q(t)) dt = +\infty$$

then a necessary and sufficient condition for (L) *to have oscillatory solutions is that for every nonoscillatory solution* $u(x) \not\equiv 0$, *there exists a number* $c \in [a, \infty)$ *such that*

$$\operatorname{sgn} u(x) = \operatorname{sgn} u'(x) = \operatorname{sgn} u''(x) \neq 0 \tag{27}$$

for $x \geqq c$, *and*

(28) $$\lim_{x\to+\infty} | u(x) | = \lim_{x\to+\infty} | u'(x) | = -\infty .$$

Proof. The necessity follows from Theorem 2.1. To prove the sufficiency we will employ the identity

(29) $$\begin{aligned} F[y(x)] &= y'(x)^2 - 2y(x)y''(x) - p(x)y^2(x) \\ &= F[y(a)] - \int_a^x (p'(t) - 2q(t))y^2(t)dt \end{aligned}$$

which holds for any solution $y(x)$ of (L). This identity, which has played an important role in most of the previous investigations of (L), is originally due to Mammana [14]. It may be verified through differentiation.

We assume that (27) and (28) hold for any nonoscillatory solution $u(x)$ of (L). Without loss of generality we may assume that $u(x) > 0$, $u'(x) > 0$, $u''(x) > 0$, and $u''' = -p(x)u'(x) - q(x)u(x) \geqq 0$ for $x > c$; otherwise consider $-u(x)$. It follows that

$$u(x) > \frac{u''(c)}{2}(x - c)^2$$

for $x > c$, and hence by (29)

$$F[u(x)] \leqq F[u(c)] - u''(c)\int_a^x \frac{(t - c)^4}{2}(p'(t) - 2q(t))dt$$

for $x > c$. Thus, by the hypothesis of the theorem,

$$\lim_{x\to+\infty} F[u(x)] = -\infty .$$

This must be true for every nontrivial, nonoscillatory solution and therefore, to prove the existence of an oscillatory solution, it is sufficient to prove the existence of a nonzero solution $y(x)$ for which

$$\lim_{x\to+\infty} F[y(x)] \neq -\infty .$$

To this end, we choose a basis of solution of (L) $z_1(x)$, $z_2(x)$, and $z_3(x)$, and consider the sequence of solutions of $(L)\{y_n(x)\}$ defined by the initial conditions

$$y_n(n) = y'_n(n) = 0 , \quad y''_n(n) \neq 0 ,$$

and the normalization

$$y_n(x) = c_{1n}z_1(x) + c_{2n}z_2(x) + c_{2n}z_2(x) + c_{3n}z_3(x) ,$$

with $c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1$. Here n is any integer greater than a. By using the same type of argument that was used in the proof of

Theorem 1.1, we can show the existence of a sequence of integers $\{n_i\}$ such that the sequences

$$\{y_{n_i}(x)\}\ ,\quad \{y'_{n_i}(x)\}\ ,\quad \text{and}\quad \{y''_{n_i}(x)\}$$

converge uniformly on any finite subinterval of $[a, \infty)$ to $y(x)$, $y'(x)$, and $y''(x)$, where $y(x)$ is a nonzero solution of L. Since $p'(x) - 2q(x) \geqq 0$, it follows from (29) that $F[y_{n_i}(x)]$ is a nonincreasing function of x. Therefore, since $F[y_{n_i}(n_i)] = 0$, $F[y_{n_i}(x)] \geqq 0$ for $x \in [a, n_i)$. Letting n_i approach infinity, we see that $F[y(x)] \geqq 0$ for all $x \in [a, \infty)$ and hence, $\lim_{x\to+\infty} F[y(x)] \neq -\infty$. By the above remark the nonzero solution $y(x)$ must be oscillatory.

We now turn to the question of when the zeros of two different oscillatory solutions separate.

LEMMA 2.3. *If $p(x) \leqq 0$ and $p'(x) - 2q(x) \leqq 0$ then the derivative of any oscillatory solution of (L) is bounded on $[a, \infty)$.*

Proof. Let us suppose that $y(x)$ is an oscillatory solution of (L) and that $b \in [a, \infty)$ is a zero of $y''(x)$. Since the function

$$\begin{aligned} F[y(x)] &= y'^2(x) - 2y(x)y''(x) - p(x)y^2(x) \\ &= F[y(a)] - \int_a^x (p'(t) - 2q(t))y^2(t)dt \end{aligned}$$

is nonincreasing and $(x) \leqq 0$, we see that

$$y'(b)^2 \leqq y'(b)^2 - p(b)y^2(b) = F[y(b)] \leqq F[y(a)]\ .$$

Thus the values of $y'(x)$ are bounded at its relative maxima and minima and furthermore, since $y(x)$ is oscillatory, $y'(x)$ vanishes for arbitrary large values of x. From these two conditions we see once that $y'(x)$ is bounded on $[a, \infty)$.

THEOREM 2.3. *If $p(x) \leqq 0$, $q(x) \leqq 0$, and $2q(x) - p'(x) \leqq 0$, then the zeros of any two linearly independent solutions of (L) separate on $[a, \infty)$.*

Proof. It is sufficient to show that if $u(x)$ and $v(x)$ are any two linearly independent solutions of (L), then their Wronskian $W(u(x), v(x))$ does not vanish for any $x \in [a, \infty)$. If we assumed on the contrary that

$$W(u(b),\ v(b)) = u(b)v'(b) - v(b)u'(b) = 0$$

for some $b \in [a, \infty)$, then there would exist constants c_1 and c_2, both not zero, such that

$$c_1u(b) + c_2v(b) = 0 ,$$
$$c_1u'(b) + c_2v'(b) = 0 ,$$

and

$$c_1u''(b) + c_2v''(b) > 0 .$$

On considering the solution $z(x) = c_1u(x) + c_2v(x)$, it would follow from Lemma 1.1 that

$$\lim_{x\to+\infty} z(x) = \lim_{x\to+\infty} z'(x) = +\infty .$$

On the other hand, the assumptions that $p'(x) - 2q(x) \geqq 0$, $p(x) \leqq 0$, and that $u(x)$ and $v(x)$ are oscillatory, would imply, by Lemma 2.3, that both $u'(x)$ and $v'(x)$ and hence $z'(x) = c_1u'(x) + c_2v'(x)$ are bounded as x tends to infinity. From this contradiction it follows that $W(u(x), v(x)) \neq 0$ for all $x \in [a, \infty)$.

THEOREM 2.4. *If $p'(x) - 2q(x) \geqq d > 0$, $p(x) \leqq 0$, and $u(x)$ is any oscillatory solution of (L), then $u(x) \in L^2[a, \infty)$ and $\lim_{x\to+\infty} u(x) = 0$.*

Proof. Since $u(x)$ is oscillatory the function

$$\begin{aligned} F[u(x)] &= u'(x)^2 - 2u(x)u''(x) - p(x)u^2(x) \\ &= F[u(a)] - \int_a^x [p'(t) - 2q(t)]u^2(t)dt \end{aligned}$$

is nonnegative for arbitrarily large values of x, namely, those values of x for which $u(x)$ vanishes.
Thus,

$$\int_a^x u^2(t)dt \leqq \frac{1}{d}\int_a^x [p'(t) - 2q(t)]u^2(t)dt < \frac{F[u(a)]}{d} \quad \text{for all} \quad x \in [a, \infty) .$$

Hence

$$\int_a^\infty u^2(t)dt < +\infty .$$

Since the conditions this theorem include those of Lemma 2.3, it follows that $u'(x)$ is bounded. Therefore, since $u(x) \in L^2[a, \infty)$, it is easy to see that

$$\lim_{x\to+\infty} u(x) = 0 .$$

THEOREM 2.5. *If $p(x) \leqq 0$, $q(x) < 0$, and $2\dfrac{p(x)}{q(x)} + \dfrac{d^2}{dx^2}(q(x))^{-1} \geqq 0$, then the zeros of any two linearly independent oscillatory solutions of (L) separate.*

Proof. If $u(x)$ and $v(x)$ are two linearly independent oscillatory solutions of (L) then by the above conditions and Lemma 1.4 the absolute values of $u(x)$ and $v(x)$ at their successive maxima and minima points form nonincreasing sequences. Since $u(x)$ and $v(x)$ vanish for arbitrarily large values of x, it is easy to see that both $u(x)$ and $v(x)$ are bounded on $[a, \infty)$. If the Wronskian $W(u(x), v(x))$ vanished at a point $b \in [a, \infty)$, then by the same argument as was used in the proof of Theorem 2.3, there would exist constants c_1 and c_2 such that

$$\lim_{x \to +\infty} c_1 u(x) + c_2 v(x) = +\infty .$$

But this is impossible if both $u(x)$ and $v(x)$ are bounded. This contradiction shows that $w(u(x), v(x)) \neq 0$ for all $x \in [a, \infty)$, and hence, the zeros of $u(x)$ and $v(x)$ separate.

We conclude this section by deriving a sufficient condition for (L) to have two linearly independent oscillatory solutions under the conditions $p(x) \leqq 0$ and $q(x) - p'(x) < 0$.

THEOREM 2.6. *If* $p'(x) - q(x) > 0$, $p(x) \leqq 0$ *and*

$$\int_a^\infty [3\sqrt{3}(p'(x) - q(x)) - 2(-p(x))^{3/2}]dx = +\infty ,$$

then (L) *has two independent oscillatory solutions.*

Proof. By Theorem 1.3, if $p'(x) - q(x) > 0$, $p(x) \leqq 0$, and

$$\int_a^\infty [3\sqrt{3}(p'(x) - q(x)) - 2(-p(x))^{3/2}]dx = +\infty ,$$

then the adjoint of (L)

$$(L') \qquad y''' + p(x)y' + (p'(x) - q(x))y = 0$$

must have some oscillatory solutions. By considering two independent solutions with a common zero and then applying Theorem 1.2 it is easy to see that (L') has two independent oscillatory solutions $u(x)$ and $v(x)$. Furthermore, by Theorem 1.1, (L) has a solution $w(x)$ which does not vanish on $[a, \infty)$. It is well known (see for example [21]) and can be easily verified, that the Wronskians

$$V(x) = w(x)v'(x) - v(x)w'(x) = w^2(x)\frac{d}{dx}\left(\frac{v(x)}{w(x)}\right) ,$$

and

$$U(x) = w(x)u'(x) - u(x)w'(x) = w^2(x)\frac{d}{dx}\left(\frac{u(x)}{w(x)}\right) ,$$

are solutions of L. Moreover, they are linearly independent and oscillatory.

3. In this final section, we will investigate properties of solutions of (L) under the conditions $p(x) \geqq 0$, $q(x) \geqq 0$. In all of our theorems, we will also require $2q(x) - p'(x) \geqq 0$, and not identically zero in any interval. The following lemma will serve as a basic tool in our investigation.

LEMMA 3.1. *If $p(x) \geqq 0$, $q(x) \geqq 0$, $2q(x) - p'(x) \geqq 0$ and not identically zero in any subinterval of $[a, \infty)$ and $y(x) \not\equiv 0$ is a nonoscillatory solution of (L) which is eventually nonnegative with*

$$0 \leqq F[y(c)] = y'(c)^2 - 2y(c)y''(c) - p(c)y^2(c) ,$$

($c \in [a, \infty)$ arbitrary) then there exists a number $d \geqq c$ such that $y(x) > 0$, $y'(x) > 0$, $y''(x) > 0$, and $y'''(x) \leqq 0$, for $x \geqq d$.

Proof. Since

$$F[y(x)] = F[y(c)] + \int_c^x (2q(t) - p'(t))y^2(t)dt$$

is strictly increasing, nonnegative at $x = c$, and vanishes at points where $y(x)$ has a double zero, it follows that if $y(x)$ is any nonoscillatory solution which is eventually nonnegative, there exists a $c_1 \geqq c$ such that $y(x) > 0$ for $x \geqq c_1$. If b is any point in $[c_1, \infty)$ such that $y'(b) = 0$, then since

$$F[y(b)] = -y''(b)y(b) - p(b)y(b)^2 > 0 , \quad y''(b) < 0 .$$

Consequently $y'(x)$ cannot vanish more than once in $[c_1, \infty)$, and there exists a $c_2 \geqq c_1$, such that $y(x) > 0$, $y'(x) \neq 0$, for $x \geqq c_2$. We will now show that $y'(x) > 0$ for $x \geqq c_2$.

Suppose on the contrary that $y'(x) < 0$ for $x \geqq c_2$. If (i), $y''(x) \leqq 0$ for $x \geqq b \geqq c_2$, then $y'(x) \leqq y'(b) < 0$ for $x \geqq b$ so that $y(x)$ would eventually become negative in $[b, \infty)$, which is a contradiction. If (ii) $y''(x) \geqq 0$ for $x \geqq b \geqq c_2$, then since $y'(x) < 0$ for $x \geqq b$, we would have $\lim_{x \to +\infty} y'(x) = 0$, and consequently

$$\begin{aligned} \lim_{x \to +\infty} F[y(x)] &= \lim_{x \to +\infty} y'(x)^2 - 2y''(x)y(x) - p(x)y(x)^2 \\ &= \lim_{x \to +\infty} - 2y(x)y''(x) - p(x)y^2(x) \leqq 0 \end{aligned}$$

which would contradict the fact that $F[y(c)]$ is nonnegative and $F[y(x)]$ is strictly increasing. Finally, suppose (iii), $y''(x)$ changed signs for arbitrarily large values of x. Since for positive $\varepsilon > 0$, there would have to exist arbitrarily large values of x for which $0 > y'(x) > -\varepsilon$, there would exist relative maxima $\bar{x}$ of $y(x)$ with

$$0 > y'(x) > -\varepsilon , \quad y''(\bar{x}) = 0 .$$

We would then have

$$F[y(\bar{x})] \leqq \varepsilon^2 - p(\bar{x})y^2(\bar{x}) \leqq \varepsilon^2$$

for arbitrarily large values of $\bar{x}$, which would imply that

$$\lim_{x \to +\infty} F[y(x)] \leqq 0 .$$

This, as in the above would be a contradiction. Thus, since the three mutually exclusive and exhaustive possibilities all lead to a contradiction when we assume $y'(x) < 0$ for $x \geqq c_2$, we must have $y'(x) > 0$ for $x \geqq c_2$.

From this it follows that

$$y'''(x) = -p(x)y'(x) - q(x)y(x) < 0$$

for $x \geqq c_2$ and hence, $y''(x)$ must eventually be of one sign. If $y''(x)$ were eventually negative, say for $x \geqq d$, then

$$y''(x) \leqq y''(d) < 0 \quad \text{for} \quad x \geqq d$$

so that $\lim_{x \to +\infty} y'(x) = -\infty$. Hence, $y''(x)$ is eventually positive and there exists a number d such that

$$y(x) > 0 , \quad y'(x) > 0 , \quad y''(x) > 0$$

and $y'''(x) < 0$, for $x \geqq d$.

LEMMA 3.2. *If $y(x) \in C^3[a, \infty)$, and $y(x) > 0$, $y'(x) > 0$, $y'''(x) \leqq 0$, for $x \geqq a$, then*

$$\liminf_{x \to +\infty} \frac{y(x)}{xy'(x)} \geqq 1/2 .$$

Proof. Consider the function

$$G(x) = (x - a)y(x) - \frac{(x - a)^2}{2} y'(x) .$$

$$\begin{aligned} G(a) = 0, \text{ and } G'(x) &= y(x) - \frac{(x - a)^2}{2} y''(x) \\ &= y(a) + y'(a)(x - a) + \frac{(x - a)^2}{2}[y''(c) - y''(x)] , \end{aligned}$$

where $a < c < x$. Since $y'''(x) \leqq 0$ for $x \geqq a$, $y''(c) \geqq y''(x)$, and hence,

$G'(x) > 0$ for $x > a$. Thus since $G(a) = 0$, $G(x) > 0$, and hence,

$$\frac{y(x)}{(x-a)y'(x)} > 1/2, \quad \text{for} \quad x > a .$$

From this, the assertion of the lemma follows immediately.

By means of the two preceding lemmas and the classical Sturm comparison theorem we shall derive an oscillation condition for (L).

THEOREM 3.1. *If* $p(x) \geqq 0$, $q(x) \geqq 0$, $2q(x) - p'(x) \geqq 0$ *and not identically zero in any interval, and there exists a number* $m < 1/2$ *such that the second-order differential equation*

$$y''(x) + [p(x) + mxq(x)]y = 0$$

is oscillatory, then (L) *has oscillatory solutions. In fact, if* $y(x)$ *is any nonzero solution of* (L) *with*

$$0 \leqq F[y(c)] = y'(x)^2 - 2y(x)y''(x) - p(x)y^2(x)$$

then $y(x)$ *is oscillatory.*

Proof. Suppose that $u(x) \not\equiv 0$ were a nonoscillatory solution of (L), with $F[u(c)] \geqq 0$. Without loss of generality, we could assume $u(x)$ to be eventually nonnegative. By Lemma 3.1, there would exist a number $d \geqq c$ such that

$$u(x) > 0, \quad u'(x) > 0, \quad u''(x) > 0, \quad \text{and} \quad u'''(x) \leqq 0 \tag{30}$$

for $x \geqq d$.

Hence, by Lemma 3.2,

$$\liminf_{x \to +\infty} \frac{u(x)}{xu'(x)} \geqq 1/2 .$$

Thus, since $m < 1/2$, there would exist a number $d_1 \geqq d$, such that $u(x)/u'(x) > mx$ for $x \geqq d_1$. By writing (L) in the form of a system

$$\begin{aligned} u' &= w \\ w'' + pw + qu &= 0 , \end{aligned} \tag{31}$$

we could write the second equation in the form

$$w'' + \left[p(x) + q(x)\frac{u(x)}{w(x)}\right]w(x) = 0 . \tag{32}$$

Since by the above,

$$\begin{aligned} p(x) + q(x)\frac{u(x)}{w(x)} &= p(x) + q(x)\frac{u(x)}{u'(x)} \\ &> p(x) + mxq(x) \quad \text{for} \quad x \geqq d_1 , \end{aligned}$$

it would follow from the Sturm comparison theorem that since

$$y'' + [p(x) + mxq(x)]y = 0$$

is oscillatory, all nonzero solutions of

$$y'' + \left[p(x) + q(x)\frac{u(x)}{w(x)}\right]y = 0 \tag{33}$$

defined for $x \geqq d_1$, would oscillate. But this contradicts (30), for the particular solution $w(x) = u'(x)$. Thus, the assumption that $u(x)$ is nonoscillatory leads to a contradiction.

Hanan [12], has shown that if $2q(x) - p'(x) \geqq 0$, and not identically zero in any interval, and (L) has one oscillatory solution, then any solution which vanishes once is oscillatory. The following theorem, which generalizes this result, will be useful in the remainder of our investigation.

THEOREM 3.2. *If $2q(x) - p'(x) \geqq 0$, and not identically zero in any interval, and (L) has one oscillatory solution, then a necessary and sufficient condition for a solution $u(x) \not\equiv 0$ to be nonoscillatory is that $F[u(x)] < 0$ for all $x \in [a, \infty)$.*

Proof. The sufficiency is trivial. Indeed, if

$$F[u(x)] = u'(x)^2 - 2u(x)u''(x) - p(x)u^2(x)$$

is negative for all $x \in [a, \infty)$ it is clear that $u(x)\, x \neq 0$ for all $x \in [a, \infty)$. To prove the necessity we will show that if (L) has one oscillatory solution and $u(x) \not\equiv 0$, $F[u(c)] \geqq 0$, $c \in [a, \infty)$ arbitrary, then $u(x)$ is oscillatory. If $u(c) = 0$, the assertion follows from Hanan's result. If $u(c) \neq 0$, we consider a second solution defined by the initial conditions

$$v(c) = 0 , \quad v'(c) = u(c) , \quad v''(c) = u'(c) .$$

Since $v(x)$ is not identically zero and vanishes at c, we see from Hanan's result that $v(x)$ is oscillatory. Furthermore, for any constants c_1 and c_2 both not zero

$$\begin{aligned} &F[c_1u(c) + c_2v(c)] \\ &\quad = c_1^2F[u(c)] \\ &\qquad + 2c_1c_2(u'(c)v'(c) - u''(c)v(c) - v''(c)u(c) - p(c)u(c)v(c)) \\ &\qquad + c_2^2F[v(c)] \end{aligned} \tag{34}$$

$$\begin{aligned} &= c_1^2 F[(u)] + 2c_1c_2(u'(c)u(c) - u'(c)u(c)) + c_2^2 u(c)^2 \\ &= c_1^2 F[u(c)] + c_2^2 u(c)^2 \geqq 0 \,. \end{aligned}$$

Consider the Wronskian

$$W(u(x),\ v(x)) = u(x)v'(x) - v(x)u'(x) \,.$$

If $W(u(x),\ v(x))$ vanished at a point $d > c$, then there would exist constants c_1 and c_2 such that

$$\begin{aligned} c_1 u(d) + c_2 v(d) = 0 \,, \\ c_2 u'(d) + c_2 v'(d) = 0 \,, \end{aligned}$$

and

$$c_1^2 + c_2^2 \neq 0 \,.$$

If $z(x)$ were the solution $c_1u(x) + c_2v(x)$, then $F[z(d)] = 0$ and by (34),

$$F[z(c)] = c_1^2 F[u(c)] + c_2^2 u(c)^2 \geqq 0 \,.$$

But

$$\begin{aligned} F[z(d)] &= F[z(c)] + \int_c^d (2q(x) - p'(x))z^2(x)dx \\ &> F[z(c)] \geqq 0 \,. \end{aligned}$$

This contradiction shows that $W(u(x),\ v(x)) \neq 0$ for $x > c$. Hence, since $v(x)$ is oscillatory, $u(x)$ is oscillatory.

The next theorem shows that solutions satisfying the conditions of Theorem 3.2 actually exist. Since the method of construction has already been given by Greguš [11], and is similar to the method used in Theorem 1.1 and Theorem 2.2, we will only sketch the proof.

THEOREM 3.3. *If* $2q(x) - p'(x) \leqq 0$, *and not identically zero in any interval then* (L) *has a solution* $u(x)$ *for which*

$$\begin{aligned} F[u(x)] &= u'(x)^2 - 2u(x)u''(x) - p(x)u^2(x) \\ &= F[u(a)] + \int_a^x (2q(t) - p'(t))u^2(t)dt \end{aligned}$$

is always negative. Consequently $u(x)$ *is nonoscillatory.*

Proof. For each integer n a, we consider the solution $y_n(x)$ defined by the initial conditions

$$y_n(n) = y_n'(n) = 0 \,,\ y_n''(n) \neq 0 \,,$$

and the normalization

$$y_n(x) = c_{1n}z_1(x) + c_{2n}z_2(x) + c_{3n}z_3(x) ,$$
$$c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1 ,$$

where $z_1(x)$, $z_2(x)$, and $z_3(x)$ are a basis of solutions of (L). As in the proof of Theorem 1.1 and Theorem 2.2, one can show the existence of a sequence of integers $\{n_i\}$ such that the sequence $\{y_{n_i}(x)\}$, $\{y'_{n_i}(x)\}$, and $\{y''_{n_i}(x)\}$ converge uniformly on any finite subinterval of $[a, \infty)$ to $u(x)$, $u'(x)$, and $u''(x)$ where $u(x)$ is a nontrivial solution of (L).

Let b be an arbitrary point in the interval $[a, \infty)$. Since $F[y_{n_i}(n_i)] = 0$, and $F[y_{n_i}(x)]$ is strictly increasing, $F[y_{n_i}(b)] < 0$ for $n_i > b$. Thus, since

$$F[u(b)] = \lim_{n_i \to +\infty} F[y_{n_i}(b)] ,$$

$F[u(b)] \leqq 0$. As b is arbitrary $F[u(x)] \leqq 0$ for all $x \in [a, \infty)$. Finally if equality cannot hold at any point c, since this would imply that $F[u(x)] > 0$ for $x > c$, as $F[u(x)]$ is strictly increasing.

In Theorem 3.5 below we will need a result due to Hanan [12], which we state as a separate theorem.

THEOREM 3.4. *If $p(x) \geqq 0$, $q(x) \geqq 0$, and the second-order differential equation*

$$y''(x) + p(x)y = 0$$

nonoscillatory, and (L) has one oscillatory solution, then any nontrivial, nonoscillatory solution does not vanish on $[a, \infty)$ and is always decreasing in absolute value.

In case the oscillation criteria of Theorem 3.1 fails, the following theorem gives a nonoscillation condition.

THEOREM 3.5. *Suppose $p(x) \geqq 0$, $q(x) \geqq 0$, $2q(x) - p'(x) \geqq 0$ and not identically zero in any interval $p(\infty) = 0$, and $\int_a^\infty q(x)dx < +\infty$, then if the second-order differential equation*

$$y'' + \left[p(x) + \frac{3}{2}\int_x^\infty q(t)dt\right]y = 0 \tag{35}$$

is nonoscillatory, (L) has no oscillatory solutions.

Proof. We will prove that if the first conditions are met, and (L) possesses oscillatory solutions, then the equation (35) is oscillatory.

If the second-order equation

$$y'' + p(x)y = 0 \tag{36}$$

is oscillatory the assertion follows trivially from the Sturm comparison theorem. If (35) is nonoscillatory then, by Theorem 3.4, any nontrivial nonoscillatory of L, and in particular the nonvanishing solution $u(x)$ of Theorem 3.3, is steadily decreasing in absolute value on the interval $[a, \infty)$, Let us suppose that $u(x) > 0$ so that $u(x)$ is asymptotic to a nonnegative constant c as x tends to infinity.

It is clear that for any positive $\varepsilon > 0$, we must have

$$0 \leqq u''(x) < \varepsilon \tag{37}$$

for arbitrarily large values of x. Indeed, if $u''(x)$ were eventually negative, then since $u'(x) \leqq 0$ for all $x \in [a, \infty)$, $u(x)$ would eventually become negative; if $u''(x) \geqq \varepsilon > 0$ from a certain point on, $u(x)$ could not be monotonically decreasing. Since $p(\infty) = 0$, for sufficiently large values of x,

$$0 \leqq p(x) < \varepsilon\ . \tag{38}$$

Thus, since $\lim_{x\to+\infty} u(x) = c$, it follows that for arbitrarily large values of x,

$$\begin{aligned} &0 \leqq p(x) < \varepsilon\ , \qquad 0 < u(x) < c + 1 \\ &0 \leqq u''(x) < \varepsilon\ , \end{aligned} \tag{39}$$

and hence, for arbitrarily large values of x,

$$\begin{aligned} F[u(x)] &= u'(x)^2 - 2u(x)u''(x) - p(x)u(x)^2 \\ &\geqq -2\varepsilon(c+1) - \varepsilon(c+1)^2\ . \end{aligned} \tag{40}$$

Since $F[u(x)]$ is always negative (see Theorem 3.3) and strictly increasing, (40) implies that

$$\lim_{x\to+\infty} F[u(x)] = 0\ ,$$

or

$$\begin{aligned} &u'(x)^2 - 2u(x)u''(x) - p(x)u^2(x) \\ &\quad = -\int_x^\infty (2q(t) - p'(t))u^2(t)dt\ . \end{aligned}$$

Thus, since $u(x) \neq 0$ for $a \leqq x < +\infty$, we see that

$$\begin{aligned} &\frac{2u''(x)}{u(x)} - \frac{u'(x)^2}{u^2(x)} \\ &\quad = -p(x) + \frac{2}{u(x)^2}\int_x^\infty (2q(t) - p'(t))u^2(t)dt\ . \end{aligned} \tag{41}$$

Using the fact, established earlier in the proof that $u(x)$ is decreasing in absolute value and the assumption that $p(\infty) = 0$, we may conclude that

$$\frac{1}{u(x)^2}\int_x^\infty (2q(t) - p'(t))u^2(t)dt$$
$$\leqq \int_x^\infty 2q(t) - p'(t)dt = 2\int_x^\infty q(t)dt + p(x) .$$

Hence, by (41)

$$\frac{2u''(x)}{u(x)} - \frac{u'(x)^2}{u(x)^2} \leqq 2\int_x^\infty q(t)dt \tag{42}$$

for all $x \in [a, \infty)$.

Let

$$y(x) = u(x)\int_a^x z(t)dt ,$$

where $z(x)$ is chosen so that $y(x)$ is a solution of (L). We have on substituting into (L),

$$z'' + \frac{3u'(x)}{u(x)}z'(x) + \left[p(x) + \frac{3u''(x)}{u(x)}\right]z(x) = 0 ,$$

and after making the substitution

$$z(x) = w(x)u(x)^{-3/2} ,$$

we obtain

$$w''(x) + \left[p(x) + \frac{3}{4}\left(\frac{2u''(x)}{u(x)} - \frac{u'(x)^2}{u(x)^2}\right)\right]w(x) = 0 . \tag{43}$$

Any nonzero solution $w(x)$ of (43) must be oscillatory; otherwise the solution

$$y(x) = u(x)\int_a^x u(t)^{-3/2}w(t)dt \tag{44}$$

would be a nontrivial, nonoscillatory solution of (L) which would vanish for $x = a$, and this would contradict the assumption that (L) has oscillatory solutions and Theorem 3.4. Since by (42), we have

$$p(x) + \frac{3}{4}\left(\frac{2u''(x)}{u(x)} - \frac{u'(x)^2}{u(x)^2}\right)$$
$$\leqq p(x) + \frac{3}{2}\int_x^\infty q(t)dt ,$$

it follows from the Sturm comparison theorem, that since the differential equation

$$y'' + \left[p(x) + \frac{3}{4}\left(\frac{2u''(x)}{u(x)} - \frac{u'(x)^2}{u(x)}\right)\right]y = 0$$

is oscillatory, the equation

$$x'' + \left[p(x) + \frac{3}{2}\int_x^\infty q(t)dt\right]y = 0$$

is oscillatory.

We conclude by proving a generalization of a theorem due to Zlamal [26]. At the same time, we will derive a sufficient condition for all nonoscillatory solutions to be constant multiples of one particular nonoscillatory solution.

Zlamal has shown that if $p(x) \geqq d > 0$, $q(x) > d$, and $q(x) - p'(x) \geqq 0$, then any nonoscillatory solution $y(x)$ of (L) has the property that

$$y(x) \in L^2[a, \infty) \quad \text{and}$$
$$\lim_{x\to+\infty} y(x) = \lim_{x\to+\infty} y'(x) = 0 .$$

We relax the condition that $p(x)$ be bounded below by a positive constant.

THEOREM 3.6. *If* $p(x) \geqq 0$, $q(x) \geqq d > 0$, $q(x) - p'(x) \geqq 0$, *and* $y(x)$ *is any nonoscillatory solution of* (L) *then*

$$y(x) \in L^2[a, \infty) \quad \textit{and}$$
$$\lim_{x\to+\infty} y(x) = \lim_{x\to+\infty} y'(x) = 0.$$

Moreover, all nonoscillatory solutions of (L) *are constant multiples of the nonoscillatory solution* $u(x)$ *whose existence was proven in Theorem* 3.3.

Proof. If $y(x)$ is any nonoscillatory solution of (L) we may assume that $y(x) \geqq 0$ for $x \geqq c$. From the inequality

$$\begin{aligned} & y''(x) + p(x)y(x) \\ &= y''(c) + p(c)y(c) - \int_c^x (q(t) - p'(t))y(t)dt \\ &< y''(c) + p(c)y(c) , \end{aligned}$$

which follows from integration of (L) and the conditions of the theorem, we see that

$$y''(x) + p(x)y(x)$$

is always bounded above and since $p(x)y(x) \geqq 0$ for $x \geqq c$, it follows that $y''(x)$ is bounded above for $x \geqq c$. Hence there exists a positive constant k such that

$$2y''(x) + p(x)y(x) \leqq k \tag{45}$$

for $x \geqq c$.

Since $q(x) \geqq d > 0$ and $p(x) \geqq 0$, there certainly exists a positive constant $m < 1/2$ such that the second-order differential equation

$$u''(x) + [p(x) + mxq(x)]u = 0$$

is oscillatory. Thus, since

$$2q(x) - p'(x) \geqq d + q(x) - p'(x) \geqq d > 0 \ ,$$

it follows from Theorem 3.1 that (L) has oscillatory solutions and hence, from Theorem 3.2, $F[y(x)] < 0$ for all $x \in [a, \infty)$ or

$$\begin{aligned} F[y(x)] &= y'(x)^2 - 2y(x)y''(x) - p(x)y^2(x) \\ &= F[y(a)] + \int_a^x (2q(t) - p'(t))y^2(t)dt < 0 \ , \quad x > a \ . \end{aligned} \tag{46}$$

Therefore

$$\begin{aligned} d\int_a^\infty y^2(t)dt \leqq \int_a^\infty q(t)y^2(t)dt &\leqq \int_a^\infty (2q(t) - p'(t))y^2(t)dt \\ &\leqq -F[y(a)] < +\infty \ , \end{aligned}$$

from which it follows that $y(x) \in L^2[a, \infty)$.
From (45) and (46), we see that

$$y'(x)^2 \leqq (2y''(x) + p(x)y(x))y(x) < ky(x) \ , \quad x > c \ . \tag{47}$$

If $y(x)$ did not tend to zero as x tended to infinity, the for some $\varepsilon > 0$, we would have $y(x) > 2\varepsilon$ for arbitrarily large values of x. On the other hand, since $y(x) \in L^2[a, \infty)$, we could also find arbitrarily large values of x for which $y(x) < \varepsilon$. Thus we could find sequences $\{x_n\}$ and $\{x_n^*\}$ with $x_n < x_n^* < x_{n+1}$ and

$$\lim_{x \to +\infty} x_n = \lim_{x \to +\infty} x_n^* = +\infty$$

such that

$$y(x_n) < \varepsilon \ , \quad y(x_n^*) > 2\varepsilon \ .$$

By elementary continuity considerations, we could then find sequences $\{z_n\}$ and $\{z_n^*\}$ such that

$$x_n < z_n < z_n^* < x_n^* \ ,$$

$$y(z_n) = \varepsilon\,, \quad y(z_n^*) = 2\varepsilon\,, \tag{48}$$

and

$$\varepsilon < y(x) < 2\varepsilon \quad \text{for} \quad x \in (z_n,\ z_n^*)\,. \tag{49}$$

Since $y(x) \geqq 0$ for $x \geqq c$ and the intervals (z_n, z_n^*) are disjoint, it would follow from (49) that

$$\begin{aligned}\sum_{n=1}^{\infty}(z_n^* - z_n)\varepsilon^2 &\leqq \sum_{n=1}^{\infty}\int_{z_n^*}^{z_n^*} y(t)^2 dt \\ &\leqq \int_c^{\infty} y(t)^2 dt < +\infty\end{aligned}$$

and hence,

$$\lim_{n\to+\infty}(z_n^* - z_n) = 0\,. \tag{50}$$

By the mean value theorem, there would exist a sequence of points c_n such that $z_n < c_n < z_n^*$ and

$$y'(c_n) = \frac{y(z_n^*) - y(z_n)}{z_n^* - z_n} = \frac{\varepsilon}{z_n^* - z_n}\,. \tag{51}$$

Hence, from (50)

$$\lim_{x\to+\infty} y'(c_n) = +\infty\,.$$

But, from (47) and (49),

$$y'(c_n)^2 < 2k\varepsilon\,. \tag{52}$$

Therefore the assumption that $\lim y(x) \neq 0$ leads to a contradiction; consequently

$$\lim_{x\to+\infty} y(x) = 0\,, \tag{53}$$

and from (47),

$$\lim_{x\to+\infty} y'(x) = 0\,. \tag{54}$$

Suppose now, that in addition to the nonvanishing solution $u(x)$, whose existence was proven in Theorem 3.3, (L) had a second independent nonoscillatory solution $w(x)$. If we chose c such that $v(a) - cu(a) = 0$, then the solution $w(x) = v(x) - cu(x)$ would have the property that

$$F[w(a)] = w'(a)^2 - 2w(a)w''(a) - p(a)w(a)^2 = w'(a)^2 \geqq 0\,.$$

By Theorem 3.2 and the fact, established earlier in the proof, that

(L) has oscillatory solutions, $w(x)$ would be oscillatory. If $\bar{x}$ were any zero of $w(x)$ with $\bar{x} > a$, then since $F[w(x)]$ is strictly increasing,

$$F[w(\bar{x})] = (w'(\bar{x}))^2 = F[w(a)] + \int_a^{\bar{x}} (2q(t) - p'(t))w^2(t)dt$$
$$> F[w(a)] \geqq 0 .$$

As $w(x)$ would vanish for arbitrarily large values of x

$$\limsup_{x \to +\infty} | w'(x) | > \sqrt{F[w(\bar{x})]} > 0 .$$

But, as $u(x)$ and $v(x)$ are nonoscillatory, it would follow from (54) that

$$\lim_{x \to +\infty} w(x) = \lim_{x \to +\infty} v'(x)\, cu'(x) = 0$$

which is a contradiction. Therefore every nonoscillatory solution must be a constant multiple of $u(x)$.

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References

1. G. D. Birkhoff, *On solutions of the ordinary linear differential equations of the third order*, Ann. of Math. **12** (1911), 103-127.
2. M. Greguš, *On some properties of the solutions of a homogeneous linear differential equation of the third order*, Mat-Fyz. Casopis Slovensk. Akad. Vied. **5** (1955), 73-78.
3. ———, *Über einige neue Eigenshaften der Losungen der Differentialgliechung* $y''' + Qy' + Q'y = 0$, Publ. Fac. Sci. Univ. Masaryk (1955), 237-254.
4. ———, *Über einige Zussammenhange zwichen der Integralen der gegenseitig adjunglerten linear differentialgliechungen der dritten Ordnung und uber ein Randwert problem*, Acta Fac. Nat. Univ. Comenian. Math. **1** (1956), 265-272.
5. ———, *Die Differentialgliechungen der dritten Ordnung* $y'' + 2Ay' + (A' + b) = 0$ *mit allen oszillatorischen Losungen*, Acta. Fac. Nat. Univ. Comenian. Math. **1** (1956), 41-47.
6. ———, *Über einige neue Randwert probleme einer Differentialgleichungen dritten Ordnung*, Czechoslovak Math. J. (82) **7** (1957), 41-47.
7. ———, *Über die lineare Differentialgliechungen der dritten Ordnung mit Konstanten Koeffizienten*, Acta. Fac. Nat. Univ. Comenian. Math. **2** (1957), 61-66.
8. ———, *Das homogeneors Randwert problem fur die Losungen einer linearen Differentialgliechungen dritten Ordnung*, Acta. Fac. Nat. Univ. Comenian. Math. **2** (1958), 219-228.
9. ———, *Bemerkungen zu den oscillatorischen Eigenshaften der Losungen der Differentialgleichungen dritten Ordnung*, Acta. Fac. Nat. Univ. Comenian. **3** (1958), 23-28.
10. ———, *Oszillatorische Eigenschaften der Losungen der linearen Differentialgleichungen dritten Ordnung* $y''' + 2Ay' + (A' + b)y = 0$, *wo* $A = A(x) \leqq 0$ *lst*, Czechoslovak Math. J. (84) **9** (1959), 416-428.
11. ———, *Über einige Eigenschaften der Losungen der Differentialgliechungen* $y''' + 2Ay' + (A' + b)y = 0$, $A \leqq 0$, Czechoslovak Math. J. **11** (86) (1961), 106-114.
12. M. Hanan, *Oscillation criteria for third-order linear Differential Equation*, Pacific J. Math. (3) **11** (1961), 919-944.

13. P. Hartman and A. Wintner, *Linear differential and difference equations with montone solutions*, Amer. J. Math., **75** (1953) 731–743.
14. G. Mammana, *Decomposizione delle espressioni lineari omogenee in prodotti di prodotti di fattori simbolici e applicazione relativa allo studio delle equazioni differenziali lineari*, Math. Zeit. **33** (1931), 186–231.
15. M. Ráb, *Oscillatorische Eigenshaften der Losungen linearer Differentialgliechungen 3 Ordnung*, Acta. Acad. Sci. Cechoslovenicae Basis Brunensis **27** (1955), 349–360.
16. ———, *Asymptotishe Eigenshaften der Losungen Linearer Differentialgliechung dritter Ordnung*, Publ. Fac. Sci. Univ. Masaryk (1956), 177–184.
17. ———, *Asymptotishe Eigenshaften der Losungen linearer Differentialgliechung dritter Ordnung*, Publ. Fac. Sci. Univ. Masaryk (1956), 441–454.
18. ———, *Über die Differentialgliechung* $y''' + 2A(x)y' + (A'(x) + w(x))y = 0$, Mat.-Fyz. Casopis. Slovensk. Akad. Vied **8** (1958), 115–122.
19. ———, *Über lineare Perturbationen eines systems von linearen Differentialgliechungen*, Czechoslovak Math. J. (83) **8** (1958), 222–229.
20. ———, *Über eine verallgemeinerung eines satzes von Sansone über Nichtoszillation der Losungen der Differentialgliechungen* $y''' + 2A(x)y' + (A'(x) + w(x))y = 0$, Mat.-Fys. Casopis. Slovensk. Acad. Vied. **10** (1960), 3–8.
21. G. Sansone, *Studi sulle equazioni differentiali lineari omogenee di terzo ordine nel campo reale*, Univ. Nac. Tucuman. Rev. **A6** (1948), 195–253.
22. M. Švec, *Sur une propriété des intégrales de l'équation* $y(n) + Q(x)y = 0$, $n = 3, 4$, Czechoslovak Math. J. (82) **7** (1957), 450–462.
23. ———, *Asymptotische Davstellung der Losungen der Differentialgliechung* $y(n) + Q(x)y = 0$, $n = 3, 4$. Czechoslovak Math. J. (87) **12** (1962), 572–581.
24. G. Villari, *Sul carattera oscillatorio delle soluzioni delle equazioni differenziali lineari omogenee del terzo ordine*, Boll. Un. Mat. Ital. (3) **13** (1958), 73–78.
25. ———, *Contributi allo studio asintotico del l equazione* $x'''(t) + p(t)x(t) = 0$, Ann. Math. Pura. Appl. (4) **51** (1960) 301–328.
26. M. Zlamal, *Asymptotic properties of the solutions of the third-order linear differential equations*, Publ. Fac. Sci. Univ. Masaryk (1951/6).

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